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Weak solutions of a two-phase Navier–Stokes model with a general slip law

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ABSTRACT

This work is devoted to a study of a gas–liquid Navier–Stokes model with a general slip law that allows the two phases to flow with unequal fluid velocity. The slip law is general enough to describe counter-current flow, i.e., a situation where gas and liquid move in opposite direction. Motivated by applications in the context of wellbore flow systems we assume that there is a free interface which separates the gas–liquid mixture region from a strongly gas dominated region which holds a specified positive pressure p^*. The slip law is incorporated in the mass and momentum equations resulting in a model where flow is described in terms of the gas velocity. However, the mixture momentum equation contains a generalized pressure law composed of the standard pressure function combined with three new terms that depend on parameters characterizing the difference between gas and liquid velocity. These new terms make the analysis challenging. By carefully exploiting the positive external pressure p^* we are able to obtain uniform lower and upper bounds on a pressure-related quantity. This in turn allows for various higher order regularity estimates which imply that global existence and uniqueness of weak solutions can be shown.

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1. Introduction

The drift-flux model is one of the commonly used models nowadays for the prediction of various two-phase flows. It was first developed by Zuber and Findlay [25]. It is used in chemical engineering to predict flow in bubble column reactors, in petroleum applications to model various wellbore operations related to drilling, production of oil and gas, and for the study of geothermal energy related problems and injection of CO_2. A one-dimensional transient drift-flux model can be written in the following form:

$$\begin{aligned}
&\partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] = 0,\\
&\partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] = 0,\\
&\partial_t[\alpha_g\rho_g u_g + \alpha_l\rho_l u_l] + \partial_x\left[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + P\right]\\
&\quad = -q + \partial_x[\varepsilon\partial_x u_M], \quad u_M = \alpha_g u_g + \alpha_l u_l,
\end{aligned} \tag{1.1}$$

where $\varepsilon \geq 0$. The model is supposed under isothermal conditions. The unknowns are $\rho_l(P)$, $\rho_g(P)$ the liquid and gas densities; α_l, α_g volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$; u_l, u_g velocities of liquid and gas; P common pressure for liquid and gas; and q representing external forces like gravity and friction. In the following we assume that the liquid is incompressible whereas the gas phase is described by the polytropic gas law

$$P = C\rho_g^\gamma, \quad \gamma > 1, \tag{1.2}$$

where, without loss of generality, we choose $C = 1$. Since the momentum is given only for the mixture, we need an additional closure law, a so-called hydrodynamical closure law, which connects the two phase velocities. Generally, this law should be able to take into account different *flow regimes*. A commonly used slip relation is in the form [25,8,1,15]

$$u_g = \hat{c}_0 u_M + \hat{c}_1. \tag{1.3}$$

Here $\hat{c}_0$ and $\hat{c}_1$ are flow dependent coefficients. $\hat{c}_0$ is referred to as the *distribution parameter* and $\hat{c}_1$ to as the *drift velocity*. Various discrete schemes have been developed for computing numerical solutions of the compressible two-phase model (1.1)–(1.3). It is well known that it is difficult to solve this model efficiently due to strong nonlinear coupling mechanisms and challenges associated with transition to single-phase regions. Therefore it is of interest to deepen the insight into the finer mechanism of this model, also from a mathematical point of view. In particular, it is desirable to obtain a better understanding of the effect from the slip law (1.3).

Motivated by applications in the context of wellbore flow systems [1] we consider a flow scenario as indicated in Fig. 1. In this flow system there is a mixture of gas and liquid which is separated by a gas-dominated region through a free interface. The

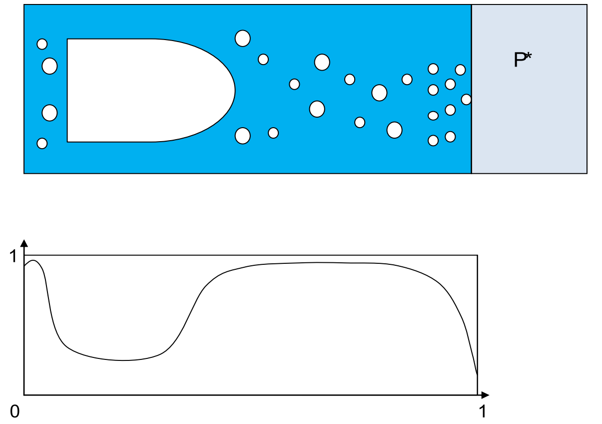

Fig. 1. Top: Schematic figure showing a gas–liquid mixture separated by a strongly gas-dominated region to the right with a free boundary at the interface and a positive pressure p^* associated with the right gas region. Bottom: Description of the above gas–liquid scenario in terms of the liquid volume fraction $\alpha_l(x,\cdot)$ in Lagrangian coordinates where $x \in [0,1]$ and the free interface corresponds to $x = 1$. Note that $\alpha_l(x,\cdot) \sim (1-x)^\alpha + \alpha_l^*$, i.e., there is a decay rate $\alpha > 0$ associated with the liquid mass at the free interface.

gas-dominated region holds a positive pressure $p^* > 0$ which plays an essential role in the analysis presented in this work. This situation is typical for certain drilling operations where the possibility to adjust the position of the free interface is used to control the pressure in the mixture region.

1.1. The model

We set $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$ in (1.1) and consider the model

$$
\begin{aligned}
\partial_t n + \partial_x[nu_g] &= 0,\\
\partial_t m + \partial_x[mu_l] &= 0,\\
\partial_t[nu_g + mu_l] + \partial_x\big[nu_g^2 + mu_l^2 + P(n,m)\big]&\\
= \partial_x\big[\varepsilon(n,m)\partial_x u_M\big], \quad u_M = \left(1 - \frac{m}{\rho_l}\right)u_g + \frac{m}{\rho_l}u_l.&
\end{aligned} \tag{1.4}
$$

Note that external forces q have been ignored. The pressure law $P(n,m)$ and viscosity term $\varepsilon(n,m)$ are given by

$$
P(n,m) = \left(\frac{n}{\rho_l - m}\right)^\gamma, \qquad \varepsilon(n,m) = \mu\frac{(m-k^*)}{(n+m-k^*)}, \quad \gamma > 1,\ \mu > 0, \tag{1.5}
$$

together with the constitutive relations

$$
\alpha_l + \alpha_g = 1, \qquad u_g - \hat{c}_0 u_M - \hat{c}_1 = 0, \qquad \rho_l = \rho_{l,0}, \qquad \rho_g = \rho_g(P), \tag{1.6}
$$

where $\hat{c}_0$, $\hat{c}_1$, and $\rho_{l,0}$ are assumed to be constants. As will be explained in the following the slip law $u_g - \hat{c}_0 u_M - \hat{c}_1 = 0$ requires that the liquid mass is above a critical lower limit k^*, i.e., $m \geq k^*$. More precisely, we assume that $\hat{c}_0 \geq 1$ and $\hat{c}_1 \geq 0$ and introduce α_g^*, α_l^* given by

$$\alpha_g^* = \frac{1}{\hat{c}_0}, \qquad \alpha_l^* = 1 - \alpha_g^*, \tag{1.7}$$

implying that $0 < \alpha_g^* \leq 1$. From the slip law, we get

$$m u_l = \rho_l u_g \big(\alpha_l - \alpha_l^*\big) - \rho_l \big(1 - \alpha_l^*\big)\hat{c}_1 = u_g \big(m - k^*\big) - \rho_l \big(1 - \alpha_l^*\big)\hat{c}_1, \tag{1.8}$$

where the constant k^* is defined by

$$k^* = \rho_l \alpha_l^*. \tag{1.9}$$

Now, we introduce the notation

$$\rho = n + m - k^*, \qquad c = \frac{m - k^*}{\rho}. \tag{1.10}$$

We then can show that the model (1.4) can be written in the form

$$\begin{aligned} \partial_t[c\rho] + \partial_x[c\rho u] &= 0, \\ \partial_t[\rho] + \partial_x[\rho u] &= 0, \\ \partial_t[\rho u] + \partial_x\big[\rho u^2\big] + \partial_x\big[P(c,\rho) - u^2 g(c\rho) - u h(c\rho) + j(c\rho)\big] &= \frac{1}{\hat{c}_0}\partial_x\big[\varepsilon(c)\partial_x u\big], \end{aligned} \tag{1.11}$$

where we use $u := u_g$. We refer to the recent work [4] for details of this derivation (see also [5]). We may absorb the constant $1/\hat{c}_0$ into the viscosity constant μ in (1.5) without loss of any generality. Moreover, the pressure law $P(n,m)$ takes the form

$$P(n,m) = \left(\frac{n}{\rho_l - m}\right)^\gamma = \left(\frac{[1-c]\rho}{[\rho_l - k^*] - c\rho}\right)^\gamma = \left(\frac{[1-c]\rho}{a^* - c\rho}\right)^\gamma := P(c,\rho), \tag{1.12}$$

where $a^* = \rho_l - k^* = \rho_l \alpha_g^*$. The functions $g(\cdot)$, $h(\cdot)$, and $j(\cdot)$ are defined as follows:

$$\begin{aligned} g(c\rho) &= k^* \frac{c\rho}{k^* + c\rho}, \\ h(c\rho) &= 2\rho_l \left(\frac{\hat{c}_1}{\hat{c}_0}\right) \frac{c\rho}{k^* + c\rho}, \\ j(c\rho) &= \rho_l^2 \left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 \frac{1}{k^* + c\rho}. \end{aligned} \tag{1.13}$$

For the viscosity term $\varepsilon(n, m)$ we have

$$\varepsilon(n, m) = \mu c = \varepsilon(c). \tag{1.14}$$

In this reformulation the slip law (1.3) has been incorporated directly in the PDE formulation and leads to the model (1.11). This allows to describe the flow by means of the gas velocity u only, however the mixture momentum equation now contains a generalized pressure function which depends on the standard pressure function P and the three new terms g, h, and j that correct for the difference in gas and liquid fluid velocity.

Remark 1.1. The basic feature of the viscosity term (1.14) is that it will vanish when the mass of the liquid phase reaches its lower limit k^*, which happens at the free interface which separates the gas–liquid mixture from the strongly gas dominated region which holds a positive pressure p^*, see Fig. 1. Moreover, if gas vanishes, then ε takes the constant value μ. This seems to be a very natural property since the viscosity of the liquid phase is typically several orders higher than for the gas phase.

1.2. Lagrangian coordinates

In view of the flow system as indicated in Fig. 1 the conduit is closed at the left boundary $x = a$ whereas a free boundary at $x = b(t)$ separates the gas–liquid mixture region from the strongly gas-dominated region associated with a specified pressure $p^* > 0$. Hence, we introduce a free boundary formulation given by

$$\begin{aligned} \partial_t[c\rho] + \partial_x[c\rho u] &= 0, \\ \partial_t[\rho] + \partial_x[\rho u] &= 0, \\ \partial_t[\rho u] + \partial_x\left[\rho u^2\right] + \partial_x\left[P(c,\rho) - u^2 g(c\rho) - uh(c\rho) + j(c\rho)\right] &= \partial_x\left[\varepsilon(c)\partial_x u\right], \end{aligned} \tag{1.2.1}$$

with $x \in (a, b(t))$ and $t > 0$. Initial data are

$$\begin{aligned} \rho(x, t = 0) &= \rho_0(x), \\ c(x, t = 0) = c_0(x) &= \frac{m_0(x) - k^*}{\rho_0(x)}, \\ u(x, t = 0) &= u_0(x), \end{aligned} \tag{1.2.2}$$

for $x \in [a, b_0]$ where $b_0 = b(t = 0)$. Boundary conditions are set to be as follows:

$$u(a, t) = 0, \qquad \rho\big(b(t), t\big) = n^*, \qquad c\big(b(t), t\big) = 0, \tag{1.2.3}$$

where $p^* = P(\rho = n^*, c = 0) = (n^*/a^*)^\gamma$. Here $b(t)$, which separates the gas–liquid mixture and the gas region corresponding to $\rho = n^*$ and $c = 0$, satisfies

$$\frac{db}{dt} = u\big(b(t), t\big), \qquad b(0) = b_0. \tag{1.2.4}$$

We can introduce Lagrangian coordinates by using the transformation $(x,t) \to (\xi,\tau)$ given by

$$\xi = \int_a^x \rho(z,t)\,dz, \qquad \tau = t, \tag{1.2.5}$$

observing that

$$\int_a^{b(t)} \rho(z,t)\,dz = \int_a^{b_0} \rho(z,t=0)\,dz = \text{constant}\,(=1,\ \text{without loss of generality}).$$

This implies that $[a,b(t)]$ is converted into the fixed interval $[0,1]$. Since $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial x} = \rho\frac{\partial}{\partial \xi}$, we can transform (1.2.1) into the following form (after replacing (ξ,τ) by (x,t)):

$$\begin{aligned} &\partial_t c = 0, \\ &\partial_t \rho + \rho^2 \partial_x u = 0, \\ &\partial_t u + \partial_x\big[P(c,\rho) - u^2 g(c\rho) - uh(c\rho) + j(c\rho)\big] = \mu\partial_x\big[E(c\rho)\partial_x u\big], \quad \text{in } 0 < x < 1, \end{aligned} \tag{1.2.6}$$

with boundary conditions

$$u(0,t) = 0, \qquad \rho(1,t) = n^*, \qquad c(1,t) = 0, \tag{1.2.7}$$

and with initial conditions

$$c(x,0) = c_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.2.8}$$

where $c(x,t) = c_0(x) = \frac{m_0(x) - k^*}{\rho_0(x)}$. Moreover, we have that

$$\begin{aligned} &P(c,\rho) = \left(\frac{[1-c]\rho}{a^* - c\rho}\right)^{\gamma}, \qquad E(c\rho) = c\rho, \\ &g(c\rho) = k^* \frac{c\rho}{k^* + c\rho}, \qquad h(c\rho) = 2\rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\frac{c\rho}{k^* + c\rho}, \\ &j(c\rho) = \rho_l^2\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 \frac{1}{k^* + c\rho}. \end{aligned} \tag{1.2.9}$$

1.3. Previous work

The drift-flux model (1.1) with zero slip, i.e. $u_g = u_l$ which corresponds to $\hat{c}_0 = 1$ and $\hat{c}_1 = 0$ in (1.3), has been studied before in Eulerian coordinates. A first work of the one-dimensional model is represented by [3] where existence of weak solutions was investigated. A 2D version of the model was studied in [21] where existence of weak solutions was proved. Some blow-up results have been obtained in [22] and [16]. In [6] global existence and uniqueness of the strong solution of a multidimensional version of the model was obtained for initial data close to a stable equilibrium. Moreover, the local-in-time existence and uniqueness of the solution with general initial data was studied in the framework of Besov spaces. In all these works the liquid is treated as a compressible fluid. This gives rise to a nonlinear pressure function which is more difficult to handle than for the classical Navier–Stokes equations. For previous studies of the 1D model (1.1) combined with a simplified slip law (1.3) with $\hat{c}_0 > 1$ and $\hat{c}_1 = 0$ considered in a free boundary setting similar to (1.2.6)–(1.2.9), we refer to [2,4,20]. Note that in these works the liquid has been assumed to be incompressible. In [2,20] local in time results were presented whereas [4] gives a local in time existence result for a general slip ($\hat{c}_0 > 1$ and $\hat{c}_1 > 0$) and a global in time result for the special case where $\hat{c}_1 = 0$ which implies that $h = j = 0$, see (1.2.9). The more general case $\hat{c}_1 > 0$ is important because it allows the model to describe e.g. counter-current flow where u_l and u_g possibly have different sign. This can be seen from (1.8).

The current work, to the authors knowledge, represents a first *global* existence result for the free boundary problem of the drift-flux model with a general slip of the form (1.3) where $\hat{c}_0 > 1$ and $\hat{c}_1 > 0$. We first perform a variable transformation where the set (c, ρ, u) is replaced by (c, Q, u) where $Q = \frac{\rho}{a^* - c\rho}$. This implies that the model (1.2.6) is converted into the model (2.1.2). Some essential points and relations reflected by the analysis that we would like to highlight are:

- The basic energy estimate can be obtained subject to appropriate smallness conditions on the initial energy and the slip parameters $\hat{c}_0$ and $\hat{c}_1$ versus the strength of the viscosity term, see Lemma 3.1. These conditions are employed to control the new terms appearing in the mixture momentum equation.
- Uniform upper and lower bounds on the pressure-related quantity Q defined in (2.1.1) can be obtained as expressed by Corollary 3.1 and Lemma 3.4. These estimates involve an interplay between the strength of the viscosity term, the slip parameters $\hat{c}_0$ and $\hat{c}_1$, and the initial energy and masses. In particular, the pressure $p^* > 0$ associated with the strongly gas dominated region at the free interface is essential for the lower limit. As a consequence, it follows that the gas mass will not vanish at any point nor the liquid mass apart from at the free interface located at $x = 1$ where the decay rate is of the order $(1 - x)^\alpha$ (for an appropriate choice of α), see Remark 3.1. Neither will there be any accumulation of masses at any point in the domain.
- In order to obtain an estimate of Q_x in L^1 as given by Lemma 3.9 we study the L^2 estimate of the related quantity $[c \ln(\frac{Q}{Q^*(1+cQ)})]_x$ combined with an appropriate weighting function. Here Q^* corresponds to the pressure p^*. This is the purpose of Lemma 3.8. However, we then need an estimate $u_t \in L^2([0, T] \times [0, 1])$ which is

provided by Lemma 3.6. This last lemma also provides an H^1 estimate as well as a uniform bound on the gas velocity u which turns out to be useful for the following reason: In order to extract the L^1 estimate of Q_x from $[c\ln(\frac{Q}{Q^*(1+cQ)})]_x$ we need more precise information about the decay rate of $Q - Q^*$ (which controls the decay rate of gas $n - n^*$) at the free interface. In other words, we must make use of information about the decay rate of the gas phase (in addition to the liquid phase) at the free interface. This is the purpose of Lemma 3.5 and Lemma 3.7. The uniform estimate of u plays a key role in Lemma 3.7.

- Other estimates needed for applying standard compactness arguments then follow directly, see Corollary 3.5, from which existence of weak solutions can be obtained as expressed by Theorem 2.1. For the uniqueness result of Theorem 2.2 it turns out to be essential that the decay rate of liquid and gas at the free interface is the same. This is required in order to control the behavior of u_x at the free interface.

The positive pressure p^* in the gas-dominated region is crucial for the above analysis. From a physical point of view this is perhaps not so surprising since the case where p^* tends to zero (i.e. a vacuum state) amounts to an extreme situation where the expansion of the gas phase will be dramatic. The lower bound of the pressure-related quantity Q is no longer guaranteed by the approach used in this paper, neither the uniform bound on the velocity at the free interface.

Finally, we would also like to point out two open problems related to the study of the model (1.1). First, the interesting case where an initial vacuum state exists within the domain $(0, 1)$, i.e., gas density ρ_g (hence pressure) becomes zero at some point, is excluded from our analysis since we rely on the assumption that $\sup c_0 < 1$. See [10] for a study of this situation in the context of the classical Navier–Stokes equations. Secondly, the model (1.2.6) where the slip law (1.3) has been accounted for with constant parameters $\hat{c}_0 > 1$ and $\hat{c}_1 > 0$ obviously predicts that the fraction of liquid mass relatively total mass remains constant, i.e., $c = c_0$. This does not seem so reasonable from a physical point of view, for example for a vertical wellbore where an initial mixture of gas and liquid (i.e. $0 < c_0 < 1$) ultimately should be separated into a liquid region and a gas region corresponding to $c = 1$ and $c = 0$. This limitation of the model seems to be related to the assumption that $\hat{c}_0$ and $\hat{c}_1$ are assumed to be constant and motivates for studies where $\hat{c}_0$ and $\hat{c}_1$ are functions. As a final comment it should be noted that the techniques employed for the current study of genuine gas–liquid flow behavior (in the sense that gas and liquid move with different fluid velocities and involve gas–liquid transition zones), are inspired by previous works on the classical Navier–Stokes equations [7,13,11,12,17,14,18,19,9,23].

The paper is organized as follows: In the next section we first present the reformulated model and then state the main result as expressed by Theorem 2.1. Section 3 contains a number of a priori estimates that are sufficient to guarantee existence of global weak solutions. Section 4 sketches the uniqueness part of Theorem 2.1.

2. Main results

2.1. Reformulation

For the analysis of the model (1.2.6), it will be convenient to introduce the function $Q(c,\rho)$ given by

$$Q(c,\rho)=\frac{\rho}{a^*-c\rho},\quad \text{which corresponds to } \rho=a^*\frac{Q}{1+cQ}. \tag{2.1.1}$$

The following relation holds for $Q(c,\rho)$:

$$\begin{aligned}Q(c,\rho)_t&=Q_c c_t+Q_\rho\rho_t=Q_\rho\rho_t\\&=\left(\frac{1}{a^*-c\rho}+\frac{c\rho}{(a^*-c\rho)^2}\right)\rho_t\\&=\frac{a^*}{(a^*-c\rho)^2}\rho_t=-\frac{a^*\rho^2}{(a^*-c\rho)^2}u_x=-a^*Q(c,\rho)^2u_x.\end{aligned}$$

Hence, the system (1.2.6) can be replaced by

$$\begin{cases}\partial_t c=0,\\ \partial_t Q+a^*Q^2\partial_x u=0,\\ \partial_t u+\partial_x\big[P(c,Q)-u^2g(cQ)-uh(cQ)+j(cQ)\big]\\ \quad=\mu\partial_x\big[E(cQ)\partial_x u\big],\quad x\in(0,1),\ t>0,\end{cases} \tag{2.1.2}$$

with

$$\begin{aligned}P(c,Q)&=\big[(1-c)Q\big]^\gamma,\qquad E(c\rho)=c\rho=a^*\frac{cQ}{1+cQ}:=E(cQ),\\ g(c\rho)&=k^*\frac{c\rho}{c\rho+k^*}=a^*\alpha_l^*\left(\frac{cQ}{\alpha_l^*+cQ}\right):=g(cQ),\\ h(c\rho)&=2\rho_l\left(\frac{\hat c_1}{\hat c_0}\right)\frac{c\rho}{c\rho+k^*}=2a^*\left(\frac{\hat c_1}{\hat c_0}\right)\left(\frac{cQ}{\alpha_l^*+cQ}\right):=h(cQ),\\ j(c\rho)&=\rho_l^2\left(\frac{\hat c_1}{\hat c_0}\right)^2\frac{1}{k^*+c\rho}=\rho_l\left(\frac{\hat c_1}{\hat c_0}\right)^2\left(\frac{1+cQ}{\alpha_l^*+cQ}\right):=j(cQ),\end{aligned} \tag{2.1.3}$$

since

$$c\rho=a^*\frac{cQ}{1+cQ},\qquad a^*=\rho_l\alpha_g^*,\qquad k^*=\rho_l\alpha_l^*.$$

Boundary conditions for our system (2.1.2)–(2.1.3) are (in view of (2.1.1) and (1.2.7)):

$$u(0,t)=0,\qquad c(1,t)=0,\qquad Q(1,t)=\frac{n^*}{a^*}:=Q^*, \tag{2.1.4}$$

where Q^* is a constant. Initial conditions are (in view of (2.1.1) and (1.2.8)):

$$c(x,0) = c_0(x), \qquad Q(x,0) = Q_0(x) = \frac{\rho_0}{a^* - c_0\rho_0},$$
$$u(x,0) = u_0(x), \quad x \in [0,1]. \tag{2.1.5}$$

Remark 2.1. For later use we note the following relations:

$$g(cQ) = a^*\alpha_l^*\left(\frac{\frac{1}{\alpha_l^*}cQ}{1+\frac{1}{\alpha_l^*}cQ}\right) \le a^*\left(\frac{cQ}{1+cQ}\right) = E(cQ) \le a^*$$
$$h(cQ) = 2a^*\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\left(\frac{cQ}{\alpha_l^*+cQ}\right) \le \frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)E(cQ)$$
$$j(cQ) = \rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\left(\frac{1+cQ}{\alpha_l^*+cQ}\right), \qquad j^*(cQ) = -\rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{a^*}{k^*}\left(\frac{cQ}{\alpha_l^*+cQ}\right) \tag{2.1.6}$$

where

$$j^*(cQ) + \frac{\rho_l}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 = j(cQ). \tag{2.1.7}$$

Hence, we may replace j by j^* in $(2.1.2)_3$.

Throughout the rest of the paper, we denote $L^p = L^p([0,1])$, $\int_0^1 f = \int_0^1 f\,dx$ when it will not cause any confuse.

2.2. Global weak solution

Assumptions: The following assumptions are made on the parameters α and γ:

$$0 < \alpha < \frac{1}{2}, \qquad \gamma > 1, \tag{2.2.1}$$

where $\phi(x) = 1 - x$. Initial data is then specified as follows:

$$\begin{cases} \tilde{c}_1\phi^\alpha \le c_0 \le \tilde{c}_2\phi^\alpha, \quad \text{with } 0 < \tilde{c}_1 \le \tilde{c}_2, \quad \sup c_0 < 1, \quad A_1 \le Q_0 \le B_1, \\ \left|\ln\left(\frac{Q_0}{Q^*(1+c_0Q_0)}\right)\right| \le \tilde{C}\phi^{\beta_1}, \quad \beta_1 \in (0,\alpha] \cap \left(0, \frac{1}{2}-\alpha\right], \quad \int\limits_0^1 G_0(x)\,dx \le M, \\ \phi(x)^{\frac{1-\alpha_1}{2}}\left[c_0\ln\left(\frac{Q_0}{Q^*(1+c_0Q_0)}\right)\right]_x \in L^2, \quad \alpha_1 \in (2\alpha, 1) \text{ such that} \\ \quad \phi^{\frac{1-\alpha_1-\alpha+2\beta_1}{2}}c_{0,x} \in L^2 \\ \sqrt{E(c_0Q_0)}u_{0,x} \in L^2, \end{cases} \tag{2.2.2}$$

with positive constants $\tilde{c}_1$, $\tilde{c}_2$, A_1, B_1, $\tilde{C}$ where G_0 represents the initial energy and is given by

$$G_0(x) = \frac{1}{2}u_0^2 + \int_{Q^*}^{Q} \frac{P(c,s) - P(c,Q^*)}{a^* s^2}\,ds + \frac{p^* - P(c,Q^*)}{a^* Q} - \frac{\rho_l}{k^* a^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 c_0 \ln\left(\frac{c_0 Q_0}{\alpha_l^* + c_0 Q_0}\right),$$

where $p^* = [Q^*]^\gamma$. Here $M > 0$ is a (sufficiently small) constant determined by different relations that depend on the upper and lower bounds A and B appearing in (2.2.8).

More precisely, the lower bound A of Q and the bound on the initial energy M must be chosen such that they obey the following relations:

$$\begin{cases} \left[\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\right]^2 \le \frac{\mu^2 \tilde{c}_1 A}{8}, \quad M \le \frac{\mu^2 \tilde{c}_1 A}{32}, \\ A < \frac{A_1}{4}, \quad e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}} < \frac{4}{3}, \quad e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}} < \frac{1}{6A} + 1. \end{cases} \tag{2.2.3}$$

Moreover, the following relation must be obeyed by the external pressure p^*

$$\left(p^*\right)^{1/\gamma} \ge \max\left\{B_1(1 - \sup c_0), 4A\right\}. \tag{2.2.4}$$

This is necessary in order to obtain the lower limit of Q as expressed by Lemma 3.3.

Similarly, the upper bound B of Q and initial energy bounded by M must be chosen such that the following relations are satisfied:

$$\begin{cases} (1 - \sup c_0)^\gamma \left(\frac{B}{2}\right)^\gamma \ge C\left(p^*, \hat{c}_0, \hat{c}_1\right) = p^* + \left(\frac{\hat{c}_1}{\hat{c}_0}\right) a^* + \rho_l \left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 \frac{a^*}{k^*}, \\ B_1 \le \frac{B}{2}, \quad B \le \frac{2 - 4\delta}{3\delta}, \\ \quad \frac{2}{\mu \tilde{c}_1}(2M)^{1/2} + \frac{12}{\mu^2}\left[\frac{2}{\tilde{c}_1 A} + 1\right]\left[1 + \left(\frac{\hat{c}_1}{\hat{c}_0}\right)\frac{1}{\alpha_l^*}\right] BM \le \ln(1 + \delta), \end{cases} \tag{2.2.5}$$

for some $\delta > 0$ and subject to the condition that

$$\sup c_0 < 1, \tag{2.2.6}$$

which follows from $(2.2.2)_1$. In addition, we ensure that A and B are chosen such that

$$2A \le Q_0 \le \frac{3}{4} B,$$

which follows from $(2.2.3)_2$, $(2.2.5)_2$ and $(2.2.2)_1$. This is employed in the classical continuity arguments that allow us to derive the bounds $A \le Q \le B$. Finally, we must also ensure that M obeys the smallness condition of Lemma 3.6 given by

$$\frac{2M}{\mu^2}\left[\frac{2}{\tilde{c}_1 A} + 1\right] \le \frac{1}{64}. \tag{2.2.7}$$

For the boundary condition (2.1.4), we require the compatibility condition $u_0(0) = 0$.

Remark 2.2. Note that the assumption $(2.2.2)_3$ puts a constraint on the choice of α_1. For example, if $c_{0,x} \sim \phi(x)^{\alpha-1}$ then $\alpha_1 < \alpha + 2\beta_1$. From $(2.2.2)_3$ it follows that $2\alpha < \alpha_1 < 1$. Clearly, these conditions are realized for example by the choice $\alpha = \beta_1 = \frac{1}{4}$ and $\alpha_1 = \frac{3}{5}$.

Remark 2.3. It is interesting to note that the external pressure p^* cannot become zero. According to (2.2.4) this pressure must be related to either the initial upper limit B_1 of the mass related term Q or its time independent lower limit A obtained in Lemma 3.3.

Remark 2.4. Note that α represents the decay rate of c at $x = 1$, i.e., how fast the liquid mass m decays towards its lower limit k^*, see Remark 2.8. On the other hand, β_1 represents the decay rate associated with Q towards its limit Q^*, i.e., how fast the gas mass n decays to its limit n^* at the free interface at $x = 1$, see Corollaries 3.3 and 3.4. Higher order regularity estimates like Lemma 3.9 require information about both these decay rates.

Remark 2.5. Note that (2.2.5) is realized by first choosing $\delta > 0$ small enough to ensure that the upper limit $B \le \frac{2-4\delta}{3\delta}$ represents no hinderance for choosing a B satisfying $(2.2.5)_1$. Finally, the last estimate of $(2.2.5)_2$ will then enforce a smallness condition on the choice of $\frac{M}{\mu^2}$.

Note that the restriction on the slip parameters $\hat{c}_0 > 1$ and $\hat{c}_1 > 0$ is related to the choice of initial data Q_0, initial energy represented by M, and the lower bound A. If A is made small, then we either have to choose the slip parameter $\hat{c}_1$ sufficiently small, see $(2.2.3)_1$, or we could make the strength of the viscosity term represented by μ larger. Similarly, the choice of the upper limit B is also sensitive for the slip parameters as expressed by (2.2.5).

We now state the main result of this paper:

Theorem 2.1 *(Existence). Under the assumptions of (2.2.1)–(2.2.7), there exists a constant $M_0 > 0$ such that (2.1.2)–(2.1.5) admits a global weak solution (c, Q, u) on $[0, 1] \times [0, T]$ for any time $T > 0$ for all $M \le M_0$ in the sense that*

(A) we have the following regularity:

$$c, Q \in L^\infty\big([0,T]; L^\infty\big) \cap C^{\frac{3}{4}}\big([0,T]; L^2\big), \qquad E(cQ)u_x \in L^\infty\big([0,T]; L^2\big),$$
$$u \in L^\infty\big([0,T]; L^\infty\big) \cap C^{\frac{1}{2}}\big([0,T]; L^2\big).$$

Moreover, the following estimates hold:

$$A \le Q \le B, \qquad \int_0^1 E u_x^2\,dx + \int_0^t \int_0^1 u_s^2\,dx\,ds \le C_2, \tag{2.2.8}$$

and

$$\|u_x\|_{L^4(0,T;L^2)} + \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2 \le C_2, \tag{2.2.9}$$

for $(x,t) \in [0,1] \times [0,T]$, *where* C_2 *depends on* A, B, M, $\tilde{c}_1$, $\tilde{c}_2$, α, β, γ, T *and the initial data.*

(B) The following equations hold:

$$\begin{cases} \partial_t c = 0, \quad \partial_t Q + a^* Q^2 \partial_x u = 0, \quad \text{for a.e. } (x,t) \in (0,1) \times (0,T], \\ (c,Q)(x,0) = \big(c_0(x), Q_0(x)\big), \quad \text{for a.e. } x \in [0,1], \\ \displaystyle\int_0^T \int_0^1 \Bigg[u\varphi_t + \Bigg(P(c,Q) - u^2 g(cQ) - uh(cQ) + j(cQ) \\ \displaystyle\quad - \left[\frac{\rho_l}{\alpha_l^*} \left(\frac{\hat{c}_1}{\hat{c}_0} \right)^2 + p^* \right] - \mu E(cQ) u_x \Bigg) \varphi_x \Bigg] dx\,dt + \int_0^1 u_0 \varphi(x,0)\,dx = 0, \end{cases}$$

for any test function $\varphi \in C_0^\infty((0,1] \times [0,T))$.

(C) Interface behaviors:

$$\big|u(x,t)\big| \le C_2 x^{\frac{r-1}{r}} \tag{2.2.10}$$

for some $r \in (1,2)$ *such that* $r(\alpha+1) < 2$, *and*

$$\big|Q(x,t) - Q^*\big| \le C_2 |x-1|^{\beta_1} \tag{2.2.11}$$

for $\beta_1 \in (0,\alpha] \cap (0, \frac{1}{2} - \alpha]$. *Here* $(x,t) \in [0,1] \times [0,T]$.

Theorem 2.2 *(Uniqueness). Under the conditions of Theorem 2.1 and by requiring that* $\beta_1 = \alpha$, *where* $0 < \alpha \le \frac{1}{4}$, *the weak solution is unique.*

Remark 2.6. Denote $\rho = a^* \frac{Q}{1+cQ}$, then from Theorem 2.1, we get a global weak solution (c, ρ, u) on $[0, 1] \times [0, T]$ to (1.2.6), (1.2.7), and (1.2.8).

Remark 2.7. Regarding Theorem 2.2, the additional assumption $\beta_1 = \alpha$ is needed to get an estimate of u_x at the free interface $x = 1$, see (4.2), which plays a very important role in the proof of uniqueness. As seen from (4.2) this requires that the decay rate of gas and liquid is the same.

Remark 2.8. Note that the interface behavior of the liquid phase is characterized by

$$\frac{a^* \tilde{c}_1 A}{1+B} \phi^\alpha \le m - k^* = c\rho \le a^* \tilde{c}_2 B \phi^\alpha,$$

in view of $(2.2.2)_1$, (2.1.1), and (2.2.8). The interface behavior of the gas phase is characterized by

$$\begin{aligned}\big|n - n^*\big| &= \big|(1-c)\rho - n^*\big| = \big|\big[(1-c) - 1\big]\rho + \rho - n^*\big| \le c\rho + a^* \left|\frac{Q}{1+cQ} - Q^*\right| \\ &\le c\rho + a^* \big|Q - Q^*\big| + a^* Q^* cQ \le C_2\big(\phi^\alpha + \phi^{\beta_1}\big) \le C_2 \phi^{\beta_1},\end{aligned}$$

in view of (2.1.1), (2.2.8), and (2.2.11).

3. Global existence of weak solutions

3.1. A priori estimates

We assume the following a priori estimate on the liquid mass related quantity $Q = \frac{\rho}{a^* - c\rho}$

$$A \le Q, \tag{3.1.1}$$

for an appropriate constant $A > 0$ that will be specified later. The aim is to show that then

$$2A \le Q. \tag{3.1.2}$$

This is proved in Lemma 3.3.

We use C to denote a generic positive constant dependent on initial data and some known constants but independent of A and T. We use C_1 to denote a generic positive constant that depends on A, initial data and some known constants but independent of T.

Lemma 3.1. *Under the assumptions of Theorem 2.1 and (3.1.1), it holds that*

$$\int_0^1 G(x,t)\,dx + \frac{\mu}{12}\int_0^t\int_0^1 Eu_x^2\,dxds \le \int_0^1 G_0(x)\,dx, \tag{3.1.3}$$

where

$$G = \frac{1}{2}u^2 + \int_{Q^*}^{Q} \frac{P(c,s)-P(c,Q^*)}{a^*s^2}\,ds + \frac{p^*-P(c,Q^*)}{a^*Q} - \frac{\rho_l}{k^*a^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 c\ln\left(\frac{cQ}{\alpha_l^*+cQ}\right), \tag{3.1.4}$$

subject to the following two conditions on initial state:

$$\left[\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\right]^2 \le \frac{\mu^2}{4C_1(A)} \le \frac{\mu^2\tilde{c}_1A}{8}, \qquad M \le \frac{\mu^2}{16C_1(A)} \le \frac{\mu^2\tilde{c}_1A}{32}, \tag{3.1.5}$$

with $C_1(A) = \frac{2}{\tilde{c}_1A} + 1$.

Proof. We have

$$\frac{d}{dt}\int_0^1 \frac{1}{2}u^2 + \frac{d}{dt}\int_0^1\left[\int_{Q^*}^{Q}\frac{P(c,s)-P(c,Q^*)}{a^*s^2}\,ds + \frac{p^*-P(c,Q^*)}{a^*Q}\right] + \mu\int_0^1 E(cQ)u_x^2$$
$$= -\int_0^1 g(cQ)u^2u_x - \int_0^1 h(cQ)uu_x + \int_0^1 j^*(cQ)u_x. \tag{3.1.6}$$

Here we have used that

$$-\int_0^1 \big[P(c,Q)\big]_x u = -\frac{d}{dt}\int_0^1\left[\int_{Q^*}^{Q}\frac{P(c,s)-P(c,Q^*)}{a^*s^2}\,ds + \frac{p^*-P(c,Q^*)}{a^*Q}\right].$$

Moreover,

$$-\int_0^1 g(cQ)u^2u_x \le \frac{\mu}{4}\int_0^1 g(cQ)u_x^2 + \frac{1}{\mu}\int_0^1 u^4g(cQ)$$
$$\le \frac{\mu}{4}\int_0^1 Eu_x^2 + \|u\|_{L^\infty}^2\|g\|_{L^\infty}\frac{1}{\mu}\int_0^1 u^2$$

$$\le \frac{\mu}{4}\int_0^1 Eu_x^2 + \frac{1}{\mu}\int_0^1\left(\frac{1}{cQ}+1\right)\int_0^1 u^2\int_0^1 Eu_x^2$$
$$\le \frac{\mu}{4}\int_0^1 Eu_x^2 + \frac{C_1(A)}{\mu}\int_0^1 u^2\int_0^1 Eu_x^2. \tag{3.1.7}$$

We have used that

$$\left|u(x,t)\right|^2 = \left|u(x,t)-u(0,t)\right|^2 = \left|\int_0^x E^{-\frac{1}{2}}\cdot E^{\frac{1}{2}}u_y(y,t)\,dy\right|^2$$
$$\le \int_0^1 E^{-1}\int_0^1 Eu_x^2 = \frac{1}{a^*}\int_0^1\left(\frac{1}{cQ}+1\right)\int_0^1 Eu_x^2$$
$$\le \frac{1}{a^*}\left[\frac{2}{\tilde{c}_1 A}+1\right]\int_0^1 Eu_x^2$$
$$= \frac{1}{a^*}C_1(A)\int_0^1 Eu_x^2, \tag{3.1.8}$$

by Hölder's inequality, $(2.1.2)_1$, (2.2.1), $(2.2.2)_1$ and (3.1.1). Moreover, using (2.1.6) and Hölder's inequality we get

$$-\int_0^1 h(cQ)uu_x \le \frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\int_0^1 |Euu_x|$$
$$\le \frac{\varepsilon_1}{4}\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\int_0^1 Eu_x^2 + \frac{1}{\varepsilon_1}\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\|u\|_{L^\infty}^2\int_0^1 E$$
$$\le \frac{\mu}{4}\int_0^1 Eu_x^2 + \frac{1}{\mu}\left[\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\right]^2 C_1(A)\int_0^1 Eu_x^2, \tag{3.1.9}$$

by an appropriate choice of ε_1 corresponding to $\varepsilon_1 = \mu\frac{\alpha_l^*}{2}(\frac{\hat{c}_0}{\hat{c}_1})$. Here we also have used that $E(cQ) = a^*\frac{cQ}{1+cQ} \le a^*$. Finally, we note that

$$\int_0^1 j^*(cQ)u_x = \frac{\rho_l}{k^*\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{d}{dt}\int_0^1 c\ln\left(\frac{cQ}{\alpha_l^*+cQ}\right) \tag{3.1.10}$$

where we have used that

$$-c\frac{\rho_l}{k^*\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\left[\ln\left(\frac{cQ}{\alpha_l^*+cQ}\right)\right]_t + j^*u_x = 0, \quad x\in(0,1),$$

which is obtained from (2.1.2)$_2$. Clearly, $\ln(\frac{cQ}{\alpha_l^*+cQ})$ is non-positive for $cQ > 0$.

Combining (3.1.6) with (3.1.7), (3.1.9), and (3.1.10) we get

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{1}{2}u^2 + \frac{d}{dt}\int_0^1\left[\int_{Q^*}^{Q}\frac{P(c,s)-P(c,Q^*)}{a^*s^2}\,ds + \frac{p^*-P(c,Q^*)}{a^*Q}\right]\\
&\quad - \frac{\rho_l}{k^*\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{d}{dt}\int_0^1 c\ln\left(\frac{cQ}{\alpha_l^*+cQ}\right) + \mu\int_0^1 E(cQ)u_x^2\\
&= -\int_0^1 g(cQ)u^2u_x - \int_0^1 h(cQ)uu_x\\
&\le \frac{\mu}{4}\int_0^1 Eu_x^2 + \frac{C_1(A)}{\mu}\int_0^1 u^2\int_0^1 Eu_x^2 + \frac{\mu}{4}\int_0^1 Eu_x^2\\
&\quad + \frac{1}{\mu}\left[\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\right]^2 C_1(A)\int_0^1 Eu_x^2.
\end{aligned} \tag{3.1.11}$$

That is,

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{1}{2}u^2 + \frac{d}{dt}\int_0^1\left[\int_{Q^*}^{Q}\frac{P(c,s)-P(c,Q^*)}{a^*s^2}\,ds + \frac{p^*-P(c,Q^*)}{a^*Q}\right]\\
&\quad - \frac{\rho_l}{k^*\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{d}{dt}\int_0^1 c\ln\left(\frac{cQ}{\alpha_l^*+cQ}\right) + \frac{\mu}{2}\int_0^1 Eu_x^2\\
&\le \frac{C_1(A)}{\mu}\int_0^1 u^2\int_0^1 Eu_x^2 + \frac{1}{\mu}\left[\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\right]^2 C_1(A)\int_0^1 Eu_x^2.
\end{aligned}$$

By the following smallness assumption

$$\left[\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\right]^2 C_1(A) \le \frac{\mu^2}{4}, \quad C_1(A) = \frac{2}{\tilde{c}_1 A} + 1, \tag{3.1.12}$$

we conclude that

$$\frac{d}{dt}\int_0^1\left[\frac{1}{2}u^2+\int_{Q^*}^{Q}\frac{P(c,s)-P(c,Q^*)}{a^*s^2}\,ds+\frac{p^*-P(c,Q^*)}{a^*Q}-\frac{\rho_l}{k^*\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 c\ln\left(\frac{cQ}{\alpha_l^*+cQ}\right)\right]$$

$$+\frac{\mu}{4}\int_0^1 Eu_x^2\le\frac{C_1(A)}{\mu}\int_0^1 u^2\int_0^1 Eu_x^2.$$

Hence, for appropriate choices of f, g and a constant $K=\frac{8C_1}{\mu^2}$ we have

$$\frac{df}{dt}+g\le Kfg. \tag{3.1.13}$$

Claim. *If*

$$f_0\le\frac{1}{2K}=\frac{\mu^2}{16C_1(A)}, \tag{3.1.14}$$

then it follows that

$$f(t)+\frac{1}{3}\int_0^t g\le f_0,\quad t>0. \tag{3.1.15}$$

This corresponds to (3.1.3).

To prove (3.1.15) we may argue as follows: In view of (3.1.14) and by continuity of $f(t)$ there is a time $\tilde{T}>0$ such that

$$f(t)\le\frac{2}{3K},\quad t\in[0,\tilde{T}]. \tag{3.1.16}$$

Let $\tilde{T}^*$ be the maximal time for (3.1.16). I.e., (3.1.16) holds for $t\in[0,\tilde{T}^*)$. If $\tilde{T}^*=\infty$ there is nothing to prove. If $\tilde{T}^*<\infty$ we use (3.1.13) and calculate as follows for $t\in[0,\tilde{T}^*)$

$$f(t)-f(0)+\int_0^t g\le K\int_0^t fg\le\frac{2}{3}\int_0^t g,$$

employing (3.1.16). That is,

$$f(t)+\frac{1}{3}\int_0^t g\le f_0\le\frac{1}{2K}<\frac{2}{3K}.$$

The continuity of $f(t)$ implies then that (3.1.16) must hold at time $\tilde{T}^*$. This contradiction leads us to conclude that $\tilde{T}^*=\infty$ and (3.1.15) has been proved. □

Next, we seek to obtain pointwise control on masses. For that purpose we recall the following lemma based on a work by Zlotnik [24].

Lemma 3.2. *Let $f \in C(\mathbb{R})$ and $y, b \in W^{1,1}(0,T)$. Let $y(t)$ satisfy the following equation*

$$\frac{dy}{dt} = f(y) + \frac{db}{dt}, \quad t \in \mathbb{R}^+ \tag{3.1.17}$$

and $|b(t_2) - b(t_1)| \leq N_0$ for any $0 \leq t_1 < t_2$. Then

(1) *if $f(z) \geq 0$, for $z \leq M_1$,*

$$\min\{y(0), M_1\} - N_0 \leq y(t), \quad t \in \mathbb{R}^+; \tag{3.1.18}$$

(2) *if $f(z) \leq 0$, for $z \geq M_2$,*

$$\max\{y(0), M_2\} + N_0 \geq y(t), \quad t \in \mathbb{R}^+; \tag{3.1.19}$$

Lemma 3.3. *Under the assumptions of Theorem 2.1 and (3.1.1), it holds that*

$$2A \leq Q, \tag{3.1.20}$$

for an appropriate choice of $A > 0$ and $M > 0$ (independent of time) dictated by the relations (3.1.27) and (3.1.29) and under the condition (3.1.28) or (3.1.31) on the external pressure $p^* > 0$.

Proof. First, we observe that

$$\int_x^1 u_t + p^* - \left(P - u^2 g - uh + j^*\right) = -\mu E u_x \tag{3.1.21}$$

and

$$\begin{aligned} -Eu_x &= c\frac{\partial}{\partial t}\left(\ln \frac{cQ}{1+cQ}\right) = c\frac{\partial}{\partial t}\left(\ln c + \ln \frac{Q}{1+cQ}\right) \\ &= -c\frac{\partial}{\partial t}\left(\ln \frac{1+cQ}{Q}\right). \end{aligned} \tag{3.1.22}$$

We consider a fixed $x \in (0,1)$, hence $c(x) > 0$, and introduce the variable $Y(t) = \ln(\frac{1+cQ}{Q}) = \ln(\frac{1}{Q} + c)$ for $Q \in (0,\infty)$ and observe that

$$Q = \frac{1}{e^Y - c}, \quad Y > \ln c \tag{3.1.23}$$

since $Q > 0$ corresponds to $e^Y - c > 0$ or $Y > \ln c$. Moreover,

$$P(c, Q) = \left[(1-c)Q\right]^\gamma = \left(\frac{1-c}{e^Y - c}\right)^\gamma = P(c, Y).$$

Clearly, P is a decreasing function relatively Y and tends to zero as Y goes to infinity. We may combine (3.1.21) and (3.1.22) and get

$$Y'(t) = \frac{d}{dt}\left(-\frac{1}{c\mu}\int_x^1 u\right) + \frac{1}{c\mu}\left[-p^* + P - u^2 g - uh + j^*\right] = \frac{db}{dt} + f(Y).$$

First we claim that

$$-u^2 g - uh + j^* \le 0, \tag{3.1.24}$$

or

$$-uh \le u^2 g - j^*.$$

Clearly,

$$\begin{aligned}
|uh| &= |u|\left(a^*\alpha_l^*\right)^{1/2} \cdot \left(a^*\alpha_l^*\right)^{-1/2} 2\left(\frac{\hat{c}_1}{\hat{c}_0}\right) a^* \left(\frac{cQ}{\alpha_l^* + cQ}\right) \\
&\le u^2\left(a^*\alpha_l^*\right)\left(\frac{cQ}{\alpha_l^* + cQ}\right) + \frac{1}{4}\left(\frac{cQ}{\alpha_l^* + cQ}\right)\frac{4(a^*)^2}{(a^*\alpha_l^*)}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 \\
&= u^2 g + \left(\frac{cQ}{\alpha_l^* + cQ}\right)\frac{a^*}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 = u^2 g - j^*.
\end{aligned}$$

Then we can claim that

$$f(Y) = \frac{1}{c\mu}\left[-p^* + P - u^2 g - uh + j^*\right] \le \frac{1}{c\mu}\left[P - p^*\right] \le 0, \tag{3.1.25}$$

for $P \le p^*$. Clearly,

$$P(c, Y) = \left(\frac{1-c}{e^Y - c}\right)^\gamma = \left(\frac{1-c_0}{e^Y - c_0}\right)^\gamma \le p^*$$

for Y large enough. More precisely,

$$Y \ge \ln\left(\frac{1-c}{(p^*)^{1/\gamma}} + c\right) \overset{\text{def}}{:=} M_2 \tag{3.1.26}$$

will ensure that (3.1.25) holds. For the b function we have

$$\begin{aligned}
\left|b(t_2)-b(t_1)\right| &\le \frac{1}{c_0\mu}\left|\int_x^1 \left[u(y,t_2)-u(y,t_1)\right]dy\right| \\
&\le \frac{2}{c_0\mu}\sup_t \int_x^1 |u|\,dy \le \frac{2}{c_0\mu}\phi(x)^{1/2}\sup_t\left(\int_0^1 u^2\,dy\right)^{1/2} \\
&\le \frac{2}{\mu\tilde{c}_1}(2M)^{1/2} \overset{\text{def}}{:=} N_0,
\end{aligned}$$

subject to the condition that A, M are chosen such

$$M \le \frac{\mu^2\tilde{c}_1 A}{32}, \qquad \left[\frac{2}{\alpha_l^*}\left(\frac{\hat{c}_1}{\hat{c}_0}\right)\right]^2 \le \frac{\mu^2\tilde{c}_1 A}{8}. \tag{3.1.27}$$

Hence, the conclusion is that

$$Y(t) \le \max\{Y(0), M_2\} + N_0.$$

Since $Y_0 = \ln(1/Q_0 + c_0)$, comparison with M_2 defined by (3.1.26) shows that if

$$\frac{(p^*)^{1/\gamma}}{1-c_0} \ge B_1 \ge Q_0, \tag{3.1.28}$$

then $M_2 \le Y_0$. Consequently, $Y(t) \le Y(0) + N_0 = \ln(1/Q_0 + c_0) + (4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}$. That is

$$\begin{aligned}
\frac{1}{Q} &\le \left(\frac{1}{Q_0}+c_0\right)e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}} - c_0 \le \frac{e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}}}{A_1} + c_0\left[e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}} - 1\right] \\
&< \frac{e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}}}{4A} + \frac{1}{6A} < \frac{1}{3A} + \frac{1}{6A} = \frac{1}{2A}.
\end{aligned}$$

Here we have chosen A and M sufficiently small such that

$$A < \frac{A_1}{4}, \qquad e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}} < \frac{4}{3}, \qquad e^{(4/\tilde{c}_1)\frac{\sqrt{M}}{\mu}} < \frac{1}{6A} + 1. \tag{3.1.29}$$

Hence, we have shown that

$$2A \le Q, \tag{3.1.30}$$

subject to the condition that A and M are chosen in accordance with both (3.1.27) and (3.1.29). On the other hand, if (3.1.28) is not satisfied than we may have that $Y(t) \le \ln(\frac{1-c}{(p^*)^{1/\gamma}} + c) + (4/\tilde{c}_1)M^{1/2}$. From this we can show that (3.1.30) holds if

$$A \le \frac{(p^*)^{1/\gamma}}{4}, \tag{3.1.31}$$

in addition to (3.1.29). □

Corollary 3.1. *Under the assumptions of Theorem 2.1, it holds that*

$$A \le Q. \tag{3.1.32}$$

Proof. In view of Lemma 3.3 and classical continuity arguments, the estimate (3.1.32) follows. □

Lemma 3.4. *Under the assumptions of Theorem 2.1, it holds that*

$$Q \le B, \tag{3.1.33}$$

for an appropriate choice of $B > 0$ and $M > 0$ (independent of time) as dictated by the relations (3.1.41), (3.1.46), (3.1.48), (3.1.45), and for $\sup c_0 < 1$, in addition to the two relations in (3.1.27).

Proof. Let us make the following a priori estimate

$$Q \le B. \tag{3.1.34}$$

We will show that we can obtain the estimate

$$Q \le \frac{3B}{4}, \tag{3.1.35}$$

for an appropriate choice of $B > 0$ and for a sufficiently small M. We have

$$\int_x^1 u_t + p^* - \left(P - u^2 g - uh + j^*\right) = -\mu E u_x = c\mu \frac{\partial}{\partial t}\left(\ln \frac{cQ}{1+cQ}\right) = c\mu \frac{\partial}{\partial t}\left(\ln \frac{Q}{1+cQ}\right).$$

We consider a fixed $x \in (0,1)$ and introduce the variable $Y(t) = \ln(\frac{Q}{1+cQ}) = \ln(\frac{1}{c}\frac{Q}{\frac{1}{c}+Q}) < \ln(\frac{1}{c})$ for $Q \in (0,\infty)$ for $c(x) > 0$ (but possibly arbitrary near 0) and observe that

$$Q = \frac{e^Y}{1-ce^Y} = \frac{1}{c}\left(\frac{e^Y}{1/c - e^Y}\right), \quad Y \in \left(-\infty, \ln(1/c)\right). \tag{3.1.36}$$

We have

$$P = (1-c)^\gamma Q^\gamma = (1-c)^\gamma \left(\frac{e^Y}{1-ce^Y}\right)^\gamma = P(c, Y). \tag{3.1.37}$$

Clearly, P is an increasing function relatively Y (here we use that $\sup c = \sup c_0 < 1$). By letting Y approach $\ln(1/c)$ from below, we can get $1 - ce^Y$ P as close to 0 as we desire. Hence, we can get $P(c, Y)$ as large as we want. In particular,

$$P(c,Y) \geq (1-c)^\gamma \left(\frac{B}{2}\right)^\gamma = (1-c_0)^\gamma \left(\frac{B}{2}\right)^\gamma,$$
$$\text{if } Y \geq M_2 = \ln\left(\frac{B/2}{1+cB/2}\right) \in \big(-\infty, \ln(1/c)\big) \tag{3.1.38}$$

for some $B \in (0, \infty)$.

$$\begin{aligned} Y'(t) &= \frac{d}{dt}\left(\frac{1}{c\mu}\int_x^1 u\right) + \frac{1}{c\mu}\left[p^* - P + u^2 g + uh - j^*\right] \\ &= \frac{d}{dt}\left(\frac{1}{c\mu}\int_x^1 u\right) + \frac{d}{dt}\left(\frac{1}{c\mu}\int_0^t \left[u^2 g + \frac{1}{2}u^2 h\right]\right) + \frac{1}{c\mu}\left[p^* - P + uh - \frac{1}{2}u^2 h - j^*\right] \\ &= \frac{db}{dt} + f(Y), \end{aligned}$$

where

$$f(Y) = \frac{1}{c\mu}\left[p^* - j^* - P + uh - \frac{1}{2}u^2 h\right],$$
$$b(t) = \frac{1}{c\mu}\int_x^1 u + \frac{1}{c\mu}\int_0^t \left[u^2 g + \frac{1}{2}u^2 h\right].$$

We have that

$$uh \leq \frac{1}{2}h + \frac{1}{2}hu^2,$$

hence

$$f(Y) \leq \frac{1}{c\mu}\left[p^* - j^* + \frac{1}{2}h - P\right]. \tag{3.1.39}$$

Moreover, for $Q \in (0, \infty)$ (or $Y \in (-\infty, \ln(1/c))$) we have

$$h = 2\left(\frac{\hat{c}_1}{\hat{c}_0}\right)a^*\left(\frac{cQ}{\alpha_l^* + cQ}\right) \leq 2\left(\frac{\hat{c}_1}{\hat{c}_0}\right)a^*$$
$$-j^* = \rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 \frac{a^*}{k^*}\left(\frac{cQ}{\alpha_l^* + cQ}\right) \leq \rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2 \frac{a^*}{k^*}. \tag{3.1.40}$$

Clearly, in view of (3.1.39) and (3.1.40) we can estimate as follows

$$
\begin{aligned}
f(Y) &\le \frac{1}{c\mu}\left[p^* - j^* + \frac{1}{2}h - P\right] \le \frac{1}{c\mu}\left[p^* + \left(\frac{\hat{c}_1}{\hat{c}_0}\right)a^* + \rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{a^*}{k^*} - P\right] \\
&= \frac{1}{c\mu}\left[C\left(p^*, \hat{c}_0, \hat{c}_1\right) - P\right] \le 0
\end{aligned}
$$

for

$$
Y \ge M_2 = \ln\left(\frac{B/2}{1 + cB/2}\right),
$$

if (compare with (3.1.38))

$$
(1 - c_0)^\gamma\left(\frac{B}{2}\right)^\gamma \ge C\left(p^*, \hat{c}_0, \hat{c}_1\right). \tag{3.1.41}
$$

Next, we have to estimate b and see that it can be bounded as a function of time. Clearly, in view of (3.1.34) we have

$$
\frac{Q}{\alpha_l^* + cQ} \le \frac{B}{\alpha_l^*}. \tag{3.1.42}
$$

Thus, we conclude that

$$
\begin{aligned}
\frac{h}{c} &= 2\left(\frac{\hat{c}_1}{\hat{c}_0}\right)a^*\left(\frac{Q}{\alpha_l^* + cQ}\right) \le 2\left(\frac{\hat{c}_1}{\hat{c}_0}\right)a^*\frac{B}{\alpha_l^*} \\
\frac{g}{c} &= a^*\alpha_l^*\left(\frac{Q}{\alpha_l^* + cQ}\right) \le a^*B.
\end{aligned} \tag{3.1.43}
$$

These estimates will be used to bound $b(t)$ as follows:

$$
\begin{aligned}
\left|b(t_2) - b(t_1)\right| &\le \frac{2}{c\mu}\sup_t\left|\int_x^1 u\right| + \frac{1}{c\mu}\int_{t_1}^{t_2}\left[u^2 g + \frac{1}{2}u^2 h\right] \\
&\le \frac{2}{c\mu}\phi(x)^{1/2}\sup_t\left(\int_0^1 u^2\,dy\right)^{1/2} + \frac{1}{c\mu}\int_{t_1}^{t_2} u^2\left[g + \frac{1}{2}h\right] \\
&\le \frac{2}{\mu\tilde{c}_1}(2M)^{1/2} + C_1(A)\frac{1}{\mu a^*}\int_{t_1}^{t_2}\left[\frac{g}{c} + \frac{1}{2}\frac{h}{c}\right]\int_0^1 Eu_x^2 \\
&\le \frac{2}{\mu\tilde{c}_1}(2M)^{1/2} + 12C_1(A)\frac{1}{\mu^2}\left[1 + \left(\frac{\hat{c}_1}{\hat{c}_0}\right)\frac{1}{\alpha_l^*}\right]BM := N_0.
\end{aligned} \tag{3.1.44}
$$

Here we have used the estimate (3.1.8) of u^2, (3.1.3), as well as estimates (3.1.43) to ensure that $\frac{g}{c}$ and $\frac{h}{c}$ are bounded, independent of c. For the chosen B, we can choose M small enough to ensure that

$$N_0 \leq \ln(1+\delta), \tag{3.1.45}$$

for some $\delta > 0$ which is specified below. Consequently,

$$Y(t) \leq \max\{Y(0), M_2\} + \ln(1+\delta).$$

Hence, we can choose initial data such that $Y(0) \leq M_2$, that is,

$$Q_0 \leq B_1 \leq \frac{B}{2}, \tag{3.1.46}$$

which gives

$$Y(t) \leq \ln\left(\frac{B/2}{1+cB/2}\right) + \ln(1+\delta) = \ln\left(\frac{[1+\delta]B/2}{1+cB/2}\right).$$

Thus, since $Q = \frac{e^Y}{1-ce^Y}$ we get

$$Q \leq \frac{\frac{1+\delta}{2}B}{1-cB\frac{\delta}{2}} \leq \frac{\frac{1+\delta}{2}B}{1-B\frac{\delta}{2}}. \tag{3.1.47}$$

According to (3.1.35), we want to show that $Q \leq \frac{3}{4}B$. In light of (3.1.47), this is obtained subject to the condition that

$$1 \leq \frac{3}{4}\left(\frac{2}{1+\delta}\right)\left(1 - B\frac{\delta}{2}\right),$$

that is

$$B \leq \frac{2}{\delta} - \frac{4}{3}\frac{1+\delta}{\delta} = \frac{2-4\delta}{3\delta}. \tag{3.1.48}$$

(3.1.48) will guarantee that (3.1.35) holds. Consequently, we choose $\delta > 0$ small enough such that the upper bound of B given by the right hand side of (3.1.48) allows us to choose a B such that (3.1.41) holds. This choice of δ and B will then define a smallness condition on M by the condition (3.1.45) where N_0 is defined in last line of (3.1.44).

Combining (3.1.34) and (3.1.35) with the classical continuation argument, we can conclude that the estimate (3.1.33) has been proved. □

Remark 3.1. Clearly, we have that since $\sup c < 1$, then $\inf n > 0$. Consequently,

$$\begin{aligned} 0 < \inf n \le n = (1-c)\rho \le \rho \\ a^* \frac{A}{1+B} \le \rho = a^* \frac{Q}{1+cQ} \le a^* B \\ a^* \frac{A}{1+B}\tilde{c}_1 \phi^\alpha \le m - k^* = c\rho \le a^* B \tilde{c}_2 \phi^\alpha, \end{aligned} \tag{3.1.49}$$

in view of (3.1.32) and (3.1.33) and the definitions of c and Q. These estimates ensure that the initial mixture of gas and liquid will remain a mixture of both phases at any point in $[0, 1)$ in the sense that the gas mass will not vanish at any point nor the liquid mass. The liquid mass will vanish at the free interface and the decay rate is $\phi(x)^\alpha$.

Remark 3.2. In light of Lemma 3.3 and Lemma 3.4 we have found positive uniform bounds on the quantity Q. However, it turns out that more detailed information about the interface behavior of Q is required for obtaining estimates of Q_x in L^1, see Lemma 3.9. More precisely, we will search for estimates of the decay rate of Q towards Q^* at the free interface. This is the purpose of Lemma 3.5 and Lemma 3.7 where an estimate of the quantity $|\ln(Q^*(1 + cQ)) - \ln(Q)|$ is obtained in two steps. In particular, we will need the uniform bound on the fluid velocity as given in (3.1.72) of Corollary 3.2 in order to obtain the estimate of the quantity $\ln \frac{Q}{Q^*(1+cQ)}$.

Lemma 3.5. *Under the assumptions of Theorem 2.1, it holds that*

$$\ln \frac{Q^*(1+cQ)}{Q} \le C_1 \phi^{\beta_1}, \tag{3.1.50}$$

where $\beta_1 \in (0, \alpha] \cap (0, \frac{1}{2} - \alpha]$.

Proof.

$$-Eu_x = -c\frac{\partial}{\partial t}\left(\ln \frac{1+cQ}{Q}\right) = -c\frac{\partial}{\partial t}\left(\ln \frac{Q^*(1+cQ)}{Q}\right). \tag{3.1.51}$$

We fix $x \in (0, 1)$ and introduce the variable $Y(t) = \ln(\frac{Q^*(1+cQ)}{Q}) = \ln(\frac{Q^*}{Q} + cQ^*)$ for $Q \in (0, \infty)$ and observe that

$$Q = \frac{Q^*}{e^Y - cQ^*}, \tag{3.1.52}$$

where $Y > \ln(cQ^*)$ since $Q > 0$. Moreover,

$$P(c, Q) = \big[(1-c)Q\big]^\gamma = \left(\frac{Q^*(1-c)}{e^Y - cQ^*}\right)^\gamma = P(c, Y).$$

Clearly, P is a decreasing function relatively Y and tends to zero as Y goes to infinity. We may combine (3.1.21) and (3.1.51) and get

$$Y'(t) = \frac{d}{dt}\left(-\frac{1}{c\mu}\int_x^1 u\right) + \frac{1}{c\mu}\left[-p^* + P - u^2 g - uh + j^*\right] = \frac{db}{dt} + f(Y).$$

From (3.1.25), we obtain

$$f(Y) \le \frac{1}{c\mu}\left[P - p^*\right]. \tag{3.1.53}$$

Furthermore,

$$f(Y) \le 0, \tag{3.1.54}$$

provided that

$$P \le p^*,$$

i.e.,

$$Y \ge \ln\left(1 + \left(Q^* - 1\right)c_0\right) := M_2.$$

For the b function we have

$$\left|b(t_2) - b(t_1)\right| \le \frac{2}{c\mu}\phi(x)^{1/2}\sup_t\left(\int_0^1 u^2\,dy\right)^{1/2} \le \frac{2\phi^{\frac{1}{2}-\alpha}}{\mu\tilde{c}_1}(2M)^{1/2} \overset{\text{def}}{:=} N_0,$$

subject to the condition that A, M are chosen according to (3.1.27). Hence, by Lemma 3.2, we get

$$Y(t) \le \max\left\{Y(0), M_2\right\} + N_0,$$

i.e.,

$$\begin{aligned}\ln\left(\frac{Q^*(1+cQ)}{Q}\right) &\le \max\left\{\ln\left(\frac{Q^*(1+c_0Q_0)}{Q_0}\right), \ln\left(1 + \left(Q^* - 1\right)c_0\right)\right\} + \frac{2\phi^{\frac{1}{2}-\alpha}}{\mu\tilde{c}_1}(2M)^{1/2} \\ &\le C_1\phi^{\beta_1},\end{aligned}$$

provided that

$$\ln\left(\frac{Q^*(1+c_0Q_0)}{Q_0}\right) \le \tilde{C}\phi^{\beta_1}$$

and $\beta_1 \in (0,\alpha] \cap (0, \frac{1}{2} - \alpha]$. $\square$

Throughout the rest of the paper, we use C_2 to denote a generic positive constant that depends on A, B, T, initial data and some other known constants. Moreover, $C_2 < \infty$ for any given $T < \infty$.

Lemma 3.6. *Under the assumptions of Theorem 2.1, it holds that*

$$\int_0^1 Eu_x^2 + \int_0^t \int_0^1 u_t^2 \le C_2, \tag{3.1.55}$$

for M satisfying the assumption (3.1.70).

Proof. Multiplying $(2.1.2)_3$ (j replaced by j^*) by u_t, integrating by parts over $[0,1]$, and using $(2.1.2)_1$, we have

$$\int_0^1 u_t^2 + \frac{\mu}{2}\frac{d}{dt}\int_0^1 Eu_x^2$$
$$= \frac{d}{dt}\int_0^1 \left[P(c,Q) - p^* - u^2 g(cQ) - uh(cQ) + j^*(cQ)\right]u_x + I, \tag{3.1.56}$$

where we have used that $u_t(1,t) = \frac{d}{dt}\int_0^1 u_x$ and

$$\begin{aligned} I &= -\int_0^1 \left[P(c,Q) - u^2 g(cQ) - uh(cQ) + j^*(cQ)\right]_t u_x + \frac{1}{2}\int_0^1 E_t u_x^2 \\ &= -\int_0^1 \left[P(c,Q)\right]_t u_x + \int_0^1 \left[u^2 g(cQ)\right]_t u_x + \int_0^1 \left[uh(cQ)\right]_t u_x - \int_0^1 \left[j^*(cQ)\right]_t u_x + \frac{1}{2}\int_0^1 E_t u_x^2 \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

For I_1, we have

$$I_1 = -\int_0^1 \frac{\partial P(c,Q)}{\partial Q} Q_t u_x = a^* \int_0^1 \frac{\partial P(c,Q)}{\partial Q} Q^2 u_x^2 \le C_1 \int_0^1 u_x^2, \tag{3.1.57}$$

where we have used $(2.1.2)_2$ and (3.1.33). By (3.1.21), (3.1.32), (3.1.8), Hölder's inequality and $c = c_0$, we have

$$
\begin{aligned}
|u_x| &\le \frac{1}{\mu a^*}\left(\frac{1}{cA}+1\right)\int_x^1 |u_t| + \frac{1}{\mu a^*}\left(\frac{1}{cA}+1\right)\left|p^*-P\right| + C_1 c\left(\frac{1}{cA}+1\right)\left(u^2+|u|+1\right) \\
&\le C_1\phi^{\frac{1}{2}-\alpha}\left(\int_0^1 u_t^2\right)^{\frac{1}{2}} + \frac{C_1}{c} + C_1\left(u^2+|u|+1\right) \\
&\le C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}} + C_1\phi^{-\alpha} + C_1|u|\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}} + C_1|u| + C_1.
\end{aligned}
\tag{3.1.58}
$$

To estimate I_1 further, we need to control $\int_0^1 u_x^2$ with the help of (3.1.58), i.e.,

$$
\begin{aligned}
\int_0^1 u_x^2 &\le C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}}\int_0^1 |u_x| + C_1\int_0^1 |u_x|\phi(x)^{-\alpha} + C_1\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}}\int_0^1 |u_x||u| \\
&\quad + C_1\int_0^1 |u_x|\left(|u|+1\right) \\
&\le C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}}\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}} + \frac{1}{4}\int_0^1 u_x^2 + C_1\int_0^1 \phi(x)^{-2\alpha} \\
&\quad + C_1\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}}\left(\int_0^1 u_x^2\right)^{\frac{1}{2}}\left(\int_0^1 u^2\right)^{\frac{1}{2}} + C_1\int_0^1\left(u^2+1\right) \\
&\le C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}}\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}} + \frac{1}{2}\int_0^1 u_x^2 + C_1\int_0^1 Eu_x^2 + C_1,
\end{aligned}
\tag{3.1.59}
$$

where we have used the Cauchy inequality, Hölder inequality and (3.1.3). (3.1.59) implies that

$$
\int_0^1 u_x^2 \le C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}}\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}} + C_1\int_0^1 Eu_x^2 + C_1. \tag{3.1.60}
$$

Putting (3.1.60) into (3.1.57), and using Cauchy's inequality again, we have

$$
I_1 \le \frac{1}{8}\int_0^1 u_t^2 + C_1\int_0^1 Eu_x^2 + C_1. \tag{3.1.61}
$$

For I_2, we have

$$\begin{aligned} I_2 &= \int_0^1 \big[2uu_t g(cQ) + u^2 g'(cQ)cQ_t\big]u_x \\ &\le \frac{1}{8}\int_0^1 u_t^2 + C_1\int_0^1 (cQ)^2u^2u_x^2 - a^*\int_0^1 u^2g'(cQ)cQ^2u_x^2 \\ &\le \frac{1}{8}\int_0^1 u_t^2 + C_1\|u\|_{L^\infty}^2\int_0^1 Eu_x^2 \\ &\le \frac{1}{8}\int_0^1 u_t^2 + C_1\bigg(\int_0^1 Eu_x^2\bigg)^2, \end{aligned} \tag{3.1.62}$$

where we have used the Cauchy inequality, (2.1.2)$_2$, (3.1.33) and (3.1.8).

Similar to I_2, we have

$$\begin{aligned} I_3 + I_4 &= \int_0^1 \big[u_t h(cQ) + uh'(cQ)cQ_t\big]u_x - \int_0^1 \big[j^*(cQ)\big]' cQ_t u_x \\ &\le \frac{1}{8}\int_0^1 u_t^2 + C_1\int_0^1 Eu_x^2 + C_1\|u\|_{L^\infty}\int_0^1 Eu_x^2 \\ &\le \frac{1}{8}\int_0^1 u_t^2 + C_1\int_0^1 Eu_x^2 + C_1\bigg(\int_0^1 Eu_x^2\bigg)^2. \end{aligned} \tag{3.1.63}$$

Note that $E = E(cQ)$ and that $c = c_0$. Then

$$\begin{aligned} I_5 &= \frac{1}{2}\int_0^1 E'cQ_t u_x^2 \\ &= -\frac{a^*}{2}\int_0^1 E'cQ^2u_xu_x^2 \le C_1\int_0^1 E|u_x|u_x^2. \end{aligned} \tag{3.1.64}$$

(3.1.58), (3.1.8) and the Cauchy inequality give that

$$|u_x| \le C_1\bigg(\int_0^1 u_t^2\bigg)^{\frac{1}{2}} + C_1\phi^{-\alpha} + C_1\int_0^1 Eu_x^2 + C_1. \tag{3.1.65}$$

Putting (3.1.65) into (3.1.64), and using Cauchy's inequality and (3.1.60), we have

$$
\begin{aligned}
I_5 &\le C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}}\int_0^1 Eu_x^2 + C_1\int_0^1 u_x^2 + C_1\left(\int_0^1 Eu_x^2\right)^2 \\
&\le C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}}\int_0^1 Eu_x^2 + C_1\left(\int_0^1 u_t^2\right)^{\frac{1}{2}}\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}} + C_1\left(\int_0^1 Eu_x^2\right)^2 + C_1 \\
&\le \frac{1}{8}\int_0^1 u_t^2 + C_1\left(\int_0^1 Eu_x^2\right)^2 + C_1. \qquad (3.1.66)
\end{aligned}
$$

With (3.1.61), (3.1.62), (3.1.63) and (3.1.66), we can deduce (3.1.56) combined with the Cauchy inequality as follows.

$$
\begin{aligned}
\frac{1}{2}\int_0^1 u_t^2 + \frac{\mu}{2}\frac{d}{dt}\int_0^1 Eu_x^2 &\le \frac{d}{dt}\int_0^1 \big[P(c,Q) - p^* - u^2 g(cQ) - uh(cQ) + j^*(cQ)\big]u_x \\
&\quad + C_1\left(\mu\int_0^1 Eu_x^2\right)^2 + C_1. \qquad (3.1.67)
\end{aligned}
$$

Integrating (3.1.67) over $[0,t]$, and using (3.1.33), we have

$$
\begin{aligned}
\int_0^t\int_0^1 u_t^2 + \mu\int_0^1 Eu_x^2 &\le 2\int_0^1 \big[P(c,Q) - p^* - u^2 g(cQ) - uh(cQ) + j^*(cQ)\big]u_x \\
&\quad + C_1\int_0^t\left(\mu\int_0^1 Eu_x^2\right)^2 + C_1(t+1) \\
&\le C_1\int_0^1 |u_x| + 2\int_0^1 u^2 E|u_x| + C_1\int_0^1 E|u||u_x| \\
&\quad + C_1\int_0^t\left(\mu\int_0^1 Eu_x^2\right)^2 + C_2. \qquad (3.1.68)
\end{aligned}
$$

Note that

$$
C_1\int_0^1 |u_x| \le C_1\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}}\left(\int_0^1 \frac{1}{E}\right)^{\frac{1}{2}} \le C_1\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}} \le C_1 + \frac{\mu}{4}\int_0^1 Eu_x^2,
$$

$$2\int_0^1 u^2E|u_x| \le 2\|u\|_{L^\infty}\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}}\left(\int_0^1 Eu^2\right)^{\frac{1}{2}}$$
$$\le \frac{2}{\mu}\left[C_1(A)\right]^{\frac{1}{2}}\left(\int_0^1 u^2\right)^{\frac{1}{2}}\mu\int_0^1 Eu_x^2 \le \frac{2}{\mu}\left[\frac{2}{\tilde{c}_1A}+1\right]^{\frac{1}{2}}(2M)^{\frac{1}{2}}\mu\int_0^1 Eu_x^2$$

and

$$C_1\int_0^1 E|u||u_x| \le C_1\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}}\left(\int_0^1 Eu^2\right)^{\frac{1}{2}} \le C_1\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}} \le C_1+\frac{\mu}{4}\int_0^1 Eu_x^2.$$

These combined with (3.1.68) and Cauchy inequality conclude

$$\int_0^t\int_0^1 u_t^2+\frac{\mu}{2}\int_0^1 Eu_x^2 \le C_1\int_0^t\left(\mu\int_0^1 Eu_x^2\right)^2 + \frac{2}{\mu}\left[\frac{2}{\tilde{c}_1A}+1\right]^{\frac{1}{2}}\sqrt{2M}\mu\int_0^1 Eu_x^2+C_2. \quad (3.1.69)$$

If we take M small enough such that

$$\frac{2}{\mu}\left[\frac{2}{\tilde{c}_1A}+1\right]^{\frac{1}{2}}\sqrt{2M} \le \frac{1}{4}, \quad (3.1.70)$$

then the second term on the right side can absorb the second one on the right side. After that, we apply Gronwall's inequality and get (3.1.55). □

Corollary 3.2. *Under the assumptions of Theorem 2.1, it holds that*

$$\int_0^1 |u_x|^r \le C_2 \quad (3.1.71)$$

for some $r\in(1,2)$ *such that* $r(\alpha+1)<2$, *and*

$$\|u\|_{L^\infty} \le C_2, \quad (3.1.72)$$

and

$$\|u_x\|_{L^4(0,T;L^2)} \le C_2. \quad (3.1.73)$$

Proof.

$$\begin{aligned}\int_0^1 |u_x|^r &= \int_0^1 E^{\frac{r}{2}}|u_x|^r E^{-\frac{r}{2}} \\ &\le \left(\int_0^1 Eu_x^2\right)^{\frac{r}{2}} \left(\int_0^1 E^{-\frac{r}{2-r}}\right)^{\frac{2-r}{2}} \le C_2\left(\int_0^1 \left[\frac{1}{cQ}+1\right]^{\frac{r}{2-r}}\right)^{\frac{2-r}{2}} \\ &\le C_2(\widetilde{c}_1, A)\left(\int_0^1 \phi(x)^{\frac{\alpha r}{r-2}}\right)^{\frac{2-r}{2}} \le C_2,\end{aligned}$$

for some $r \in (1,2)$ such that $r(\alpha+1) < 2$.

(3.1.72) can be obtained by using (3.1.55) and (3.1.8). With (3.1.60) and (3.1.55), we get (3.1.73). □

Armed with the uniform bound of the fluid velocity (3.1.72) we can proceed and get a bound that characterizes the decay rate of Q towards Q^* in terms of the quantity $\ln \frac{Q}{Q^*(1+cQ)}$.

Lemma 3.7. *Under the assumptions of Theorem 2.1, it holds that*

$$\ln \frac{Q}{Q^*(1+cQ)} \le C_2\phi^{\beta_1}, \tag{3.1.74}$$

where $\beta_1 \in (0, \alpha] \cap (0, \frac{1}{2} - \alpha]$.

Proof. We consider a fixed $x \in (0,1)$, hence, $c(x) = c_0(x) > 0$. Next, introduce the variable $Y(t) = \ln(\frac{Q}{Q^*(1+cQ)}) < \ln(\frac{1}{c_0 Q^*})$ for $Q \in (0, \infty)$ and observe that

$$Q = \frac{Q^* e^Y}{1 - cQ^* e^Y}, \quad Y \in \left(-\infty, \ln\left(1/c_0 Q^*\right)\right). \tag{3.1.75}$$

We have

$$P = (1-c)^\gamma Q^\gamma = (1-c)^\gamma \left(\frac{Q^* e^Y}{1 - cQ^* e^Y}\right)^\gamma = P(c, Y). \tag{3.1.76}$$

Then

$$\begin{aligned}Y'(t) &= \frac{d}{dt}\left(\frac{1}{c\mu}\int_x^1 u\right) + \frac{1}{c\mu}\left[p^* - P + u^2 g + uh - j^*\right] \\ &= \frac{db}{dt} + f(Y),\end{aligned}$$

where

$$f(Y) = \frac{1}{c\mu}\big[p^* - P - j^* + u^2 g + uh\big],$$
$$b(t) = \frac{1}{c\mu}\int_x^1 u.$$

For $Q \in (0,\infty)$ (or $Y \in (-\infty, \ln\frac{1}{cQ^*})$), we have

$$\begin{aligned} g &= a^*\alpha_l^*\left(\frac{cQ}{\alpha_l^* + cQ}\right) \le a^* B\tilde{c}_2\phi^\alpha \\ h &= 2\left(\frac{\hat{c}_1}{\hat{c}_0}\right)a^*\left(\frac{cQ}{\alpha_l^* + cQ}\right) \le \frac{2\hat{c}_1 a^*\tilde{c}_2 B}{\hat{c}_0\alpha_l^*}\phi^\alpha \\ -j^* &= \rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{a^*}{k^*}\left(\frac{cQ}{\alpha_l^* + cQ}\right) \le \rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{a^*\tilde{c}_2 B}{k^*\alpha_l^*}\phi^\alpha. \end{aligned} \tag{3.1.77}$$

Using (3.1.72) and (3.1.77), we get

$$\begin{aligned} f(Y) &\le \frac{1}{c\mu}\left[p^* - P + \left(\rho_l\left(\frac{\hat{c}_1}{\hat{c}_0}\right)^2\frac{a^*\tilde{c}_2 B}{k^*\alpha_l^*} + C_2(C_2+1)\left(a^* B\tilde{c}_2 + \frac{2\hat{c}_1 a^*\tilde{c}_2 B}{\hat{c}_0\alpha_l^*}\right)\right)\phi^\alpha\right] \\ &= \frac{1}{c\mu}\big[p^* + \tilde{B}\phi^\alpha - P\big] \le 0, \end{aligned} \tag{3.1.78}$$

where $\tilde{B} = \rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2\frac{a^*\tilde{c}_2 B}{k^*\alpha_l^*} + C_2(C_2+1)(a^* B\tilde{c}_2 + \frac{2\hat{c}_1 a^*\tilde{c}_2 B}{\hat{c}_0\alpha_l^*})$, for

$$Y \ge M_2 := \ln\frac{(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}}{(1-c_0)Q^* + c_0 Q^*(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}}.$$

Next, we have to estimate b and see that it can be bounded as a function of time. Clearly, we have that

$$\big|b(t_2) - b(t_1)\big| \le \frac{2}{c\mu}\phi(x)^{1/2}\sup_t\left(\int_0^1 u^2\,dy\right)^{1/2} \le C_1\phi^{1/2-\alpha} := N_0. \tag{3.1.79}$$

Consequently, we apply Lemma 3.2 to get

$$Y(t) \le \max\big\{Y(0), M_2\big\} + N_0 \le \max\big\{Y(0), M_2\big\} + C_1\phi^{1/2-\alpha}. \tag{3.1.80}$$

Note that

- $\frac{1}{2} \le c_0 \le 1$. For this case, $\phi^\alpha \ge \frac{1}{2\tilde{c}_2}$. Then

$$M_2 = \ln \frac{(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}}{(1-c_0)Q^* + c_0 Q^*(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}}$$
$$\le \ln \frac{(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}}{c_0 Q^*(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}} = \ln \frac{1}{c_0 Q^*} \le \ln \frac{2}{Q^*} \le 2\tilde{c}_2 \max\left\{\ln \frac{2}{Q^*}, 0\right\}\phi^\alpha.$$

- $0 \le c_0 < \frac{1}{2}$. For this case, we get

$$M_2 = \ln \frac{(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}}{(1-c_0)Q^* + c_0 Q^*(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}} \le \ln \frac{(p^* + \tilde{B}\phi^\alpha)^{\frac{1}{\gamma}}}{(1-c_0)Q^*} \le \ln \frac{(p^* + \frac{\tilde{B}}{\tilde{c}_1}c_0)^{\frac{1}{\gamma}}}{(1-c_0)Q^*}$$
$$= \int_0^{c_0} \frac{d}{dx} \ln \frac{(p^* + \frac{\tilde{B}}{\tilde{c}_1}x)^{\frac{1}{\gamma}}}{(1-x)Q^*}\, dx$$
$$= \int_0^{c_0} \frac{(1-x)Q^*}{(p^* + \frac{\tilde{B}}{\tilde{c}_1}x)^{\frac{1}{\gamma}}} \frac{\frac{1}{\gamma}\frac{\tilde{B}}{\tilde{c}_1}(p^* + \frac{\tilde{B}}{\tilde{c}_1}x)^{\frac{1}{\gamma}-1}(1-x) + (p^* + \frac{\tilde{B}}{\tilde{c}_1}x)^{\frac{1}{\gamma}}}{(1-x)^2 Q^*}\, dx$$
$$= \int_0^{c_0} \frac{\frac{1}{\gamma}\frac{\tilde{B}}{\tilde{c}_1}(p^* + \frac{\tilde{B}}{\tilde{c}_1}x)^{-1}(1-x) + 1}{1-x}\, dx \le \frac{1}{\gamma p^*}\frac{\tilde{B}}{\tilde{c}_1}c_0 + \frac{c_0}{1-c_0}$$
$$\le \left(\frac{1}{\gamma p^*}\frac{\tilde{B}}{\tilde{c}_1} + 2\right)\tilde{c}_2\phi^\alpha.$$

Consequently, for any $c_0 \in [0, 1]$, we have

$$M_2 \le \max\left\{2\tilde{c}_2 \max\left\{\ln \frac{2}{Q^*}, 0\right\},\ \left(\frac{1}{\gamma p^*}\frac{\tilde{B}}{\tilde{c}_1} + 2\right)\tilde{c}_2\right\}\phi^\alpha. \qquad (3.1.81)$$

By (3.1.80), (3.1.81) and the initial assumption

$$Y(0) = \ln\left(\frac{Q_0}{Q^*(1 + c_0 Q_0)}\right) \le \tilde{C}\phi^{\beta_1},$$

we have

$$Y(t) \le C_2\phi^{\beta_1}$$

for $\beta_1 \in (0, \alpha] \cap (0, \frac{1}{2} - \alpha]$. □

Combining Lemma 3.5 with Lemma 3.7, we get a corollary as follows:

Corollary 3.3. *Under the assumptions of Theorem 2.1, it holds that*

$$\left|\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right| \le C_2\phi^{\beta_1} \tag{3.1.82}$$

for $\beta_1 \in (0,\alpha] \cap (0,\frac{1}{2}-\alpha]$.

Based on (3.1.82), the behavior of $|Q-Q^*|$ as x goes to 1 will be obtained. This is the purpose of the next corollary.

Corollary 3.4. *Under the assumptions of Theorem 2.1, it holds that*

$$\left|u(x,t)\right| \le C_2 x^{\frac{r-1}{r}} \tag{3.1.83}$$

for some $r \in (1,2)$ *such that* $r(\alpha+1) < 2$, *and*

$$\left|Q-Q^*\right| \le C_2\phi^{\beta_1} \tag{3.1.84}$$

for $\beta_1 \in (0,\alpha] \cap (0,\frac{1}{2}-\alpha]$.

Proof. (3.1.83) can be obtained as follows:

$$\left|u(x,t)\right| = \left|u(x,t)-u(0,t)\right| = \left|\int_0^x u_y(y,t)\,dy\right| \le x^{\frac{r-1}{r}}\left(\int_0^x |u_y|^r\,dy\right)^{\frac{1}{r}} \le C_2 x^{\frac{r-1}{r}},$$

where we have used the boundary condition, Hölder's inequality and (3.1.71).

By (3.1.82), we get

$$\exp\{-C_2\phi^{\beta_1}\} \le \frac{Q}{Q^*(1+cQ)} \le \exp\{C_2\phi^{\beta_1}\}.$$

This concludes

$$Q^*\exp\{-C_2\phi^{\beta_1}\} \le Q \le Q^*\left(1+\tilde{c}_2B\phi^{\alpha}\right)\exp\{C_2\phi^{\beta_1}\}, \tag{3.1.85}$$

where we have used (3.1.33). With (3.1.85), we easily get

$$Q^*\left(\exp\{-C_2\phi^{\beta_1}\}-1\right) \le Q-Q^* \le \left(\exp\{C_2\phi^{\beta_1}\}-1\right)Q^* + Q^*\tilde{c}_2B\phi^{\alpha}\exp\{C_2\phi^{\beta_1}\},$$

which combined with the fact that

$$-B_1y \le \exp\{-y\}-1, \qquad \exp\{y\}-1 \le B_2y$$

for y in a bounded interval $[0, C_2]$ and some positive constants $B_1(C_2)$ and $B_2(C_2)$, deduces (3.1.84). □

Lemma 3.8. *Under the assumptions of Theorem 2.1, it holds that*

$$\int_0^1 \phi(x)^{1-\alpha_1}\left|\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x\right|^2 \le C_2, \tag{3.1.86}$$

for some $\alpha_1 \in (0,1)$ such that $\phi^{\frac{1-\alpha_1-\alpha+2\beta_1}{2}}c_{0,x} \in L^2$ and

$$\phi(x)^{\frac{1-\alpha_1}{2}}\left[c_0\ln\left(\frac{Q_0}{Q^*(1+c_0Q_0)}\right)\right]_x \in L^2.$$

Proof. We have from (3.1.21) and (3.1.22) that

$$\left(u + \mu\left[c\ln\left(\frac{cQ}{1+cQ}\right)\right]_x\right)_t + \left(P - u^2g - uh + j^*\right)_x = 0.$$

This also corresponds to

$$\left(u + \mu\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x\right)_t + \left(P - u^2g - uh + j^*\right)_x = 0, \tag{3.1.87}$$

where we have used $c = c_0$ due to $(1.2.6)_1$.

Multiplying (3.1.87) by $\phi(x)^{1-\alpha_1}[c\ln(\frac{Q}{Q^*(1+cQ)})]_x$, and integrating by parts over $[0,1]$, we have

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_0^1 \phi(x)^{1-\alpha_1}\left|\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x\right|^2 \\
&= -\int_0^1 \phi(x)^{1-\alpha_1}u_t\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \\
&\quad - \int_0^1 \phi(x)^{1-\alpha_1}\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \left[P(c,Q)\right]_x \\
&\quad + \int_0^1 \phi(x)^{1-\alpha_1}\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \left[u^2g(cQ)\right]_x \\
&\quad + \int_0^1 \phi(x)^{1-\alpha_1}\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \left[uh(cQ)\right]_x
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^1 \phi(x)^{1-\alpha_1} \left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \left[j^*(cQ)\right]_x \\
&= \sum_{i=1}^5 II_i.
\end{aligned} \tag{3.1.88}
$$

For II_1, using the Cauchy inequality, we have

$$
II_1 \le C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left|\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x\right|^2 + C_2 \int_0^1 u_t^2. \tag{3.1.89}
$$

For II_2, we have

$$
\begin{aligned}
II_2 &= -\gamma \int_0^1 \phi(x)^{1-\alpha_1} \left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \left[(1-c)Q\right]^{\gamma-1} \left[Q_x(1-c) - c_x Q\right] \\
&= -\gamma \int_0^1 \phi(x)^{1-\alpha_1} (1-c)^\gamma Q^{\gamma-1} Q_x \left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \\
&\quad + \gamma \int_0^1 \phi(x)^{1-\alpha_1} \left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \left[(1-c)Q\right]^{\gamma-1} c_x Q.
\end{aligned} \tag{3.1.90}
$$

One can easily figure out how Q_x is linked to $[c\ln(\frac{Q}{Q^*(1+cQ)})]_x$ by using some direct calculations, i.e.,

$$
\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x = \left[\ln\left(\frac{Q}{Q^*(1+cQ)}\right) - \frac{cQ}{1+cQ}\right] c_x + \frac{c}{Q(1+cQ)} Q_x.
$$

Hence

$$
\begin{aligned}
Q_x &= (1+cQ)\frac{Q}{c}\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x \\
&\quad - \left[\frac{1}{c}(1+cQ)\ln\left(\frac{Q}{Q^*(1+cQ)}\right) - Q\right] c_x Q.
\end{aligned} \tag{3.1.91}
$$

Putting (3.1.91) into (3.1.90), we have

$$
\begin{aligned}
II_2 &= -\gamma \int_0^1 \phi(x)^{1-\alpha_1} \frac{1+cQ}{c} \left[(1-c)Q\right]^\gamma \left|\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x\right|^2 \\
&\quad + \gamma \int_0^1 \phi(x)^{1-\alpha_1} \left[(1-c)Q\right]^\gamma
\end{aligned}
$$

$$\times\left[\frac{1}{c}(1+cQ)\ln\left(\frac{Q}{Q^*(1+cQ)}\right)-Q\right]c_x\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x$$
$$+\gamma\int_0^1\phi(x)^{1-\alpha_1}\left[c\ln\left(\frac{Q}{Q^*(1+cQ)}\right)\right]_x\left[(1-c)Q\right]^{\gamma-1}c_xQ$$
$$=-II_{2,0}+II_{2,1}+II_{2,2}, \tag{3.1.92}$$

where $II_{2,1}$ and $II_{2,2}$ can be handled by using Cauchy's inequality combined with the first term of the right side of (3.1.92), i.e.,

$$\begin{aligned}II_{2,1}\le{}&\frac{II_{2,0}}{8}\\&+C_2\int_0^1\phi(x)^{1-\alpha_1}\left[(1-c)Q\right]^{\gamma}\frac{c}{1+cQ}\left[\frac{1}{c}(1+cQ)\ln\left(\frac{Q}{Q^*(1+cQ)}\right)-Q\right]^2|c_x|^2\\\le{}&\frac{II_{2,0}}{8}+C_2\int_0^1\phi(x)^{1-\alpha_1+\alpha}\left[\phi^{\beta_1-\alpha}+1\right]^2|c_{0,x}|^2\\\le{}&\frac{II_{2,0}}{8}+C_2\int_0^1\phi(x)^{1-\alpha_1-\alpha+2\beta_1}|c_{0,x}|^2,\end{aligned}\tag{3.1.93}$$

and

$$\begin{aligned}II_{2,2}&\le\frac{II_{2,0}}{8}+C_2\int_0^1\phi(x)^{1-\alpha_1}\left[(1-c)Q\right]^{\gamma-2}|c_{0,x}|^2Q^2\frac{c}{1+cQ}\\&\le\frac{II_{2,0}}{8}+C_2\int_0^1\phi(x)^{1-\alpha_1+\alpha}|c_{0,x}|^2\\&\le\frac{II_{2,0}}{8}+C_2\int_0^1\phi(x)^{1-\alpha_1-\alpha+2\beta_1}|c_{0,x}|^2,\end{aligned}\tag{3.1.94}$$

where we have used $\alpha\ge\beta_1$. Putting (3.1.93) and (3.1.94) into (3.1.92), we have

$$II_2\le-\frac{3II_{2,0}}{4}+C_2\int_0^1\phi(x)^{1-\alpha_1-\alpha+2\beta_1}|c_{0,x}|^2. \tag{3.1.95}$$

For II_3, we have

$$II_3 \le C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right| \left[cQ|u_x| + |c_x| + |cQ_x| \right], \tag{3.1.96}$$

where we have used (3.1.72) and (3.1.33). (3.1.91) implies that

$$|cQ_x| \le C_2 \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right| + C_2 |c_{0,x}|. \tag{3.1.97}$$

Putting (3.1.97) into (3.1.96), we have

$$\begin{aligned} II_3 &\le C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2 + C_2 \int_0^1 E u_x^2 + \frac{II_{2,0}}{8} \\ &\quad + C_2 \int_0^1 \phi(x)^{1-\alpha_1} \frac{c}{1+cQ} \left[(1-c)Q \right]^{-\gamma} c_x^2 \\ &\le C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2 + \frac{II_{2,0}}{8} \\ &\quad + C_2 \int_0^1 \phi(x)^{1-\alpha_1+\alpha} c_{0,x}^2 + C_2, \end{aligned} \tag{3.1.98}$$

where we have used the Cauchy inequality, (3.1.55), (3.1.33), (3.1.20) and $c = c_0$. Similar to II_3, for II_4 and II_5, we have

$$\begin{aligned} II_4 + II_5 &\le C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2 \\ &\quad + \frac{II_{2,0}}{8} + C_2 \int_0^1 \phi(x)^{1-\alpha_1+\alpha} c_{0,x}^2 + C_2. \end{aligned} \tag{3.1.99}$$

Putting (3.1.89), (3.1.95), (3.1.98) and (3.1.99) into (3.1.88), we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2 \\ &\le C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2 \end{aligned}$$

$$+ C_2 \int_0^1 u_t^2 + C_2 \int_0^1 \phi(x)^{1-\alpha_1-\alpha+2\beta_1} c_{0,x}^2 + C_2$$

$$\leq C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2 + C_2 \int_0^1 u_t^2 + C_2,$$

where we have used

$$\phi(x)^{\frac{1-\alpha_1-\alpha+2\beta_1}{2}} c_{0,x} \in L^2.$$

This combined with (3.1.55) and Gronwall inequality concludes (3.1.86). □

Lemma 3.9. *Under the assumptions of Theorem 2.1, it holds that*

$$\int_0^1 |Q_x| \leq C_2. \tag{3.1.100}$$

Proof. Based on (3.1.91) which is combined with (3.1.33), (3.1.82), (3.1.86) and Cauchy inequality, (3.1.100) can be obtained as follows.

$$\int_0^1 |Q_x| \leq C_2 \int_0^1 \phi^{\frac{1-\alpha_1}{2}} \phi^{-\alpha-\frac{1-\alpha_1}{2}} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right| + C_2 \int_0^1 \left[\phi^{\beta_1-\alpha} + 1 \right] |c_{0,x}|$$

$$\leq C_2 \int_0^1 \phi(x)^{1-\alpha_1} \left| \left[c \ln\left(\frac{Q}{Q^*(1+cQ)} \right) \right]_x \right|^2$$

$$+ C_2 \int_0^1 \phi^{-2\alpha-1+\alpha_1} + C_2 \int_0^1 \phi^{\beta_1-\alpha} |c_{0,x}|$$

$$\leq C_2 + C_2 \int_0^1 \phi(x)^{1-\alpha_1-\alpha+2\beta_1} |c_{0,x}|^2 + C_2 \int_0^1 \phi^{-1+\alpha_1-\alpha} \leq C_2,$$

where we have additionally used $\alpha \geq \beta_1$, $\alpha_1 > 2\alpha$ and

$$\phi(x)^{\frac{1-\alpha_1-\alpha+2\beta_1}{2}} c_{0,x} \in L^2. \qquad \square$$

Corollary 3.5. *Under the assumptions of Theorem 2.1, it holds that*

$$\int_0^t \left(\int_0^1 Q_t^2 \right)^2 \leq C_2, \tag{3.1.101}$$

$$\int_0^1 \left|Q(x,t)-Q(x,s)\right|^2 \le C_2|t-s|^{\frac{3}{2}}, \tag{3.1.102}$$

$$\int_0^1 \left|u(x,t)-u(x,s)\right|^2 \le C_2|t-s|. \tag{3.1.103}$$

Proof. To obtain (3.1.101)–(3.1.103), we use $(2.1.2)_2$, (3.1.33), (3.1.73), Hölder's inequality and (3.1.55). Specifically, we have

$$\int_0^t \left(\int_0^1 Q_t^2\,dx\right)^2 d\tau \le C_2 \int_0^t \left(\int_0^1 u_x^2\,dx\right)^2 d\tau \le C_2,$$

$$\int_0^1 \left|Q(x,t)-Q(x,s)\right|^2 dx = \int_0^1 \left|\int_s^t \left[Q(x,\tau)\right]_\tau d\tau\right|^2 dx \le (t-s)\int_s^t \int_0^1 \left|\left[Q(x,\tau)\right]_\tau\right|^2 dx\,d\tau$$
$$\le (t-s)^{\frac{3}{2}}\left[\int_s^t \left(\int_0^1 \left|\left[Q(x,\tau)\right]_\tau\right|^2 dx\right)^2 d\tau\right]^{\frac{1}{2}} \le C_2|t-s|^{\frac{3}{2}},$$

and

$$\int_0^1 \left|u(x,t)-u(x,s)\right|^2 dx = \int_0^1 \left|\int_s^t \left[u(x,t)\right]_\tau d\tau\right|^2 dx$$
$$\le (t-s)\int_s^t \int_0^1 \left|\left[u(x,t)\right]_\tau\right|^2 dx\,d\tau \le C_2|t-s|. \qquad \square$$

The existence part of Theorem 2.1. Based on Lemmas 3.1, 3.4–3.9 and Corollaries 3.1–3.5, we can construct a weak solution to the initial boundary value problem (2.1.2)–(2.1.5) by using the finite difference approximation similar to that in [4]. Then we complete the proof of Theorem 2.1 except the uniqueness.

4. Uniqueness

Next, we seek to prove the uniqueness. A new challenge here is to handle the pressure function such that we can produce an estimate of the form $\sqrt{c}\bar{Q}\bar{u}_x$ (see (4.8)–(4.11)). As a first step, we need

$$\int_0^T \|u_x\|_{L^\infty}^2 \le C_2. \tag{4.1}$$

In fact, using (3.1.58) and (3.1.72), we have

$$|u_x|^2 \le C_2 \int_0^1 u_t^2 + \frac{|p^* - P|^2}{c^2} + C_2 \le C_2 \int_0^1 u_t^2 + C_2 \phi^{2\beta_1 - 2\alpha} + C_2, \tag{4.2}$$

where we have used the conclusion due to (3.1.84)

$$\begin{aligned} |P - p^*| &= \left|(1-c)^\gamma Q^\gamma - (Q^*)^\gamma\right| \\ &\le \left|\left[(1-c)^\gamma - 1\right] Q^\gamma\right| + \left|Q^\gamma - (Q^*)^\gamma\right| \\ &\le C_2\left(\phi^\alpha + \phi^{\beta_1}\right). \end{aligned}$$

To ensure (4.1), we need to assume $\beta_1 = \alpha$ (we cannot assume $\beta_1 > \alpha$ due to $\beta_1 \in (0, \alpha] \cap (0, \frac{1}{2} - \alpha]$). Then (4.1) follows from (4.2) and (3.1.55).

As far as the proof of the uniqueness is concerned, suppose that (c_1, Q_1, u_1) and (c_2, Q_2, u_2) are two solutions to (2.1.2) with the same initial-boundary conditions. To prove the uniqueness, it suffices to get

$$(c_1, Q_1, u_1) = (c_2, Q_2, u_2). \tag{4.3}$$

Since $c_1 = c_0$ and $c_2 = c_0$ due to $(2.1.2)_1$ and (2.1.5), we get $c_1 = c_2$ (denoted by c). Denote $\bar{Q} = Q_1 - Q_2$ and $\bar{u} = u_1 - u_2$ which satisfy the following system due to (2.1.2)–(2.1.5):

$$\begin{cases} \bar{Q}_t + a^* \bar{Q}(Q_1 + Q_2)(u_1)_x + a^* Q_2^2 \bar{u}_x = 0, \quad x \in (0,1),\ t > 0, \\ \bar{u}_t + L_1 + L_2 + L_3 + L_4 = \mu\left[E(cQ_1)\bar{u}_x\right]_x + \mu\left[\left(E(cQ_1) - E(cQ_2)\right)(u_2)_x\right]_x, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} L_1 &= \left[P(c, Q_1) - P(c, Q_2)\right]_x, \\ L_2 &= -\left[u_1^2 g(cQ_1) - u_2^2 g(cQ_2)\right]_x = -\left[\bar{u}(u_1 + u_2) g(cQ_1)\right]_x - \left[\left(g(cQ_1) - g(cQ_2)\right) u_2^2\right]_x, \\ L_3 &= -\left[u_1 h(cQ_1) - u_2 h(cQ_2)\right]_x = -\left[\bar{u} h(cQ_1)\right]_x - \left[\left(h(cQ_1) - h(cQ_2)\right) u_2\right]_x, \\ L_4 &= \left[j(cQ_1) - j(cQ_2)\right]_x. \end{aligned}$$

The initial-boundary conditions are stated as follows:

$$\begin{aligned} \bar{Q}(x,0) = 0, \qquad \bar{u}(x,0) = 0, \quad x \in [0,1], \\ \bar{u}(0,t) = 0, \qquad \tilde{Q}(1,t) = 0, \quad t \ge 0. \end{aligned} \tag{4.5}$$

Multiplying (4.4)$_2$ by $\bar{u}$, and integrating by parts over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \bar{u}^2 + \mu\int_0^1 E(cQ_1)\bar{u}_x^2 \\
&\quad = -\mu\int_0^1 \big[E(cQ_1)-E(cQ_2)\big](u_2)_x\bar{u}_x - \int_0^1 L_1\bar{u} - \int_0^1 L_2\bar{u} - \int_0^1 L_3\bar{u} - \int_0^1 L_4\bar{u} \\
&\quad = R_1+R_2+R_3+R_4+R_5. \qquad (4.6)
\end{aligned}$$

For R_1, using the Cauchy inequality and (3.1.32), we have

$$\begin{aligned}
R_1 &\le C_2\int_0^1 c\big|\bar{Q}(u_2)_x\bar{u}_x\big| \le \frac{\mu}{8}\int_0^1 E(cQ_1)\bar{u}_x^2 + C_2\int_0^1 c|\bar{Q}|^2\big|(u_2)_x\big|^2 \\
&\le \frac{\mu}{8}\int_0^1 E(cQ_1)\bar{u}_x^2 + C_2\big\|(u_2)_x\big\|_{L^\infty}^2\int_0^1 |\bar{Q}|^2. \qquad (4.7)
\end{aligned}$$

For R_2, we have

$$\begin{aligned}
R_2 &= \int_0^1 (1-c)^\gamma\big[Q_1^\gamma - Q_2^\gamma\big]\bar{u}_x\,dx \\
&= \int_0^1 (1-c)^\gamma\bar{u}_x(Q_1-Q_2)\int_0^1 \gamma\big[\xi Q_1+(1-\xi)Q_2\big]^{\gamma-1}\,d\xi\,dx \\
&= \int_0^1 (1-c_0)^\gamma\bar{u}_x\bar{Q}\bigg(\int_0^1 \gamma\big[\xi Q_1+(1-\xi)Q_2\big]^{\gamma-1}\,d\xi \\
&\quad - \int_0^1 \gamma\big[\xi Q^*+(1-\xi)Q^*\big]^{\gamma-1}\,d\xi\bigg)\,dx \\
&\quad + \gamma\big[Q^*\big]^{\gamma-1}\int_0^1 (1-c_0)^\gamma\bar{u}_x\bar{Q}\,dx \\
&\le C_2\int_0^1 |\bar{u}_x||\bar{Q}|\big(\big|Q_1-Q^*\big|+\big|Q_2-Q^*\big|\big)\,dx + \frac{\gamma}{a^*}\big[Q^*\big]^{\gamma-3}\int_0^1 (1-c_0)^\gamma a^* Q_2^2\bar{u}_x\bar{Q}\,dx \\
&\quad + \frac{\gamma}{a^*}\big[Q^*\big]^{\gamma-3}\int_0^1 (1-c_0)^\gamma a^*\big(\big[Q^*\big]^2 - Q_2^2\big)\bar{u}_x\bar{Q}\,dx
\end{aligned}$$

$$\le C_2 \int_0^1 \phi^{\beta_1} |\bar{u}_x||\bar{Q}|\,dx + \frac{\gamma}{a^*}\big[Q^*\big]^{\gamma-3} \int_0^1 (1-c_0)^\gamma a^* Q_2^2 \bar{u}_x \bar{Q}\,dx$$
$$= R_{2,1} + R_{2,2}, \tag{4.8}$$

where we have used (3.1.33), (3.1.32) and (3.1.84).

For $R_{2,1}$, using the Cauchy inequality and (3.1.32), we have

$$R_{2,1} \le \frac{\mu}{8}\int_0^1 E(cQ_1)\bar{u}_x^2 + C_2\int_0^1 \phi^{2\beta_1-\alpha}|\bar{Q}|^2 \le \frac{\mu}{8}\int_0^1 E(cQ_1)\bar{u}_x^2 + C_2\int_0^1 |\bar{Q}|^2, \tag{4.9}$$

since $\beta_1 = \alpha$. For $R_{2,2}$, using (4.4), we have

$$R_{2,2} = -\frac{\gamma}{2a^*}\big[Q^*\big]^{\gamma-3}\frac{d}{dt}\int_0^1 (1-c_0)^\gamma \bar{Q}^2 - \frac{\gamma}{a^*}\big[Q^*\big]^{\gamma-3}\int_0^1 (1-c_0)^\gamma a^*(Q_1+Q_2)(u_1)_x\bar{Q}^2$$
$$\le -\frac{\gamma}{2a^*}\big[Q^*\big]^{\gamma-3}\frac{d}{dt}\int_0^1 (1-c_0)^\gamma \bar{Q}^2 + C_2\big\|(u_1)_x\big\|_{L^\infty}\int_0^1 \bar{Q}^2. \tag{4.10}$$

Putting (4.9) and (4.10) into (4.8), we have

$$R_2 \le \frac{\mu}{8}\int_0^1 E(cQ_1)\bar{u}_x^2$$
$$- \frac{\gamma}{2a^*}\big[Q^*\big]^{\gamma-3}\frac{d}{dt}\int_0^1 (1-c_0)^\gamma \bar{Q}^2 + C_2\big(\big\|(u_1)_x\big\|_{L^\infty}+1\big)\int_0^1 \bar{Q}^2. \tag{4.11}$$

For R_3, we have

$$R_3 = -\int_0^1 \big(\bar{u}(u_1+u_2)g(cQ_1) + \big[g(cQ_1)-g(cQ_2)\big]u_2^2\big)\bar{u}_x \le C_2\int_0^1 c\big(|\bar{u}|+|\bar{Q}|\big)|\bar{u}_x|$$
$$\le \frac{\mu}{8}\int_0^1 E(cQ_1)\bar{u}_x^2 + C_2\int_0^1 \bar{u}^2 + C_2\int_0^1 \bar{Q}^2. \tag{4.12}$$

Similar to R_3, we have

$$R_4 + R_5 \le \frac{\mu}{8}\int_0^1 E(cQ_1)\bar{u}_x^2 + C_2\int_0^1 \bar{u}^2 + C_2\int_0^1 \bar{Q}^2. \tag{4.13}$$

Putting (4.7), (4.11), (4.12) and (4.13) into (4.6), we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \left(\bar{u}^2 + \frac{\gamma}{a^*}\left[Q^*\right]^{\gamma-3}(1-c_0)^{\gamma}\bar{Q}^2\right) + \mu\int_0^1 E(cQ_1)\bar{u}_x^2 \\
&\le C_2\left(\left\|(u_2)_x\right\|_{L^\infty}^2 + \left\|(u_1)_x\right\|_{L^\infty} + 1\right)\int_0^1 \bar{Q}^2 + C_2\int_0^1 \bar{u}^2 \\
&\le C_2\left(\left\|(u_2)_x\right\|_{L^\infty}^2 + \left\|(u_1)_x\right\|_{L^\infty} + 1\right)\int_0^1 \left(\bar{u}^2 + \frac{\gamma}{a^*}\left[Q^*\right]^{\gamma-3}(1-c_0)^{\gamma}\bar{Q}^2\right), \qquad (4.14)
\end{aligned}$$

where we have used $\sup c_0 < 1$. (4.14) combined with (4.1), Gronwall inequality and (4.5) turns out that

$$\int_0^1 \left(\bar{u}^2 + \frac{\gamma}{a^*}\left[Q^*\right]^{\gamma-3}(1-c_0)^{\gamma}\bar{Q}^2\right) = 0.$$

This combined with the fact $c_1 = c_2 = c_0$ gives (4.3).

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