ON A RELATION BETWEEN PRESSURE-BASED SCHEMES AND CENTRAL SCHEMES FOR HYPERBOLIC CONSERVATION LAWS

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Abstract. The purpose of this paper is to explain in more detail some ideas used recently by the authors to construct a class of hybrid explicit-implicit schemes for solving the two-fluid model relevant for well and pipe flow simulations [6, 7, 8]. We here propose a framework which allows us to implement these ideas for a general system of hyperbolic conservation laws $u_t + f(u)_x = 0$. Main ingredients in this construction are (i) a splitting $f(u) = g(u) + h(u)$ of the given flux function f; (ii) a corresponding decomposition of the original set of equations into two subsystems, one set of equations associated with the g flux, another with the h flux; (iii) inclusion of a set of flux evolution equations associated with the flux component h.

We demonstrate that a sound and consistent discretization of this extended system gives rise to a class of central schemes which contains as a special case, corresponding to the splitting $g = h = \frac{1}{2}f$, the FORCE scheme studied by Toro [20]. This justifies referring to the proposed class as eXtended FORCE (X-FORCE). We discuss basic properties of the X-FORCE class for nonlinear scalar conservation laws. By exploiting that the X-FORCE schemes can be interpreted through Riemann solutions, we construct higher order X-FORCE schemes by closely following along the line of the non-staggered NT scheme presented in [14]. Characteristic behavior of various X-FORCE schemes is demonstrated through calculation of numerical examples.

Key words. hyperbolic system of conservation laws, central scheme, flux-splitting, second-order, isothermal Euler equations, isothermal two-fluid model, explicit, implicit.

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1. Introduction.

We focus on a general system of conservation laws in the form

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{1.1}$$

We are concerned with methods for numerical computation of solutions of hyperbolic conservation laws. One simple, but important special case of (1.1), is the isothermal Euler model given by

$$\begin{aligned} \partial_t \rho + \partial_x m &= 0, \\ \partial_t m + \partial_x \Big(m^2/\rho + p(\rho) \Big) &= 0, \end{aligned} \tag{1.2}$$

where ρ is density, v fluid velocity, $\rho v = m$ momentum, and $p = p(\rho)$ scalar pressure. Multiplying the continuity equation by $p'(\rho)$ we get the pressure evolution equation

$$\partial_t p + p'(\rho)\partial_x m = 0. \tag{1.3}$$

A natural approach could then be to work directly with the pressure variable p and the corresponding pressure evolution equation (1.3), instead of the continuity equation (first equation) of (1.2) and the density variable ρ. A major difficulty with this approach is that the resulting model is written in a non-conservative form. Combined with use of donor-cell techniques a discretization of the model in this form typically leads to loss of masses as well as incorrect shock speed when the solution involves shocks [13]. Nevertheless, many simulators currently in use for studying complex phenomena involving equations similar to (1.2), rely on precisely such a donor-cell pressure-based discretization. Examples include for instance the OLGA [1] and PeTra [16] two-phase computer codes developed for the petroleum industry. A motivation for the pressure-based approach in the context of two-phase flow seems to be (i) a desire to more directly apply a discretization that is implicit enough to ensure stability for larger timesteps than those dictated by the highest wave speed (classical CFL condition); (ii) complicated source terms as well as additional equations,

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typically needed for modeling of complex phenomena, can more easily be incorporated in this approach.

The questions we deal with in this paper are:

(i) How can we generalize the above pressure-based approach, often used for models involving equations of the generic form (1.3), to a general conservation law (1.1)?
(ii) Can we relate such a "generalized" pressure-based approach to the theory for conservative and non-oscillatory methods as developed in the 80s?

Hence, our main aim is not to develop new schemes, but, in view of existing theory for conservative methods for general conservation laws, to try to shed some light on the classical idea of using a pressure evolution equation. Before we focus more on the implementation of this programme, we review briefly one class of such classical conservative schemes, so-called centred schemes.

1.1. Centred schemes. The most accurate methods for solving hyperbolic equations are upwind schemes. However, these require the explicit provision of wave propagation information, which is normally achieved via local solutions of the Riemann problem, approximate or exact.

For more complicated hyperbolic systems it may not be desirable or possible to rely on the solutions of corresponding Riemann problems. In this case we have no option but to use a centred type of scheme in which no explicit information regarding wave propagation is used in the scheme, apart from a more or less rough estimate of the maximal wave speed needed for determining the stability criteria associated with an explicit time discretization.

Toro and coworkers have proposed the so-called FORCE scheme as a basic centred scheme. It is known that this scheme possesses various good properties [20, 19, 2, 4]. It has been shown to be monotone, to possess the optimal stability condition, and to have the smallest numerical viscosity when it is considered for a scalar linear conservation law. Moreover, entropy consistence has also been shown for a general nonlinear system of conservation laws and convergence results have been obtained for special systems like the isentropic Euler equations and shallow water equations [4]. In [19] the FORCE flux was used to construct simple and general upwind numerical fluxes in a multi-stage predictor-corrector fashion for a general nonlinear hyperbolic conservation law. Numerical results were provided for the Euler equations demonstrating that the resulting scheme gave results similar to the best of upwind schemes, using a reasonable number of stages. Second-order extensions of FORCE have also been explored for the one-dimensional Euler equations [2] and FORCE-based schemes can also be extended to solve multi-dimensional problems in a straightforward way following established approaches [4, 19].

1.2. X-FORCE. In a series of papers [6, 7, 8] we derived explicit as well as hybrid explicit-implicit schemes for solving the 4-equation isothermal two-fluid model relevant to well and pipe flow simulations. The main ingredients in this construction are

(i) a splitting of the flux into a convective flux component and a pressure flux component;
(ii) development and inclusion of a pressure evolution equation;
(iii) consistent discretization of the convective flux and the pressure flux;
(iv) development of hybrid central-upwind mass fluxes for accurate resolution of mass fronts.

The points (i)–(iii) lead to central type schemes which are stable and robust (both explicit and implicit variants are considered), but they might give an inaccurate resolution of the slow mass waves. Point (iv) gives a recipe for improving the resolution of these waves by carefully incorporating an upwind component in the convective flux without affecting the robustness.

The aim of this paper is to see how the points (i)–(iii) can be put into a general framework where we consider an arbitrary conservation law (1.1). Main ingredients in this framework are

(i) a splitting of the flux f into two components g and h such that $f(u) = g(u) + h(u)$, and a corresponding decomposition of the original model into two sub-systems where one of them is associated with the g flux, the other with the h flux;
(ii) development and inclusion of flux evolution equations associated with the h flux;
(iii) consistent discretization of this extended model.

It turns out, as a somewhat surprising result, that the proposed class of schemes based on the above steps (i)–(iii), is closely linked to the FORCE scheme. More precisely, the special choice of flux splitting where $g = h = 1/2f$ reproduces the FORCE scheme, whose numerical flux is the sum

of a half Lax-Friedrich flux component and a half Lax-Wendroff flux component. Consequently, we propose to refer to this class as eXtended FORCE (X-FORCE).

This relation is then used to give an interpretation of the X-FORCE schemes through Riemann solutions, similar to what has been done for the FORCE scheme [4]. For a nonlinear scalar conservation law basic properties of the X-FORCE schemes are verified. In particular, we show that these schemes are TVD under natural CFL conditions and we identify conditions which ensure that the schemes are monotone. Entropy consistency is verified by making use of the interpretation via Riemann solutions. Equipped with the Riemann-based formulation of X-FORCE we describe an extension to higher order spatial accuracy by using linear interpolants in the evaluation of integrals, similar to what is done in the construction of the non-staggered NT scheme presented in [14]. In particular, the second-order X-FORCE class reproduces the non-staggered NT scheme by using the special splitting $g = f$ and $h = 0$. We also obtain a second-order type of FORCE scheme, different from the one presented in [2], by using the splitting $g = h = \frac{1}{2}f$. We demonstrate numerically that the TVD property possessed by the first order X-FORCE class seems to carry over to the second order class (under appropriate CFL conditions).

We also demonstrate the possibility which the X-FORCE class offers regarding construction of simple semi-implicit central schemes for some special (but important) systems of conservation laws, without making use of any Jacobian matrix calculation. For complicated systems where some of the eigenvalues may take very large values, it sometimes is desirable to avoid the strict CFL stability criterion associated with an explicit time discretization. In order to achieve this goal one must employ a partial or full implicit treatment of the fluxes. Typically, this requires information about the Jacobian of the flux. However, this information may not be so easy to obtain for the models we have in mind - complicated models where the explicit FORCE scheme potentially is an interesting candidate. The proposed X-FORCE class seems to offer some flexibility in that respect.

This aspect is illustrated by considering two concrete models. In particular, we explore X-FORCE schemes for two systems of conservation laws, the single phase isothermal Euler equations and the 4-equations isothermal two-fluid model. For the isothermal Euler model we compare the performance of the FORCE scheme with that of an X-FORCE scheme whose splitting is such that the g component is associated with the convective part whereas the h component is associated with the pressure part. We consider both an explicit and a semi-implicit variant of this X-FORCE scheme. We also identify corresponding X-FORCE schemes for the isothermal two-fluid model and thereby place the central schemes studied in [6, 7, 8] into our current framework.

The rest of this paper is organized as follows. In Section 2 we construct the X-FORCE class. Section 3 describes how the X-FORCE class can be interpreted through Riemann solutions. Section 4 contains analysis of basic properties of the X-FORCE schemes as well as construction of higher order X-FORCE schemes. In Section 5 we focus on X-FORCE for two concrete systems, the isothermal Euler model and the 4-equations isothermal two-fluid model. Explicit and hybrid explicit-implicit variants are discussed. Section 6 is devoted to some simple numerical investigations where the performance of various X-FORCE schemes is explored. Section 7 contains a brief summary of the content of this paper and some concluding remarks are also given.

2. X-FORCE: a class of centred schemes. In this section we describe a framework for constructing discrete approximations to a general conservation law (1.1). The proposed approach represents a generalization of ideas used in [6, 7, 8] for construction of certain numerical schemes for the 4-equations two-fluid model. A main ingredient in this construction is the use of a *flux evolution equation*.

To illustrate the idea, let us recall that the Godunov scheme is defined as

$$u_j^{k+1} = u_j^k - \lambda\Big(f_{j+1/2} - f_{j-1/2}\Big), \qquad \lambda = \frac{\Delta t}{\Delta x},$$

where the numerical flux $f_{j+1/2}$ is defined by

$$f_{j+1/2} = f(u_{j+1/2}), \qquad u_{j+1/2} = u^{\mathrm{R}}\left(x_{j+1/2}, t; u_j, u_{j+1}\right),$$

where $u^{\mathrm{R}}\left(x_{j+1/2}, t; u_j, u_{j+1}\right)$ represents the solution of the Riemann problem for (1.1) in the time period $[t^k, t^{k+1}]$ with respect to the initial data

$$u(x, t^k) = \begin{cases} u_j, & \text{if } x < x_{j+1/2}, \\ u_{j+1}, & \text{if } x > x_{j+1/2}. \end{cases}$$

Alternatively, we may try to find an evolution equation for $f(u)$, that is, a flux evolution equation. Armed with this, we can seek to directly obtain an expression for the flux $f(u)_{j+1/2}$ at the cell-interface $x_{j+1/2}$ without solving any Riemann problem. First, we describe how this approach can be used to reproduce the classical Lax-Wendroff scheme. Although the presentation in this section is given for a scalar conservation law, its extension to systems of conservation laws is direct and straightforward. Two examples are discussed in Section 5.

2.1. Lax-Wendroff scheme obtained via a flux evolution equation. Assume that we are given an approximation $u^k(x) \approx u(x, t^k)$ where u is the solution of (1.1). We now want to calculate an approximation $u^{k+1}(x) \approx u(x, t^{k+1})$. For that purpose, locally in the time period $[t^k, t^{k+1}]$, we want to approximate

$$v_t + f(v)_x = 0, \qquad v(\cdot, 0) = u^k(\cdot), \tag{2.1}$$

by making use of a flux evolution equation for f. Multiplying (2.1) by $f'(v)$ we obtain the following equation for $\widetilde{f}(x,t) = f(v(x,t))$

$$\widetilde{f}_t + a(x,t) f(v)_x = 0, \qquad a(x,t) = f'(v(x,t)). \tag{2.2}$$

Thus, we may approximate (2.1) by the following *extended* model with $v(x,t)$ and $\widetilde{f}(x,t)$ as unknown variables.

$$\begin{aligned} v_t + \widetilde{f}_x = 0, \qquad & v(\cdot, 0) = u^k(\cdot), \qquad t \in (0, \Delta t] \\ \widetilde{f}_t + a(x,t) f(v)_x = 0, \qquad & \widetilde{f}(\cdot, 0) = f(u^k)(\cdot), \qquad t \in (0, \Delta t]. \end{aligned} \tag{2.3}$$

Finally, we set $u^{k+1}(x) = v(x, \Delta t)$.

Method of lines. We want to construct an approximation to the original model (2.1) by defining an approximation to the extended model (2.3). As a first step to build an approximation we apply a "method of lines" approach where we assume that the computational domain is divided into N cells of grid size Δx, indexed by $j = \{1, \ldots, N\}$ and then approximate (2.3) with a system of $2N$ ODEs. More precisely, we propose to consider a semi-discrete version of (2.3) where the first equation is discretized at the cell center x_j whereas the second equation is discretized at the cell interface $x_{j+1/2}$. That is,

$$\begin{aligned} \dot{v}_j + \frac{1}{\Delta x}\Big(\widetilde{f}_{j+1/2} - \widetilde{f}_{j-1/2}\Big) = 0, \qquad & v_j(0) = u_j^k, \\ \dot{\widetilde{f}}_{j+1/2} + a_{j+1/2}\frac{1}{\Delta x}\Big(f(v_{j+1}) - f(v_j)\Big) = 0, \qquad & \widetilde{f}_{j+1/2}(0) = \frac{f(u_j^k) + f(u_{j+1}^k)}{2}, \end{aligned} \tag{2.4}$$

for $t \in (0, \Delta t]$ where $a_{j+1/2}$ is an appropriate defined average. In particular, the second equation provides a flux $\widetilde{f}_{j+1/2}$ to be used in the first equation. Then, we set $u_j^{k+1} = v_j(\Delta t)$

Fully discrete form. Finally, we propose to consider a fully discrete approximation of (2.4). For the first equation in (2.4) we consider a discrete scheme of the form

$$\frac{v_j^1 - v_j^0}{\Delta t} + \frac{1}{\Delta x}\Big(F_{j+1/2}^{1/2} - F_{j-1/2}^{1/2}\Big) = 0, \qquad v_j^0 = u_j^k, \tag{2.5}$$

where the flux $F^{1/2}_{j+1/2} \approx \widetilde{f}(x_{j+1/2}, \Delta t/2)$ is obtained by solving the second equation of (2.4) a half timestep $\Delta t/2$ forward in time. More precisely, by applying a Lax-Friedrichs discretization for this equation at the cell interface $j+1/2$, we get

$$\frac{F^{1/2}_{j+1/2} - \frac{f^0_j + f^0_{j+1}}{2}}{\Delta t/2} + \frac{a^0_{j+1/2}}{\Delta x}\Big(f^0_{j+1} - f^0_j\Big) = 0, \qquad f^0_j = f(v^0_j) = f(u^k_j), \tag{2.6}$$

where $a^0_{j+1/2}$ is either the derivative of f evaluated at some average state, for instance

$$a^0_{j+1/2} = f'\left(\frac{v^0_j + v^0_{j+1}}{2}\right), \quad \text{or} \quad a^0_{j+1/2} = \frac{f'(v^0_j) + f'(v^0_{j+1})}{2}, \tag{2.7}$$

or

$$a^0_{j+1/2} = \overline{a}(v^0_j, v^0_{j+1}), \tag{2.8}$$

where $\overline{a}(u,v)$ is a function satisfying

$$\begin{cases} f(v) - f(u) = \overline{a}(u,v)(v-u), \\ \overline{a}(u,u) = f'(u). \end{cases} \tag{2.9}$$

Finally, we set $u^{k+1}_j = v^1_j$. From (2.6) we observe that the flux evolution equation directly gives us a numerical flux $F_{j+1/2}$

$$F_{j+1/2}(u^k_j, u^k_{j+1}) = \frac{f(u^k_j) + f(u^k_{j+1})}{2} - a^k_{j+1/2}\frac{\Delta t}{2\Delta x}\Big(f(u^k_{j+1}) - f(u^k_j)\Big)$$

which is consistent with the physical flux f in the usual sense, i.e., $F(u,u) = f(u)$. Combining (2.5) and (2.6) we get the scheme

$$u^{k+1}_j - u^k_j + \frac{\lambda}{2}\Big(f^k_{j+1} - f^k_{j-1}\Big) = \frac{\lambda^2}{2}\left(a^k_{j+1/2}[f^k_{j+1} - f^k_j] - a^k_{j-1/2}[f^k_j - f^k_{j-1}]\right), \tag{2.10}$$

where $f^k_j = f(u^k_j)$ and $a^k_{j+1/2}$ is defined by (2.7) or (2.8)–(2.9) (note that $u^k_j = v^0_j$). This scheme is recognized as the classical Lax-Wendroff scheme, see for instance [11]. This scheme is known to be second order accurate, however, it is not TVD since its viscosity coefficient $\lambda^2 a^k_{j+1/2}$ does not satisfy Harten's criteria.

REMARK 1. *A slightly different approach (which avoids the use of the flux evolution equation) consists of replacing the above choice of $F^{1/2}_{j+1/2}$ by the following: Define*

$$F^{1/2}_{j+1/2} = f(v^{1/2}_{j+1/2}),$$

where $v^{1/2}_{j+1/2}$ is obtained by applying a Lax-Friedrich discretization of the equation (2.1) *at the cell interface $j+1/2$, i.e.,*

$$\frac{v^{1/2}_{j+1/2} - \frac{v^0_j + v^0_{j+1}}{2}}{\Delta t/2} + \frac{1}{\Delta x}\Big(f(v^0_{j+1}) - f(v^0_j)\Big) = 0, \qquad v^0_j = u^k_j. \tag{2.11}$$

The resulting scheme, which reads

$$\begin{aligned} u^{k+1/2}_{j+1/2} = v^{1/2}_{j+1/2} &= \frac{1}{2}(u^k_{j+1} + u^k_j) - \frac{\lambda}{2}\Big(f(u^k_{j+1}) - f(u^k_j)\Big), \\ u^{k+1}_j &= u^k_j - \lambda\Big(f(u^{k+1/2}_{j+1/2}) - f(u^{k+1/2}_{j-1/2})\Big), \end{aligned}$$

is nothing but the two-step Richtmyer's version [11]. Note that this scheme avoids the computation of "a" and the product "af" which might be of particular interest when we study system of equations and the quantity "a" is the Jacobian matrix associated with the flux vector "f".

REMARK 2. *At the discrete level, we have demonstrated that a sound discretization of the extended model* (2.3) *produces the Lax-Wendroff scheme associated with the original model* (2.1). *Convergence and consistency of the LW scheme were established for nonlinear scalar conservation laws in [3]. This may serve to justify the relevance of applying the extended model* (2.3) *for construction of approximate solutions to the original model* (2.1). *In particular, at the continuous level we might expect that for an appropriate viscous approximation to* (2.3), *one could demonstrate convergence to the entropy solution of* (2.1).

2.2. A family of schemes based on a generalized "pressure-based" approach. In this section we want to derive a family of schemes by building upon and refining the above approach which reproduced the classical Lax-Wendroff scheme. More precisely, we decompose the flux f into two components, let's denote them g and h, such that

$$f(u) = g(u) + h(u), \qquad \forall u. \tag{2.12}$$

Assume that we have given an approximation $u^k(x) \approx u(x, t^k)$. We now want to calculate an approximation $u^{k+1}(x) \approx u(x, t^{k+1})$. For that purpose, locally in the time period $[t^k, t^{k+1}]$, we consider a decomposition of (1.1) into the two subsystems

$$\begin{aligned} v_t + g(v+w)_x &= 0, \qquad & v(\cdot,0) &= u^k(\cdot), \\ w_t + h(v+w)_x &= 0, \qquad & w(\cdot,0) &= 0, \end{aligned} \tag{2.13}$$

where v and w now are the unknown variables. Setting

$$u = v + w, \tag{2.14}$$

adding the two equations in (2.13), and invoking (2.12), we see that u is a solution of $u_t + f(u)_x = 0$ with initial data $u(\cdot, 0) = u^k(\cdot)$. Thus, (2.13) represents a reformulation of the original model (1.1). We also observe that the eigenstructure of the original model (1.1) is preserved in a certain sense since the Jacobian associated with (2.13) is

$$J = \begin{pmatrix} g'(u) & g'(u) \\ h'(u) & h'(u) \end{pmatrix},$$

whose eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = f'(u)$.

Now, we follow along the line of the previous section and propose to approximate (2.13) by making use of a flux evolution equation. First, we introduce the notation

$$\widetilde{h} = h(v+w)|_{w=\text{const}}, \tag{2.15}$$

and define

$$a := \frac{\partial \widetilde{h}}{\partial v} = h'(u). \tag{2.16}$$

Multiplying the first equation of (2.13) by a, we obtain the following equation for $\widetilde{h}(x,t)$

$$\widetilde{h}_t + a(x,t) g(v+w)_x = 0, \qquad a(x,t) = \partial_v \widetilde{h}(x,t) = h'(u). \tag{2.17}$$

Thus, we approximate (2.13) by the following *extended* model with $v(x,t)$, $w(x,t)$, and $\widetilde{h}(x,t)$ as unknown variables.

$$\begin{aligned} v_t + g(v+w)_x &= 0, \qquad & v(\cdot,0) &= u^k(\cdot), \\ w_t + \widetilde{h}_x &= 0, \qquad & w(\cdot,0) &= 0, \\ \widetilde{h}_t + a(x,t) g(v+w)_x &= 0, \qquad & \widetilde{h}(\cdot,0) &= h(u^k)(\cdot), \end{aligned} \tag{2.18}$$

for $t \in (0, \Delta t]$. Finally, in view of (2.14), we set $u^{k+1}(x) = v(x, \Delta t) + w(x, \Delta t)$.

Method of lines. Now, we want to construct an approximation to the model (2.13), which in turn is a reformulation of (1.1), by defining an approximation to the extended model (2.18). Again we consider a "method of lines" approach and approximate (2.18) with a system of $3N$ ODEs. More precisely, we consider a semi-discrete version of (2.18) where we solve via a two-step procedure as follows.

Step 1: Since v and $\widetilde{h}$ are closely inter-related (the equation for $\widetilde{h}$ was derived from equation for v) we solve the first and third equation of (2.18) at the cell-interfaces $x_{j+1/2}$ by the standard Lax-Friedrichs type discretization:

$$\begin{aligned} &\dot{v}_{j+1/2} + \frac{1}{\Delta x}\Big(g_{j+1} - g_j\Big) = 0, \qquad v_{j+1/2}(0) = \frac{u_j^k + u_{j+1}^k}{2}, \\ &\dot{\widetilde{h}}_{j+1/2} + \frac{a_{j+1/2}}{\Delta x}\Big(g_{j+1} - g_j\Big) = 0, \qquad \widetilde{h}_{j+1/2}(0) = \frac{h(u_j^k) + h(u_{j+1}^k)}{2}, \end{aligned} \tag{2.19}$$

for $t \in (0, \Delta t]$ where $g_j = g(v_j + w_j)$ and $a_{j+1/2}$ is an appropriate defined average.

Step 2: Equipped with $v_{j+1/2}$ and $\widetilde{h}_{j+1/2}$, we find w_j through an evolution via the second equation of (2.18), whereas v_j is determined through a projection. That is,

$$\begin{aligned} &\dot{w}_j + \frac{1}{\Delta x}\Big(\widetilde{h}_{j+1/2} - \widetilde{h}_{j-1/2}\Big) = 0, \qquad w_j(0) = 0, \\ &\dot{v}_j = \frac{\dot{v}_{j-1/2} + \dot{v}_{j+1/2}}{2}, \end{aligned} \tag{2.20}$$

for $t \in (0, \Delta t]$. Finally, in light of (2.14), we set

$$u_j^{k+1} = v_j(\Delta t) + w_j(\Delta t). \tag{2.21}$$

Fully discrete form. First, we rewrite as follows. Combining (2.19) and (2.20), we have a semi-discrete scheme of the form

$$\begin{aligned} &\dot{v}_j + \frac{1}{\Delta x}\Big(g_{j+1/2} - g_{j-1/2}\Big) = 0, \qquad v_j(0) = \frac{1}{4}u_{j-1}^k + \frac{1}{2}u_j^k + \frac{1}{4}u_{j+1}^k, \\ &\qquad \text{with} \qquad g_{j+1/2} = \frac{g(v_j + w_j) + g(v_{j+1} + w_{j+1})}{2}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\dot{w}_j + \frac{1}{\Delta x}\Big(\widetilde{h}_{j+1/2} - \widetilde{h}_{j-1/2}\Big) = 0, \qquad w_j(0) = 0, \\ &\qquad \text{with} \qquad \dot{\widetilde{h}}_{j+1/2} + \frac{a_{j+1/2}}{\Delta x}\Big(g_{j+1} - g_j\Big) = 0, \qquad \widetilde{h}_{j+1/2}(0) = \frac{h(u_j^k) + h(u_{j+1}^k)}{2}. \end{aligned} \tag{2.23}$$

Finally, we apply a forward Euler discretization in time in (2.22). In order to obtain a numerical flux $\widetilde{h}_{j+1/2}$ for the first equation of (2.23), we solve the second equation in (2.23) forward in time a timestep Δt, again, by applying a forward Euler discretization. This yields the following fully discrete approximation.

$$\begin{aligned} &\frac{v_j^1 - v_j^0}{\Delta t} + \frac{1}{\Delta x}\Big(g_{j+1/2}^0 - g_{j-1/2}^0\Big) = 0, \qquad v_j^0 = \frac{1}{4}u_{j-1}^k + \frac{1}{2}u_j^k + \frac{1}{4}u_{j+1}^k, \\ &\qquad \text{with} \qquad g_{j+1/2}^0 = \frac{g(u_j^k) + g(u_{j+1}^k)}{2}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} &\frac{w_j^1 - w_j^0}{\Delta t} + \frac{1}{\Delta x}\Big(\widetilde{h}_{j+1/2}^1 - \widetilde{h}_{j-1/2}^1\Big) = 0, \qquad w_j^0 = 0, \\ &\text{with} \qquad \frac{\widetilde{h}_{j+1/2}^1 - \widetilde{h}_{j+1/2}^0}{\Delta t} + \frac{a_{j+1/2}^0}{\Delta x}\Big(g_{j+1}^0 - g_j^0\Big) = 0, \qquad \widetilde{h}_{j+1/2}^0 = \frac{h(u_j^k) + h(u_{j+1}^k)}{2}, \end{aligned} \tag{2.25}$$

where $g_j^0 = g(v_j^0) = g(u_j^k)$ and $a_{j+1/2}^0$ is either the derivative of h evaluated at some average state, for instance

$$a_{j+1/2}^0 = h'\left(\frac{v_j^0 + v_{j+1}^0}{2}\right), \quad \text{or} \quad a_{j+1/2}^0 = \frac{h'(v_j^0) + h'(v_{j+1}^0)}{2}, \tag{2.26}$$

or

$$a^0_{j+1/2} = \overline{a}(v^0_j, v^0_{j+1}), \tag{2.27}$$

where $\overline{a}(u,v)$ is a function similar to the one defined by (2.9) (f is replaced by h). Finally, in light of (2.21), u^{k+1}_j is obtained by setting

$$u^{k+1}_j = v^1_j + w^1_j. \tag{2.28}$$

The class of X-FORCE schemes. Combining (2.24), (2.25), and (2.28) we obtain the following scheme for an arbitrary splitting (2.12):

$$\begin{aligned} u^{k+1}_j =& u^k_j - \lambda\Big(G^k_{j+1/2} - G^k_{j-1/2}\Big) - \lambda\Big(H^{k+1}_{j+1/2} - H^{k+1}_{j-1/2}\Big) \\ =& u^k_j - \frac{\lambda}{2}\Big(f^k_{j+1} - f^k_{j-1}\Big) + \frac{1}{4}[u^k_{j+1} - u^k_j] - \frac{1}{4}[u^k_j - u^k_{j-1}] \\ &+ a^k_{j+1/2}\lambda^2\Big[g^k_{j+1} - g^k_j\Big] - a^k_{j-1/2}\lambda^2\Big[g^k_j - g^k_{j-1}\Big], \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} G^k_{j+1/2} &= \frac{g^k_j + g^k_{j+1}}{2} - \frac{\Delta x}{4\Delta t}(u^k_{j+1} - u^k_j), \qquad g^k_j = g(u^k_j) \\ H^{k+1}_{j+1/2} &= \widetilde{h}^1_{j+1/2} = \frac{h^k_j + h^k_{j+1}}{2} - \lambda a^k_{j+1/2}\Big(g^k_{j+1} - g^k_j\Big), \qquad h^k_j = h(u^k_j), \end{aligned} \tag{2.30}$$

where $f^k_j = f(u^k_j)$ and $a^k_{j+1/2}$ is defined by (2.26) or (2.27) (note that $u^k_j = v^0_j$). Note that the G flux corresponds to the *modified* Lax-Friedrich scheme [18].

REMARK 3. *A slightly different approach, as far as the discretization of* (2.13) *is concerned, would be to replace the above choice of* $\widetilde{h}^1_{j+1/2}$ *appearing in* (2.25) *by the following: Define*

$$\widetilde{h}^1_{j+1/2} = h(v^1_{j+1/2}), \tag{2.31}$$

where $v^1_{j+1/2}$ *is obtained by applying a Lax-Friedrich discretization to the equation* $v_t + g(v+w)_x = 0$ *at the cell interface* $j+1/2$*, i.e.,*

$$\frac{v^1_{j+1/2} - \frac{v^0_j + v^0_{j+1}}{2}}{\Delta t} + \frac{1}{\Delta x}\Big(g(v^0_{j+1}) - g(v^0_j)\Big) = 0, \qquad v^0_j = u^k_j. \tag{2.32}$$

The resulting scheme now reads

$$\begin{aligned} u^{k+1}_{j+1/2} = v^1_{j+1/2} &= \frac{1}{2}(u^k_{j+1} + u^k_j) - \lambda(g(u^k_{j+1}) - g(u^k_j)), \\ u^{k+1}_j &= u^k_j - \lambda\Big(G^k_{j+1/2} - G^k_{j-1/2}\Big) - \lambda\Big(h(u^{k+1}_{j+1/2}) - h(u^{k+1}_{j-1/2})\Big), \end{aligned} \tag{2.33}$$

where $G^k_{j+1/2}$ *is given by* (2.30)*. This way of obtaining a flux* $\widetilde{h}^1_{j+1/2}$ *for the second equation of* (2.13) *corresponds to a two-step Richtmyer's approach as described for the pure Lax-Wendroff scheme, see Remark 1. Note again that the resulting scheme avoids the computation of "a" and the product "ag" which might be of particular interest when we study system of equations and the quantity "a" is the Jacobian matrix associated with the flux vector "h".*

We have the following proposition:

PROPOSITION 2.1. *The class of schemes given by* (2.29) *and* (2.30) *or the alternative formulation given by* (2.33)*, for an arbitrary splitting* $f(u) = g(u) + h(u)$*, contains the FORCE scheme*

as a special case corresponding to the splitting $g = h = \frac{1}{2}f$.

Proof. For the special choice that $g = h = \frac{1}{2}f$, the scheme (2.29) takes the form

$$\begin{aligned} u_j^{k+1} =& u_j^k - \lambda\Big(G_{j+1/2}^k - G_{j-1/2}^k\Big) - \lambda\Big(H_{j+1/2}^{k+1} - H_{j-1/2}^{k+1}\Big) \\ =& u_j^k - \frac{\lambda}{2}\Big(f_{j+1}^k - f_{j-1}^k\Big) + \frac{1}{4}[u_{j+1}^k - u_j^k] - \frac{1}{4}[u_j^k - u_{j-1}^k] \\ &+ a_{j+1/2}^k \frac{\lambda^2}{4}\Big[f_{j+1}^k - f_j^k\Big] - a_{j-1/2}^k \frac{\lambda^2}{4}\Big[f_j^k - f_{j-1}^k\Big], \end{aligned} \tag{2.34}$$

where

$$a_{j+1/2}^k = f'\left(\frac{u_j^k + u_{j+1}^k}{2}\right),$$

or

$$a_{j+1/2}^k = \overline{a}(u_j^k, u_{j+1}^k),$$

where $\overline{a}(u, v)$ is a function satisfying

$$\begin{cases} f(v) - f(u) = \overline{a}(u, v)(v - u), \\ \overline{a}(u, u) = f'(u). \end{cases}$$

This corresponds to a scheme of the form

$$u_j^{k+1} = u_j^k - \lambda\Big(F_{j+1/2}^k - F_{j-1/2}^k\Big),$$

where

$$F_{j+1/2}^k = \frac{1}{2}F_{j+1/2}^{k,\mathrm{LF}} + \frac{1}{2}F_{j+1/2}^{k,\mathrm{LW}}, \tag{2.35}$$

i.e., the flux is composed of a half component from the classical Lax-Friedrich flux and a half component of the Lax-Wendroff flux as given by (2.10). This is precisely the FORCE scheme.

Similarly, the same result holds when we consider the two-step Richtmyers variant of the scheme given by (2.33). For that case, we see that (2.33) takes the form

$$\begin{aligned} u_{j+1/2}^{k+1} &= \frac{1}{2}(u_{j+1}^k + u_j^k) - \frac{\lambda}{2}(f(u_{j+1}^k) - f(u_j^k)), \\ u_j^{k+1} &= u_j^k - \lambda\Big(G_{j+1/2}^k - G_{j-1/2}^k\Big) - \lambda\Big(h(u_{j+1/2}^{k+1}) - h(u_{j-1/2}^{k+1})\Big), \\ &= u_j^k - \frac{\lambda}{2}\Big(F_{j+1/2}^{LF,k} - F_{j-1/2}^{LF,k}\Big) - \frac{\lambda}{2}\Big(f(u_{j+1/2}^{k+1}) - f(u_{j-1/2}^{k+1})\Big), \end{aligned}$$

which corresponds exactly to the FORCE scheme where the Lax-Wendroff component now is written in the two-step Richtmyer's form, see Remark 1. □

In view of Proposition 2.1, we introduce the following definition:

Definition 1. *We shall use the term* **eXtended FORCE (X-FORCE)** *to denote numerical schemes constructed within the above semi-discrete two-step procedure* (2.19)–(2.21) *leading to the scheme given by* (2.29) *and* (2.30) *(or the alternative variant* (2.33)*).*

Remark 4. *Alternatively, we could replace the forward Euler discretization used in the first equation of* (2.22) *and the second equation of* (2.23) *by a backward Euler discretization leading to*

a scheme of the form

$$\begin{aligned}
&\frac{v_j^1 - v_j^0}{\Delta t} + \frac{1}{\Delta x}\Big(g_{j+1/2}^1 - g_{j-1/2}^1\Big) = 0, \qquad v_j^0 = \frac{1}{4}u_{j-1}^k + \frac{1}{2}u_j^k + \frac{1}{4}u_{j+1}^k, \\
&\qquad \textit{with} \qquad g_{j+1/2}^1 = \frac{g(u_j^{k+1}) + g(u_{j+1}^{k+1})}{2},
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
&\frac{w_j^1 - w_j^0}{\Delta t} + \frac{1}{\Delta x}\Big(\widetilde{h}_{j+1/2}^1 - \widetilde{h}_{j-1/2}^1\Big) = 0, \qquad w_j^0 = 0, \\
&\textit{with} \qquad \frac{\widetilde{h}_{j+1/2}^1 - \widetilde{h}_{j+1/2}^0}{\Delta t} + \frac{a_{j+1/2}^0}{\Delta x}\Big(g_{j+1}^1 - g_j^1\Big) = 0, \qquad \widetilde{h}_{j+1/2}^0 = \frac{h(u_j^k) + h(u_{j+1}^k)}{2},
\end{aligned} \tag{2.37}$$

where $g_j^1 = g(v_j^1 + w_j^1) = g(u_j^{k+1})$. *We shall return to this issue in more details in Section 5 and 6 where we explore explicit and semi-implicit X-FORCE schemes for the isothermal Euler equations and 4-equations two-fluid model.*

REMARK 5. *At the discrete level, we have seen that a sound discretization of the extended model* (2.18) *produces the X-FORCE class, which contains the FORCE scheme associated with* (1.1) *as a special case. Convergence and consistency of the FORCE scheme has been discussed, for instance, in [2, 4]. Basic convergence properties of X-FORCE for a scalar nonlinear conservation law is discussed in this paper. These results may indicate the relevance of applying the extended model* (2.18) *to construct an approximation to the original model* (1.1). *At the continuous level one might also expect that by introducing an appropriate viscous approximation to* (2.18), *one could demonstrate convergence to the entropy solution of* (1.1).

3. An interpretation via Riemann solutions. The purpose of this section is to give an interpretation of the class of extended FORCE (X-FORCE) schemes proposed in Section 2 through Riemann solutions. First, we note that for the model

$$u_t + f(u)_x = 0, \tag{3.1}$$

due to the presence of discontinuous solutions, one uses the integral form. For example, integration in a control volume $V = [x_L, x_R] \times [t_B, t_T]$ in the $x - t$ plane, leading to

$$\int_{x_L}^{x_R} u(x, t_T)\, dx = \int_{x_L}^{x_R} u(x, t_B)\, dx - \Big(\int_{t_B}^{t_T} f(u(x_R, t))\, dt - \int_{t_B}^{t_T} f(u(x_L, t))\, dt\Big). \tag{3.2}$$

It is instructive first to describe how the FORCE scheme can be obtained as solutions of Riemann problems. The FORCE flux is a more recent centred flux which can be derived from a deterministic interpretation of the staggered-grid version of Glimm's method [10]. We now briefly describe this following the line of [4].

3.1. FORCE. The method is described by the following two steps:

1) Let $\hat{u}_{j-1/2}(x,t) = \mathrm{RP}(u_{j-1}^k, u_j^k)$ denote the solution of the following Riemann problem

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x) = \begin{cases} u_{j-1}^k, & x < x_{j-1/2}, \\ u_j^k, & x > x_{j-1/2} \end{cases}, \qquad t \in (0, \Delta t/2]. \tag{3.3}$$

We then calculate an approximation $u_{j-1/2}^{k+1/2}$ at the half time step $\Delta t/2$ by averaging the Riemann solution $\hat{u}_{j-1/2}(x, \Delta t/2)$ over $[x_L, x_R] = [x_{j-1/2} - \frac{1}{2}\Delta x, x_{j-1/2} + \frac{1}{2}\Delta x]$, i.e.,

$$u_{j-1/2}^{k+1/2} = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \hat{u}_{j-1/2}(x, \Delta t/2)\, dx.$$

Then, applying (3.2) we get

$$\begin{aligned}\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\hat{u}_{j-1/2}(x,\Delta t/2)\,dx&\\ &= \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\hat{u}_{j-1/2}(x,0)\,dx - \frac{1}{\Delta x}\Big(\int_0^{\Delta t/2} f(u_j^k)\,dt - \int_0^{\Delta t/2} f(u_{j-1}^k)\,dt\Big)\\ &= \frac{1}{2}\Big(u_{j-1}^k + u_j^k\Big) - \frac{\Delta t}{2\Delta x}\Big(f(u_j^k) - f(u_{j-1}^k)\Big)\end{aligned}$$

A similar approach is taken for the calculation of $u_{j+1/2}^{k+1/2}$ by averaging the Riemann solution $\hat{u}_{j+1/2}(x,\Delta t/2) = \mathrm{RP}(u_j^k, u_{j+1}^k)$ over $[x_L, x_R] = [x_{j+1/2} - \frac{1}{2}\Delta x, x_{j+1/2} + \frac{1}{2}\Delta x]$, i.e.,

$$u_{j+1/2}^{k+1/2} = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\hat{u}_{j+1/2}(x,\Delta t/2)\,dx.$$

Thus, we obtain

$$\begin{aligned}u_{j-1/2}^{k+1/2} &= \frac{1}{2}\Big(u_{j-1}^k + u_j^k\Big) - \frac{\Delta t}{2\Delta x}\Big(f(u_j^k) - f(u_{j-1}^k)\Big)\\ u_{j+1/2}^{k+1/2} &= \frac{1}{2}\Big(u_j^k + u_{j+1}^k\Big) - \frac{\Delta t}{2\Delta x}\Big(f(u_{j+1}^k) - f(u_j^k)\Big).\end{aligned} \tag{3.4}$$

2) Let $\hat{u}_j(x,t) = \mathrm{RP}(u_{j-1/2}^{k+1/2}, u_{j+1/2}^{k+1/2})$ denote the solution of the following Riemann problem

$$u_t + f(u)_x = 0, \qquad u(x,\Delta t/2) = \begin{cases} u_{j-1/2}^{k+1/2}, & x < x_j,\\ u_{j+1/2}^{k+1/2}, & x > x_j\end{cases}, \qquad t \in (\Delta t/2, \Delta t]. \tag{3.5}$$

We then calculate an approximation u_j^{k+1} at the complete time step Δt by averaging the Riemann solution $\hat{u}_j(x,\Delta t/2)$ over $[x_L, x_R] = [x_j - \frac{1}{2}\Delta x, x_j + \frac{1}{2}\Delta x]$, i.e.,

$$u_j^{k+1} = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\hat{u}_j(x,\Delta t/2)\,dx.$$

Then, applying (3.2) we get

$$\begin{aligned}\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\hat{u}_j(x,\Delta t/2)\,dt&\\ &= \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\hat{u}_j(x,0)\,dt - \frac{1}{\Delta x}\Big(\int_0^{\Delta t/2} f(u_{j+1/2}^{k+1})\,dt - \int_0^{\Delta t/2} f(u_{j-1/2}^{k+1/2})\,dt\Big)\\ &= \frac{1}{2}\Big(u_{j-1/2}^{k+1/2} + u_{j+1/2}^{k+1/2}\Big) - \frac{\Delta t}{2\Delta x}\Big(f(u_{j+1/2}^{k+1/2}) - f(u_{j-1/2}^{k+1/2})\Big)\end{aligned}$$

Thus, we obtain a scheme of the form

$$u_j^{k+1} = \frac{1}{2}\Big(u_{j-1/2}^{k+1/2} + u_{j+1/2}^{k+1/2}\Big) - \frac{\lambda}{2}\Big(F_{j+1/2}^{LW,k} - F_{j-1/2}^{LW,k}\Big),$$

where $F_{j+1/2}^{LW,k}$, in view of (3.4) and Remark 1, represents nothing but the two-step Lax-Wendroff flux. This again corresponds to the scheme

$$u_j^{k+1} = u_j^k - \frac{\lambda}{2}\Big(F_{j+1/2}^{LF,k} - F_{j-1/2}^{LF,k}\Big) - \frac{\lambda}{2}\Big(F_{j+1/2}^{LW,k} - F_{j-1/2}^{LW,k}\Big), \tag{3.6}$$

where $F_{j+1/2}^{LF,k}$ corresponds to the classical Lax-Friedrich flux.

3.2. The class of extended FORCE (X-FORCE) schemes. In a similar manner we now describe the proposed class of generalized FORCE schemes through Riemann solutions. The main difference lies in the fact that we solve for the complete time step Δt, however, we solve over this time step twice. In the first step we account for the flux component g, then in the second step we take into account the h flux component. First, we assume that we have introduced a splitting of the flux f such that

$$f(u) = g(u) + h(u), \quad \forall u.$$

The method is then described by the following two steps:

1) Let $\hat{v}_{j-1/2}(x,t) = \mathrm{RP}(u^k_{j-1}, u^k_j)$ denote the solution of the following Riemann problem

$$v_t + g(v)_x = 0, \qquad v(x,0) = \begin{cases} u^k_{j-1}, & x < x_{j-1/2}, \\ u^k_j, & x > x_{j-1/2} \end{cases}, \qquad t \in (0, \Delta t]. \tag{3.7}$$

We then calculate an approximation $v^1_{j-1/2}$ at the complete time time step Δt by averaging the Riemann solution $\hat{v}_{j-1/2}(x, \Delta t)$ over $[x_L, x_R] = [x_{j-1/2} - \frac{1}{2}\Delta x, x_{j-1/2} + \frac{1}{2}\Delta x]$, i.e.,

$$v^1_{j-1/2} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \hat{v}_{j-1/2}(x, \Delta t)\, dx. \tag{3.8}$$

Then, applying (3.2) over the control volume $V = [x_{j-1/2} - \frac{1}{2}\Delta x, x_{j-1/2} + \frac{1}{2}\Delta x] \times [0, \Delta t]$ we get

$$\begin{aligned} &\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \hat{v}_{j-1/2}(x, \Delta t)\, dx \\ &\quad = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \hat{v}_{j-1/2}(x, 0)\, dx - \frac{1}{\Delta x}\Big(\int_0^{\Delta t} g(\hat{v}_{j-1/2}(x_j, t))\, dt - \int_0^{\Delta t} g(\hat{v}_{j-1/2}(x_{j-1}, t))\, dt \Big) \\ &\quad = \frac{1}{2}\Big(u^k_{j-1} + u^k_j\Big) - \frac{\Delta t}{\Delta x}\Big(g(u^k_j) - g(u^k_{j-1})\Big) \end{aligned} \tag{3.9}$$

A similar approach is taken for the calculation of $v^1_{j+1/2}$ by averaging the Riemann solution $\hat{v}_{j+1/2}(x, \Delta t)$ over $[x_L, x_R] = [x_{j+1/2} - \frac{1}{2}\Delta x, x_{j+1/2} + \frac{1}{2}\Delta x]$, i.e.,

$$v^1_{j+1/2} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \hat{v}_{j+1/2}(x, \Delta t)\, dx.$$

Thus, we obtain

$$\begin{aligned} v^1_{j-1/2} &= \frac{1}{2}\Big(u^k_{j-1} + u^k_j\Big) - \frac{\Delta t}{\Delta x}\Big(g(u^k_j) - g(u^k_{j-1})\Big) \\ v^1_{j+1/2} &= \frac{1}{2}\Big(u^k_j + u^k_{j+1}\Big) - \frac{\Delta t}{\Delta x}\Big(g(u^k_{j+1}) - g(u^k_j)\Big) \end{aligned} \tag{3.10}$$

2) Let $\hat{w}_j(x,t) = \mathrm{RP}(v^1_{j-1/2}, v^1_{j+1/2})$ denote the solution of the following Riemann problem

$$w_t + h(w)_x = 0, \qquad w(x,0) = \begin{cases} v^1_{j-1/2}, & x < x_j, \\ v^1_{j+1/2}, & x > x_j \end{cases}, \qquad t \in (0, \Delta t]. \tag{3.11}$$

We then calculate an approximation w^1_j at the complete time step Δt by averaging the Riemann solution $\hat{w}_j(x, \Delta t)$ over $[x_L, x_R] = [x_j - \frac{1}{2}\Delta x, x_j + \frac{1}{2}\Delta x]$, i.e.,

$$w^1_j = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \hat{w}_j(x, \Delta t)\, dx. \tag{3.12}$$

Then, applying (3.2) we get

$$\begin{aligned}
&\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \hat{w}_j(x,\Delta t)\,dt \\
&\quad= \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \hat{w}_j(x,0)\,dt - \frac{1}{\Delta x}\Big(\int_0^{\Delta t} h(\hat{w}_j(x_{j+1/2},t))\,dt - \int_0^{\Delta t} h(\hat{w}_j(x_{j-1/2},t))\,dt\Big) \\
&\quad= \frac{1}{2}\Big(v^1_{j-1/2}+v^1_{j+1/2}\Big) - \frac{\Delta t}{\Delta x}\Big(h(v^1_{j+1/2}) - h(v^1_{j-1/2})\Big)
\end{aligned} \tag{3.13}$$

Setting $u_j^{k+1} = w_j^1$, and in view of (3.10), we obtain a scheme of the form

$$\begin{aligned}
u_j^{k+1} = w_j^1 &= \frac{1}{2}\Big(v^1_{j-1/2}+v^1_{j+1/2}\Big) - \lambda\Big(h(v^1_{j+1/2}) - h(v^1_{j-1/2})\Big) \\
&= u_j^k - \lambda\Big(G^k_{j+1/2} - G^k_{j-1/2}\Big) - \lambda\Big(H^{k+1}_{j+1/2} - H^{k+1}_{j-1/2}\Big)
\end{aligned} \tag{3.14}$$

where $G^k_{j+1/2}$ is the modified Lax-Friedrich flux and $H^{k+1}_{j+1/2} = h(u^{k+1}_{j+1/2})$ is the flux which coincides with (2.31) and (2.32), i.e.

$$\begin{aligned}
&G^k_{j+1/2} = \frac{g(u_j^k)+g(u_{j+1}^k)}{2} - \frac{1}{4\lambda}(u^k_{j+1}-u^k_j), \\
&H^{k+1}_{j+1/2} = h(u^{k+1}_{j+1/2}), \qquad u^{k+1}_{j+1/2} = \frac{u_j^k+u_{j+1}^k}{2} - \lambda\Big(g(u^k_{j+1}) - g(u^k_j)\Big)
\end{aligned} \tag{3.15}$$

Consequently, we see that the class of X-FORCE schemes on the two-step Richtmyer's form (2.33) can be interpreted through Riemann solutions.

4. Some basic properties of the X-FORCE schemes: The scalar case. We let Δ denote the centred difference operator defined by $\Delta v_{j+1/2} = v_{j+1} - v_j$ and $\Delta v_j = v_{j+1/2} - v_{j-1/2}$. From (2.29) we see that the scheme can be written in the form

$$u_j^{k+1} = u_j^k - \frac{\lambda}{2}(f^k_{j+1} - f^k_{j-1}) + \frac{1}{2}\Big(Q^k_{j+1/2}\Delta u^k_{j+1/2} - Q^k_{j-1/2}\Delta u^k_{j-1/2}\Big), \tag{4.1}$$

where

$$Q^k_{j+1/2} = \frac{\frac{1}{2}\Delta u^k_{j+1/2} + 2\lambda^2 a^k_{j+1/2}\Delta g^k_{j+1/2}}{\Delta u^k_{j+1/2}} = \frac{1}{2} + 2\lambda^2 a^k_{j+1/2} b^k_{j+1/2}, \tag{4.2}$$

where $a^k_{j+1/2}$ and $b^k_{j+1/2}$ is defined according to (2.8) and (2.9), i.e.,

$$a^k_{j+1/2} = \frac{\Delta h^k_{j+1/2}}{\Delta u^k_{j+1/2}}, \qquad b^k_{j+1/2} = \frac{\Delta g^k_{j+1/2}}{\Delta u^k_{j+1/2}}. \tag{4.3}$$

Moreover, we note that the scheme can be written in the form

$$u_j^{k+1} = u_j^k + C^{k,+}_{j+1/2}\Delta u^k_{j+1/2} - C^{k,-}_{j-1/2}\Delta u^k_{j-1/2}, \tag{4.4}$$

where

$$\begin{aligned}
C^+_{j+1/2} &= \frac{1}{2}\Big(Q_{j+1/2} - \lambda\frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}}\Big) \\
C^-_{j+1/2} &= \frac{1}{2}\Big(Q_{j+1/2} + \lambda\frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}}\Big).
\end{aligned} \tag{4.5}$$

REMARK 6. *Note that* **Monotone** *schemes - those for which the RHS of a scheme written in the form*

$$u_j^{k+1} = H(u_{j-1}^k, u_j^k, u_{j+1}^k), \tag{4.6}$$

is nondecreasing in all its u arguments - is a subclass of **Monotonicity Preserving** *schemes. Monotone schemes obey both the compactness and entropy requirements. The limit solutions of monotonicity-preserving schemes are not necessarily the physically relevant ones. Murman's scheme is an example of this (we refer to [17] for more details and a concrete example).*

We may state this more precisely as follows:

DEFINITION 2. *Schemes of the form* (4.6) *for the scalar, nonlinear conservation law are said to be Monotonicity Preserving Schemes if whenever the data* $\{u_j^n\}$ *is monotone, the solution set* $\{u_j^{n+1}\}$ *is monotone in the same sense. That is, if* $\{u_j^n\}$ *is monotone increasing, so is* $\{u_j^{n+1}\}$ *and if* $\{u_j^n\}$ *is monotone decreasing, so is* $\{u_j^{n+1}\}$.

It is well known that monotone, TVD, and monotonicity preserving schemes for nonlinear scalar conservation laws are related as follows (see [12] and [20]): The set S_{MON} of monotone schemes is contained in the set S_{TVD} of TVD schemes, and this in turn again is contained in the set S_{MPR} of monotonicity preserving schemes.

4.1. Monotonicity and TVD-property. We recall the following lemma according to Tadmor [17] (see also [12]) for a scheme written on the incremental form (4.4).

LEMMA 4.1. *Three-point monotonicity-preserving schemes are exactly those whose total variation is nonincreasing. They are characterized by the set of inequalities*

$$C_{j+1/2}^+ \geq 0, \qquad C_{j+1/2}^- \geq 0, \qquad 1 - C_{j+1/2}^+ - C_{j+1/2}^- \geq 0. \tag{4.7}$$

COROLLARY 4.2. *The scheme* (4.1)–(4.3) *is monotonicity preserving and total variation nonincreasing under the CFL condtions*

$$\lambda |a_{j+1/2}^k| \leq \frac{1}{2}, \qquad \lambda |b_{j+1/2}^k| \leq \frac{1}{2}, \tag{4.8}$$

Proof. We simply note that

$$\begin{aligned} C_{j+1/2}^{k,+} &= \frac{1}{2}\Big(\frac{1}{2} + 2\lambda^2 a_{j+1/2}^k b_{j+1/2}^k - \lambda[a_{j+1/2}^k + b_{j+1/2}^k]\Big) \\ &= \Big(\frac{1}{2} - \lambda a_{j+1/2}^k\Big)\Big(\frac{1}{2} - \lambda b_{j+1/2}^k\Big), \end{aligned}$$

$$\begin{aligned} C_{j+1/2}^{k,-} &= \frac{1}{2}\Big(\frac{1}{2} + 2\lambda^2 a_{j+1/2}^k b_{j+1/2}^k + \lambda[a_{j+1/2}^k + b_{j+1/2}^k]\Big) \\ &= \Big(\frac{1}{2} + \lambda a_{j+1/2}^k\Big)\Big(\frac{1}{2} + \lambda b_{j+1/2}^k\Big), \end{aligned}$$

and

$$1 - C_{j+1/2}^{k,+} - C_{j+1/2}^{k,-} = 1 - Q_{j+1/2}^k = \Big(\frac{1}{2} - 2\lambda^2 a_{j+1/2}^k b_{j+1/2}^k\Big) = 2\Big(\frac{1}{4} - \lambda^2 a_{j+1/2}^k b_{j+1/2}^k\Big).$$

From this, the result of the corollary follows. □

In view of the general expression (4.5) and Lemma 4.1 we have the following well-known result [17].

LEMMA 4.3. *A three-point conservative scheme of the form* (4.1) *is monotonicity preserving and total variation nonincreasing if, and only if, its numerical viscosity coefficient* $Q_{j+1/2}$ *satisfies*

$$\lambda \left| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \right| \leq Q_{j+1/2} \leq 1. \tag{4.9}$$

Particularly, in view of the Corollary 4.2 we have the following result.

COROLLARY 4.4. *If the CFL conditions* (4.8) *is satisfied, the relation* (4.9) *holds for the scheme* (4.1)–(4.3).

As noted in [18] we have the following result which ensures a maximum principle.

LEMMA 4.5. *A three-point conservative scheme of the form* (4.1) *with a numerical viscosity coefficient* $Q_{j+1/2}$ *satisfying*

$$\lambda \left| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \right| \leq Q_{j+1/2} \leq \frac{1}{2}, \tag{4.10}$$

obeys the following maximum principle

$$\inf_j u_j^k \leq u_j^{k+1} \leq \sup u_j^k. \tag{4.11}$$

COROLLARY 4.6. *In view of Lemma 4.5 and the form* (4.2) *of the numerical viscosity coefficient* $Q_{j+1/2}$, *we* **cannot** *conclude that the scheme* (4.1)–(4.3) *satisfies the maximum principle* (4.11).

Below we will see that the scheme (4.1)–(4.3), written in the two-step Richtmyer's form as given by (2.33), satisfies a maximum principle under a stronger CFL condition than (4.8).

REMARK 7. *The modified LxF scheme [18] corresponds to* (4.1) *with the choice* $Q^k_{j+1/2} = \frac{1}{2}$. *The corresponding CFL condition which ensures that this scheme is TVD reads (in view of Lemma 4.3)*

$$\lambda \Big| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \Big| \leq \frac{1}{2}.$$

For the special choice $g = h = 1/2f$, *in view of* (4.8), *the resulting FORCE scheme is TVD under the CFL condition*

$$\lambda \Big| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \Big| \leq 1.$$

The price to pay for this weakening of the CFL condition is that a Lax-Wendroff (LW) type of numerical viscosity term has been added.

4.2. The class of X-FORCE schemes considered as monotone schemes. In the following we consider the X-FORCE scheme obtained by applying the two-step Richtmyer's approach for the H-flux component as described in Remark 3, and also reproduced in Section 3 through an interpretation via Riemann solutions. The resulting scheme is of the form

$$u_j^{k+1} = u_j^k - \lambda \Big(G^k_{j+1/2} - G^k_{j-1/2} \Big) - \lambda \Big(H^{k+1}_{j+1/2} - H^{k+1}_{j-1/2} \Big) \tag{4.12}$$

where

$$\begin{aligned} G^k_{j+1/2}(u_j^k, u_{j+1}^k) &= \frac{g(u_j^k) + g(u_{j+1}^k)}{2} - \frac{1}{4\lambda}(u_{j+1}^k - u_j^k) \\ H^{k+1}_{j+1/2}(u_j^k, u_{j+1}^k) &= h(u^{k+1}_{j+1/2}), \end{aligned} \tag{4.13}$$

where

$$u_{j+1/2}^{k+1} = \frac{u_j^k + u_{j+1}^k}{2} - \lambda\Big(g(u_{j+1}^k) - g(u_j^k)\Big). \tag{4.14}$$

Monotone schemes, see Remark 6, is a classical subclass of monotonicity preserving schemes, capturing both the compactness and the entropy requirements, see e.g. [5]. A standard way is to verify the entropy condition is by constructing a discrete entropy pair satisfying an entropy inequality. First, we recall a well-known result for monotone schemes, see for instance [20], then we state a corollary relevant for the X-FORCE schemes.

THEOREM 4.7. *A three-point scheme of the form*

$$u_j^{k+1} = u_j^k - \lambda\Big(F_{j+1/2}^k - F_{j-1/2}^k\Big), \qquad F_{j+1/2}^k = F_{j+1/2}(u_j^k, u_{j+1}^k), \tag{4.15}$$

is monotone if

$$\frac{\partial}{\partial u_j} F_{j+1/2}(u_j, u_{j+1}) \geq 0, \qquad \textit{and} \qquad \frac{\partial}{\partial u_{j+1}} F_{j+1/2}(u_j, u_{j+1}) \leq 0. \tag{4.16}$$

It is well known that for a given data set $\{u_j^k\}$, if the solution set $\{u_j^{k+1}\}$ is obtained with a monotone method then a maximum principle holds, i.e., $\inf_j u_j^k \leq u_j^{k+1} \leq \sup_j u_j^k$.

COROLLARY 4.8. *The scheme given by* (4.12)–(4.14) *is monotone and satisfies a maximum principle under the CFL condition*

$$\lambda\|h'\|_\infty \leq \frac{1}{8}, \qquad \lambda\|g'\|_\infty \leq \frac{1}{8}. \tag{4.17}$$

Proof. The scheme can be written in the form (4.15) where

$$F_{j+1/2}(u_j^k, u_{j+1}^k) = G_{j+1/2}(u_j^k, u_{j+1}^k) + H_{j+1/2}(u_j^k, u_{j+1}^k).$$

Consequently,

$$\begin{aligned} 4\lambda\frac{\partial}{\partial u_j^k} F_{j+1/2}(u_j^k, u_{j+1}^k) &= 2\lambda g'(u_j^k) + 1 + 4\lambda h'(u_{j+1/2}^{k+1})\Big[\frac{1}{2} + \lambda g'(u_j^k)\Big] \\ &= \Big[2\lambda g'(u_j^k) + \frac{1}{4}\Big] + \Big[2\lambda h'(u_{j+1/2}^{k+1}) + \frac{1}{4}\Big] + \Big[4\lambda^2 h'(u_{j+1/2}^{k+1}) g'(u_j^k) + \frac{1}{2}\Big], \end{aligned}$$

where $u_{j+1/2}^{k+1}$ is given by (4.14). Similarly,

$$\begin{aligned} 4\lambda\frac{\partial}{\partial u_{j+1}^k} F_{j+1/2}(u_j^k, u_{j+1}^k) &= 2\lambda g'(u_{j+1}^k) - 1 + 4\lambda h'(u_{j+1/2}^{k+1})\Big[\frac{1}{2} - \lambda g'(u_{j+1}^k)\Big] \\ &= -\Big[\frac{1}{4} - 2\lambda g'(u_j^k)\Big] - \Big[\frac{1}{4} - 2\lambda h'(u_{j+1/2}^{k+1})\Big] - \Big[\frac{1}{2} + 4\lambda^2 h'(u_{j+1/2}^{k+1}) g'(u_j^k)\Big] \end{aligned}$$

Thus, we may conclude that the above inequalities (4.16) are satisfied under the CFL condition (4.17).
□

REMARK 8. *Note that for the special case that f is linear, i.e., $f(u) = cu$ where c is a constant, and for the special splitting $g = h = \frac{1}{2}f$, corresponding to the FORCE scheme, we see that*

$$4\lambda\frac{\partial}{\partial u_j^k} F_{j+1/2}(u_j^k, u_{j+1}^k) = \lambda c + 1 + 2\lambda c\Big[\frac{1}{2} + \frac{\lambda c}{2}\Big] = (1 + \lambda c)^2,$$

and

$$4\lambda\frac{\partial}{\partial u_{j+1}^k}F_{j+1/2}(u_j^k,u_{j+1}^k)=\lambda c-1+2\lambda c\Big[\frac{1}{2}-\frac{\lambda c}{2}\Big]=-(1-\lambda c)^2.$$

Hence, it follows that the CFL condition (4.17) *takes the weaker form*

$$\lambda|c|\le 1,$$

as pointed out in [2, 4].

REMARK 9. *Since the scheme given by* (4.12)–(4.14) *is monotone under the CFL condition* (4.17)*, it follows from [5] that the scheme will converge to the entropy solution.*

4.3. Entropy consistency for the class of X-FORCE schemes. In this section we explore the entropy consistency of the class of X-FORCE schemes without referring to monotonicity properties (see Remark 9). We basically follow along the line of [18] where entropy consistency is shown for the modified LxF scheme. However, we introduce a suitable modification due to the fact that the X-FORCE scheme is obtained as the solution of two sets of non-interacting Riemann problems as described in Section 3.

Assume that we have given an entropy-entropy flux pair (Φ,Ψ), i.e., $\Phi''\ge 0$ and $\Phi' f'=\Psi'$ for the scalar conservation law $u_t+f(u)_x=0$. Clearly, we can express the entropy flux Ψ as

$$\Psi(w)=\int_{w_0}^{w}\Phi'(z)f'(z)\,dz, \tag{4.18}$$

for a suitable choice of w_0. For a splitting $f(u)=g(u)+h(u)$ we may define the two scalar functions Ψ_g and Φ_h

$$\Psi_g(w)=\int_{w_0}^{w}\Phi'(z)g'(z)\,dz, \tag{4.19}$$

$$\Psi_h(w)=\int_{w_0}^{w}\Phi'(z)h'(z)\,dz. \tag{4.20}$$

Clearly, (Φ,Ψ_g) and (Φ,Ψ_h) are entropy-entropy flux pairs, respectively for the two conservation laws

$$v_t+g(v)_x=0, \tag{4.21}$$

$$w_t+h(w)_x=0, \tag{4.22}$$

For entropy solutions v and w we know that the following entropy conditions are satisfied (in a weak sense)

$$\Phi(v)_t+\Psi_g(v)_x\le 0, \tag{4.23}$$

$$\Phi(w)_t+\Psi_h(w)_x\le 0. \tag{4.24}$$

Moreover, we observe that

$$\Psi_g(w)+\Psi_h(w)=\Psi(w). \tag{4.25}$$

We want to show that the X-FORCE schemes described by (3.14) and (3.15) are entropy satisfying, i.e., that

$$\Phi(u_j^{k+1})\le\Phi(u_j^k)-\lambda\Big(\Psi_{j+1/2}^{GF,k}-\Psi_{j-1/2}^{GF,k}\Big). \tag{4.26}$$

with a numerical entropy flux $\Psi_{j+1/2}^{GF,k}=\Psi^{GF}(u_j^k,u_{j+1}^k)$ which is consistent with the differential one, i.e., $\Psi^{GF}(v,v)=\Psi(v)$.

First, we observe in view of (3.8), Jensen's inequality, and the integral form of (4.23) that

$$\begin{aligned}\Phi(v^1_{j-1/2}) &\le \frac{1}{\Delta x}\int_{\Delta x/2}^{\Delta x/2}\Phi(\hat v_{j-1/2}(x,\Delta t))\,dx\\ &\le \frac{1}{\Delta x}\int_{\Delta x/2}^{\Delta x/2}\Phi(\hat v_{j-1/2}(x,0))\,dx - \frac{1}{\Delta x}\Big(\int_0^{\Delta t}\Psi_g(\hat v_{j-1/2}(x_j,t))\,dt - \int_0^{\Delta t}\Psi_g(\hat v_{j-1/2}(x_{j-1},t))\,dt\Big)\\ &= \frac{1}{2}\Big(\Phi(u^k_{j-1})+\Phi(u^k_j)\Big) - \frac{\Delta t}{\Delta x}\Big(\Psi_g(u^k_j)-\Psi_g(u^k_{j-1})\Big)\end{aligned} \tag{4.27}$$

Here we have used the fact that the Riemann solution $\hat v_{j-1/2}(x,t)$ is entropy satisfying and the CFL condition

$$\lambda\cdot\max_v|b(v)|\le\frac{1}{2},\qquad b(v)=g'(v),$$

which ensures that no wave interaction have taken place. Similarly, we have

$$\begin{aligned}\Phi(v^1_{j+1/2}) &\le \frac{1}{\Delta x}\int_{\Delta x/2}^{\Delta x/2}\Phi(\hat v_{j+1/2}(x,\Delta t))\,dx\\ &\le \frac{1}{\Delta x}\int_{\Delta x/2}^{\Delta x/2}\Phi(\hat v_{j+1/2}(x,0))\,dx - \frac{1}{\Delta x}\Big(\int_0^{\Delta t}\Psi_g(\hat v_{j+1/2}(x_{j+1},t))\,dt - \int_0^{\Delta t}\Psi_g(\hat v_{j+1/2}(x_j,t))\,dt\Big)\\ &= \frac{1}{2}\Big(\Phi(u^k_j)+\Phi(u^k_{j+1})\Big) - \frac{\Delta t}{\Delta x}\Big(\Psi_g(u^k_{j+1})-\Psi_g(u^k_j)\Big).\end{aligned} \tag{4.28}$$

Next, in view of (3.12), Jensen's inequality, the integral form of (4.24) (since the Riemann solution $\hat w_j(x,t)$ is entropy satisfying) we have

$$\begin{aligned}\Phi(w^1_j) &\le \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\Phi(\hat w_j(x,\Delta t))\,dt\\ &\le \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\Phi(\hat w_j(x,0))\,dt - \frac{1}{\Delta x}\Big(\int_0^{\Delta t}\Psi_h(\hat w_j(x_{j+1/2},t))\,dt - \int_0^{\Delta t}\Psi_h(\hat w_j(x_{j-1/2},t))\,dt\Big)\\ &= \frac{1}{2}\Big(\Phi(v^1_{j-1/2})+\Phi(v^1_{j+1/2})\Big) - \frac{\Delta t}{\Delta x}\Big(\Psi_h(v^1_{j+1/2})-\Psi_h(v^1_{j-1/2})\Big),\end{aligned} \tag{4.29}$$

where we have used the CFL condition

$$\lambda\cdot\max_v|a(v)|\le\frac{1}{2},\qquad a(v)=h'(v).$$

Hence, the following lemma is obtained.

LEMMA 4.9. *Assume the CFL condition*

$$\lambda\cdot\max_v|a(v)|\le\frac{1}{2},\qquad \lambda\cdot\max_v|b(v)|\le\frac{1}{2},\qquad a(v)=h'(v),\quad b(v)=g'(v)$$

holds. The X-FORCE scheme described by (3.14) *and* (3.15)*, satisfies the entropy inequality* (4.26) *with a numerical entropy flux* $\Psi^{GF,k}_{j+1/2}$ *of the form*

$$\Psi^{GF,k}_{j+1/2}=\Psi^{GF}(u^k_j,u^k_{j+1})=\Psi^{G,k}_{j+1/2}+\Psi^{H,k}_{j+1/2}, \tag{4.30}$$

where

$$\begin{aligned}\Psi^{G,k}_{j+1/2} &= \frac{\Psi_g(u^k_j)+\Psi_g(u^k_{j+1})}{2} - \frac{1}{4\lambda}(\Phi(u^k_{j+1})-\Phi(u^k_j))\\ \Psi^{H,k}_{j+1/2} &= \Psi_h(u^{k+1}_{j+1/2}), \qquad u^{k+1}_{j+1/2} = \frac{u^k_j+u^k_{j+1}}{2} - \lambda\Big(g(u^k_{j+1})-g(u^k_j)\Big),\end{aligned} \tag{4.31}$$

which is consistent with the differential one, $\Psi^{GF}(v,v)=\Psi(v)$.

Proof. Combining (4.27), (4.28), (4.29), and the fact that $w^1_j = u^{k+1}_j$ and $v^1_{j-1/2}$ and $v^1_{j+1/2}$ are given by (3.10), we see that (4.26) holds with $\Psi^{GF,k}_{j+1/2}$ defined by (4.30)–(4.31). Clearly, in view of (4.25), the numerical entropy flux $\Psi^{GF,k}_{j+1/2}$ is consistent with Ψ. □

Remark 10. *Note that we cannot simply refer to Theorem 6.1 in [18] where entropy consistency is ensured through* **comparison** *of the size of the numerical dissipation* $Q_{j+1/2}$. *This theorem requires that the scheme under consideration, given in the general form*

$$u^{k+1}_j = \frac{\Big[u^k_j - \lambda(f^k_j - f^k_{j-1}) - Q^k_{j-1/2}\Delta u^k_{j-1/2}\Big] + \Big[u^k_j - \lambda(f^k_{j+1} - f^k_j) + Q^k_{j+1/2}\Delta u^k_{j+1/2}\Big]}{2},$$

satisfies the CFL condition

$$|Q_{j+1/2}| \le \frac{1}{2}. \tag{4.32}$$

In view of the numerical dissipation term (4.2) *corresponding to the X-FORCE schemes, it's clear that* (4.32) *does not hold in general.*

4.4. Construction of higher order X-FORCE schemes. The purpose of this section is to describe a procedure for obtaining higher order spatial accuracy for the X-FORCE class. Equipped with the Riemann-based interpretation of Section 3 the idea is simply to replace piecewise constant interpolants used in Step 1 and 2 for the evaluation of spatial and temporal integrals by appropriate piecewise linear interpolants. Here we closely follow the line of [14]. In particular, by using the special splitting $g = f$ and $h = 0$, the resulting higher-order X-FORCE class reproduces the non-staggered NT scheme presented in [14].

We introduce the spatial grid Δx and consider the average

$$\overline{u}(x,t) := \frac{1}{|I_x|}\int_{I_x} u(\xi,t)\,d\xi, \qquad I_x = \{\xi \ : \ |\xi - x| \le \frac{\Delta x}{2}\}.$$

Introducing a time step Δt we can rewrite the equation $u_t + f(u)_x = 0$ and end up with the following equivalent formulation

$$\overline{u}(x,t+\Delta t) = \overline{u}(x,t) - \frac{1}{\Delta x}\Big[\int_t^{t+\Delta t} f(u(x+\frac{\Delta x}{2},\tau))\,d\tau - \int_t^{t+\Delta t} f(u(x-\frac{\Delta x}{2},\tau))\,d\tau\Big]. \tag{4.33}$$

This relation between the average $\overline{u}(\cdot,t)$ and their underlying pointvalues $u(\cdot,t)$ is used repeatedly in the following. Typically, we start with some approximate solution $w(\cdot,t^n)$ at time level t^n where it is described as a piecewise polynomial written in the form

$$w(x,t^n) = \sum_j p_j(x)\chi_j(x), \qquad \chi_j(x) = 1_{I_j}, \qquad I_j = \{\xi \ : \ |\xi - x_j| \le \frac{\Delta x}{2}\}.$$

An exact evolution $w(\cdot,t^n)$ based on (4.33) gives us

$$\overline{w}(x,t^{n+1}) = \overline{w}(x,t^n) - \frac{1}{\Delta x}\Big[\int_{t^n}^{t^{n+1}} f(w(x+\frac{\Delta x}{2},\tau))\,d\tau - \int_{t^n}^{t^{n+1}} f(w(x-\frac{\Delta x}{2},\tau))\,d\tau\Big]. \tag{4.34}$$

4.4.1. Construction of the scheme. As before we assume that we have introduced a splitting of the flux f such that $f(u) = g(u) + h(u)$. Assume that we have given a piecewise constant approximation to $u(x,t^n)$ given by $\overline{u}(x,t^n) = \sum_j \bar{u}_j^n \chi_j(x)$, where $\chi_j(x) = 1_{I_j}$ and $I_j := \{|x - x_j| \le \frac{\Delta x}{2}\}$. We now describe how to construct an approximation to $u(x,t^{n+1})$ via an appropriate modification of the two steps described in Section 3.

1) We first introduce a piecewise linear interpolant

$$v_0(x) = \sum_j \Big[v_j^0 + v_j'\Big(\frac{x-x_j}{\Delta x}\Big)\Big]\chi_j(x) = \sum_j L_j(x)\chi_j(x), \qquad v_j^0 = \bar{u}_j^n. \tag{4.35}$$

Here $\frac{1}{\Delta x}v_j'$ is a numerical derivative determined by

$$\frac{1}{\Delta x}v_j' = \frac{1}{\Delta x} MM\Big(\theta\Delta v_{j+1/2}^0, \frac{\Delta v_{j-1/2}^0 + \Delta v_{j+1/2}^0}{2}, \theta\Delta v_{j-1/2}^0\Big), \tag{4.36}$$

for $\theta \in [1,2]$ where MM denotes the min-mod nonlinear limiter and $\Delta v_{j+1/2} = v_{j+1} - v_j$. Next, we perform an exact evolution of the piecewise linear interpolant (4.35) according to the problem

$$v_t + g(v)_x = 0, \qquad v(x,0) = v_0(x), \qquad t \in (0,\Delta t]. \tag{4.37}$$

Let $v(x,\Delta t)$ denote this solution. The resulting solution is then projected back into the space of staggered piecewise-constant gridfunctions, i.e.,

$$v_{j-1/2}(\Delta t) = \overline{v}(x,\Delta t) = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} v(y,\Delta t)\,dy, \qquad x \in (x_{j-1},x_j) = I_{j-1/2}. \tag{4.38}$$

In view of (4.34), considered over the control volume $V = [x_{j-1},x_j]\times[0,\Delta t]$, we get

$$\begin{aligned} v_{j-1/2}(\Delta t) &= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} v(y,\Delta t)\,dy \\ &= \frac{1}{\Delta x}\left[\int_{x_{j-1}}^{x_{j-1/2}} L_{j-1}(x)\,dx + \int_{x_{j-1/2}}^{x_j} L_j(x)\,dx\right] \\ &\qquad - \frac{1}{\Delta x}\left[\int_0^{\Delta t} g(v(x_j,\tau))\,d\tau - \int_0^{\Delta t} g(v(x_{j-1},\tau))\,d\tau\right]. \end{aligned} \tag{4.39}$$

The two first integrals on the RHS can be integrated exactly. Moreover, if the CFL condition $\lambda\cdot\max_{x_{j-1}\le x\le x_j}\rho(A_g(v(x,0))) < \frac{1}{2}$, is met (here A_g refers to the Jacobian of g), then the two last integrands on the RHS of (4.39), $g(v(x_{j-1},\tau))$ and $g(v(x_j,\tau))$ are integrated approximately by the midpoint rule resulting in the following scheme

$$v_{j-1/2}(\Delta t) = \frac{1}{2}\Big[v_{j-1}^0 + v_j^0\Big] + \frac{1}{8}\Big[v_{j-1}' - v_j'\Big] - \lambda\Big[g(v(x_j,\frac{\Delta t}{2})) - g(v(x_{j-1},\frac{\Delta t}{2}))\Big], \tag{4.40}$$

where v_j' is given by (4.36). Following [14], we use the approximation

$$v(x_j,\frac{\Delta t}{2}) \approx v(x_j,0) - \frac{\lambda}{2}g_j',$$

where g_j' represents a numerical derivative of the flux $g(v(x_j,0))$,

$$\frac{1}{\Delta x}g_j' = \frac{1}{\Delta x} MM\Big(\theta\Delta g(v_{j+1/2}^0), \frac{\Delta g(v_{j-1/2}^0) + \Delta g(v_{j+1/2}^0)}{2}, \theta\Delta g(v_{j-1/2}^0)\Big). \tag{4.41}$$

Consequently, we obtain a sequence of discrete values $\{\bar{v}_{j+1/2}^1\}$ by solving

$$v_j^{1/2} = v_j^0 - \frac{\lambda}{2}g_j', \tag{4.42}$$

$$\bar{v}_{j+1/2}^1 = \frac{1}{2}\Big[v_j^0 + v_{j+1}^0\Big] + \frac{1}{8}\Big[v_j' - v_{j+1}'\Big] - \lambda\Big[g(v_{j+1}^{1/2}) - g(v_j^{1/2})\Big]. \tag{4.43}$$

2) Next, from the discrete values $\{\bar{v}^1_{j+1/2}\}$ we introduce a piecewise linear interpolant

$$w_0(x) = \sum_j \Big[w^0_{j+1/2} + w'_{j+1/2}\Big(\frac{x - x_{j+1/2}}{\Delta x}\Big)\Big]\chi_{j+1/2}(x) = \sum_j L_{j+1/2}(x)\chi_{j+1/2}(x), \quad (4.44)$$

where

$$w^0_{j+1/2} = \bar{v}^1_{j+1/2}, \qquad \chi_{j+1/2}(x) = 1_{I_{j+1/2}}, \qquad I_{j+1/2} = \{\xi \ : \ |\xi - x_{j+1/2}| \le \frac{\Delta x}{2}\}.$$

Here $\frac{1}{\Delta x} w'_{j+1/2}$ is a numerical derivative similar to $\frac{1}{\Delta x} v'_j$ given by

$$\frac{1}{\Delta x} w'_{j+1/2} = \frac{1}{\Delta x} MM\{\Delta w^0_{j+1}, \Delta w^0_j\}, \quad (4.45)$$

where $\Delta w_j = w_{j+1/2} - w_{j-1/2}$ and MM is the min-mod limiter. An exact evolution of $w_0(x)$ is now performed according to (4.34) where we now consider the h flux, i.e., we solve

$$w_t + h(w)_x = 0, \qquad w(x,0) = w_0(x), \qquad t \in (0, \Delta t]. \quad (4.46)$$

This gives us

$$\begin{aligned}
\bar{w}^1_j &= \overline{w}(x, \Delta t) \\
&= \overline{w}(x,0) - \frac{1}{\Delta x}\Big[\int_0^{\Delta t} h(w(x + \frac{\Delta x}{2}, \tau))\, d\tau - \int_0^{\Delta t} h(w(x - \frac{\Delta x}{2}, \tau))\, d\tau\Big] \\
&= \frac{1}{\Delta x}\Big[\int_{x_{j-1/2}}^{x_j} L_{j-1/2}(x)\, dx + \int_{x_j}^{x_{j+1/2}} L_{j+1/2}(x)\, dx\Big] \\
&\qquad - \frac{1}{\Delta x}\Big[\int_0^{\Delta t} h(w(x_{j+1/2}, \tau))\, d\tau - \int_0^{\Delta t} h(w(x_{j-1/2}, \tau))\, d\tau\Big].
\end{aligned} \quad (4.47)$$

The two first integrals on the RHS can be integrated exactly, i.e.

$$\begin{aligned}
&\frac{1}{\Delta x}\Big[\int_{x_{j-1/2}}^{x_j} L_{j-1/2}(x)\, dx + \int_{x_j}^{x_{j+1/2}} L_{j+1/2}(x)\, dx\Big] \\
&= \frac{1}{2}\Big[\bar{v}^1_{j-1/2} + \bar{v}^1_{j+1/2}\Big] + \frac{1}{8}\Big[w'_{j-1/2} - w'_{j+1/2}\Big] \\
&= \frac{1}{4}(v^0_{j-1} + 2v^0_j + v^0_{j+1}) - \frac{1}{16}(v'_{j+1} - v'_{j-1}) - \frac{\lambda}{2}\Big[g(v^{1/2}_{j+1}) - g(v^{1/2}_{j-1})\Big] - \frac{1}{8}\Big[w'_{j+1/2} - w'_{j-1/2}\Big]
\end{aligned} \quad (4.48)$$

Moreover, if the CFL condition $\lambda \cdot \max_{x_{j-1/2} \le x \le x_{j+1/2}} \rho(A_h(w(x,0))) < \frac{1}{2}$, is met (here A_h refers to the Jacobian of h), then the two last integrands on the RHS of (4.47), $h(w(x_{j-1/2}, \tau))$ and $h(w(x_{j+1/2}, \tau))$ are approximated as in Step 1. Thus we arrive at

$$\begin{aligned}
w_j(\Delta t) = \frac{1}{4}(v^0_{j-1} + 2v^0_j + v^0_{j+1}) &- \frac{1}{16}(v'_{j+1} - v'_{j-1}) \\
&- \frac{\lambda}{2}\Big[g(v^{1/2}_{j+1}) - g(v^{1/2}_{j-1})\Big] - \frac{1}{8}\Big[w'_{j+1/2} - w'_{j-1/2}\Big] \\
&- \lambda\Big[h(w(x_{j+1/2}, \frac{\Delta t}{2})) - h(w(x_{j-1/2}, \frac{\Delta t}{2}))\Big],
\end{aligned} \quad (4.49)$$

where $w'_{j+1/2}$ is given by (4.45). Now the pointwise values at the half-time steps are evaluated by the approximation

$$w^{1/2}_{j+1/2} = w(x_{j+1/2}, \frac{\Delta t}{2}) \approx w(x_{j+1/2}, 0) = w^0_{j+1/2}.$$

To sum up, we obtain a sequence of discrete values $\{\bar{w}_j^1\}$ by solving

$$v_j^{1/2} = v_j^0 - \frac{\lambda}{2} g_j', \tag{4.50}$$

$$w_{j+1/2}^{1/2} = w_{j+1/2}^0, \tag{4.51}$$

$$\begin{aligned} \bar{w}_j^1 = \frac{1}{4}(v_{j-1}^0 + 2v_j^0 + v_{j+1}^0) - \frac{1}{16}(v_{j+1}' - v_{j-1}') \\ - \frac{\lambda}{2}\Big[g(v_{j+1}^{1/2}) - g(v_{j-1}^{1/2})\Big] - \frac{1}{8}\Big[w_{j+1/2}' - w_{j-1/2}'\Big] \\ - \lambda\Big[h(w_{j+1/2}^{1/2}) - h(w_{j-1/2}^{1/2})\Big], \end{aligned} \tag{4.52}$$

where

$$w_{j+1/2}^0 = \frac{1}{2}\Big[v_j^0 + v_{j+1}^0\Big] + \frac{1}{8}\Big[v_j' - v_{j+1}'\Big] - \lambda\Big[g(v_{j+1}^{1/2}) - g(v_j^{1/2})\Big]. \tag{4.53}$$

The final step is to obtain the piecewise constant approximation to $u(x, t^{n+1})$ given by

$$\overline{u}(x, t^{n+1}) = \sum_j \bar{w}_j^1 \chi_j(x).$$

REMARK 11. *We observe the following:*

1. *The special choice where $v_j' = g_j' = 0$ and $w_{j+1/2}' = 0$, recovers the X-FORCE scheme given by* (4.12), (4.13), *and* (4.14).
2. *The special case where $g = f$ and $h = 0$ gives us the following scheme:*

$$v_j^{1/2} = v_j^0 - \frac{\lambda}{2} g_j', \tag{4.54}$$

$$\begin{aligned} \bar{w}_j^1 = \frac{1}{4}(v_{j-1}^0 + 2v_j^0 + v_{j+1}^0) - \frac{1}{16}(v_{j+1}' - v_{j-1}') \\ - \frac{\lambda}{2}\Big[f(v_{j+1}^{1/2}) - f(v_{j-1}^{1/2})\Big] - \frac{1}{8}\Big[w_{j+1/2}' - w_{j-1/2}'\Big]. \end{aligned} \tag{4.55}$$

This scheme corresponds exactly to the nonstaggered NT central scheme developed in [14]. It remains to be explored whether the good properties possessed by this scheme carry over to the second-order type X-FORCE class given by (4.50)– (4.53). *Some numerical experiments are provided in Section 6.*

5. X-FORCE schemes for two systems of conservation laws. The purpose of this section is to derive two concrete X-FORCE schemes for the isothermal Euler equations. One of them corresponds to the FORCE scheme of Toro, the other results from a splitting of the flux into a convective flux and a pressure flux. In particular, in light of Remark 4 in Section 2 we derive a semi-implicit variant of this second X-FORCE scheme. In Section 6 we demonstrate through some numerical experiments how these various X-FORCE schemes are inter-related. In this section we also describe how we, within the framework of Section 2, can reproduce the central schemes used in [6, 7, 8].

5.1. X-FORCE schemes for the isothermal Euler model. We now describe two different X-FORCE schemes in more detail for the isothermal Euler equations given by (1.2). An X-FORCE scheme takes the general form

$$u_j^{k+1} = u_j^k - \lambda\Big(G_{j+1/2}^k - G_{j-1/2}^k\Big) - \lambda\Big(H_{j+1/2}^{k+1} - H_{j-1/2}^{k+1}\Big). \tag{5.1}$$

Different choices of g and h such that $f = g + h$ where $f = (m, m^2/\rho + p(\rho))^T$ will then produce different X-FORCE schemes.

Scheme 1: FORCE. One choice is simply to consider $g = h = \frac{1}{2}f = \frac{1}{2}(m, m^2/\rho + p(\rho))^T$. In view of the two-step form given by (2.33), the G and H fluxes then take the form

$$G^k_{j+1/2} = \begin{pmatrix} G^k_{1,j+1/2} \\ G^k_{2,j+1/2} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} m^k_j + m^k_{j+1} \\ (m^2/\rho + p(\rho))^k_j + (m^2/\rho + p(\rho))^k_{j+1} \end{pmatrix} - \frac{1}{4\lambda}\begin{pmatrix} \rho^k_{j+1} - \rho^k_j \\ m^k_{j+1} - m^k_j \end{pmatrix} \tag{5.2}$$

and

$$H^{k+1}_{j+1/2} = \begin{pmatrix} H^{k+1}_{1,j+1/2} \\ H^{k+1}_{2,j+1/2} \end{pmatrix} = \frac{1}{2}f(\rho^{k+1}_{j+1/2}, m^{k+1}_{j+1/2}) = \frac{1}{2}\begin{pmatrix} m^{k+1}_{j+1/2} \\ (m^2/\rho)^{k+1}_{j+1/2} + p(\rho^{k+1}_{j+1/2}) \end{pmatrix}, \tag{5.3}$$

where

$$\begin{aligned} \rho^{k+1}_{j+1/2} &= \frac{1}{2}(\rho^k_j + \rho^k_{j+1}) - \frac{\lambda}{2}\Big(m^k_{j+1} - m^k_j\Big), \\ m^{k+1}_{j+1/2} &= \frac{1}{2}(m^k_j + m^k_{j+1}) - \frac{\lambda}{2}\Big((m^2/\rho + p(\rho))^k_{j+1} - (m^2/\rho + p(\rho))^k_j\Big). \end{aligned} \tag{5.4}$$

Remark 12. *The above FORCE scheme for the Euler model* (1.2) *has been analyzed in [4] for an isentropic pressure function $p(\rho)$, for instance, in the form $p(\rho) = K\rho^\gamma$ where $\gamma > 1$ (polytropic gas). By using the fact that FORCE can be interpreted through Riemann solutions, and using the solvability of the Riemann problems for any Riemann data with nonnegative density, an L^∞ bound as well as a bound on the numerical dissipation are found for ρ_Δ and m_Δ when these are the approximations generated by FORCE. Based on these estimates strong convergence to an entropy solution is established.*

Scheme 2: X-FORCE1 (explicit variant). Another choice is to consider $g = (m, m^2/\rho)^T$ and $h = (0, p(\rho))^T$. In view of the two-step form given by (2.33), the G and H fluxes then take the form

$$G^k_{j+1/2} = \begin{pmatrix} G^k_{1,j+1/2} \\ G^k_{2,j+1/2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} m^k_j + m^k_{j+1} \\ (m^2/\rho)^k_j + (m^2/\rho)^k_{j+1} \end{pmatrix} - \frac{1}{4\lambda}\begin{pmatrix} \rho^k_{j+1} - \rho^k_j \\ m^k_{j+1} - m^k_j \end{pmatrix} \tag{5.5}$$

and

$$H^{k+1}_{j+1/2} = \begin{pmatrix} H^{k+1}_{1,j+1/2} \\ H^{k+1}_{2,j+1/2} \end{pmatrix} = h(\rho^{k+1}_{j+1/2}, m^{k+1}_{j+1/2}) = \begin{pmatrix} 0 \\ p(\rho^{k+1}_{j+1/2}) \end{pmatrix}, \tag{5.6}$$

where

$$\rho^{k+1}_{j+1/2} = \frac{1}{2}(\rho^k_j + \rho^k_{j+1}) - \lambda\Big(m^k_{j+1} - m^k_j\Big). \tag{5.7}$$

Note that the alternative form of the $H^{k+1}_{j+1/2}$ flux, according to (2.29) and (2.30), where the coefficient $a(x,t)$ now refers to the Jacobian matrix $D_u h$

$$a = D_u h = \begin{pmatrix} 0 & 0 \\ p'(\rho) & 0 \end{pmatrix},$$

is given by

$$H^{k+1}_{j+1/2} = \begin{pmatrix} H^{k+1}_{1,j+1/2} \\ H^{k+1}_{2,j+1/2} \end{pmatrix} = \begin{pmatrix} 0 \\ P^{k+1}_{j+1/2} \end{pmatrix}, \tag{5.8}$$

where

$$P^{k+1}_{j+1/2} = \frac{1}{2}(p^k_j + p^k_{j+1}) - \kappa^k_{j+1/2}\lambda\Big(m^k_{j+1} - m^k_j\Big). \tag{5.9}$$

We follow (2.26) and use the average

$$\kappa^k_{j+1/2} = \frac{1}{2}(p'(\rho^k_j) + p'(\rho^k_{j+1})). \tag{5.10}$$

Scheme 3: X-FORCE2 (implicit variant). In view of Remark 4, we may replace $G^k_{j+1/2}$ given by (5.5) by an implicit variant $G^{k+1}_{j+1/2}$ of the form

$$G^{k+1}_{j+1/2} = \frac{1}{2}\begin{pmatrix} m^{k+1}_j + m^{k+1}_{j+1} \\ (m^2/\rho)^{k+1}_j + (m^2/\rho)^{k+1}_{j+1} \end{pmatrix} - \frac{1}{4\lambda}\begin{pmatrix} \rho^k_{j+1} - \rho^k_j \\ m^k_{j+1} - m^k_j \end{pmatrix}, \tag{5.11}$$

whereas $P^{k+1}_{j+1/2}$ given by (5.9) is replaced by the implicit variant

$$P^{k+1}_{j+1/2} = \frac{1}{2}(p^k_j + p^k_{j+1}) - \kappa^k_{j+1/2}\lambda\Big(m^{k+1}_{j+1} - m^{k+1}_j\Big). \tag{5.12}$$

In order to obtain a linearly implicit scheme which avoids nonlinear iterations, we propose to modify the $G^{k+1}_{j+1/2}$ component associated with the momentum flux such that it is treated on the kth time level, i.e., we consider

$$G^{k+1}_{j+1/2} = \frac{1}{2}\begin{pmatrix} m^{k+1}_j + m^{k+1}_{j+1} \\ (m^2/\rho)^{k}_j + (m^2/\rho)^{k}_{j+1} \end{pmatrix} - \frac{1}{4\lambda}\begin{pmatrix} \rho^k_{j+1} - \rho^k_j \\ m^k_{j+1} - m^k_j \end{pmatrix}. \tag{5.13}$$

The resulting X-FORCE scheme is then given in the form

- Mass equation

$$\rho^{k+1}_j = \rho^k_j - \lambda\Big(M^{k+1/2}_{j+1/2} - M^{k+1/2}_{j-1/2}\Big) \tag{5.14}$$

with

$$M^{k+1/2}_{j+1/2} = \frac{1}{2}(m^{k+1}_j + m^{k+1}_{j+1}) - \frac{1}{4\lambda}(\rho^k_{j+1} - \rho^k_j). \tag{5.15}$$

- Momentum equation

$$\begin{aligned} m^{k+1}_j + \lambda\Big(P^{k+1}_{j+1/2} - P^{k+1}_{j-1/2}\Big) &= m^k_j - \lambda\Big([Mv]^k_{j+1/2} - [Mv]^k_{j-1/2}\Big) \\ P^{k+1}_{j+1/2} + \kappa^k_{j+1/2}\lambda\Big(m^{k+1}_{j+1} - m^{k+1}_j\Big) &= \frac{1}{2}(p^k_j + p^k_{j+1}), \end{aligned} \tag{5.16}$$

with

$$[Mv]^k_{j+1/2} = \frac{1}{2}([m^2/\rho]^k_j + [m^2/\rho]^k_{j+1}) - \frac{1}{4\lambda}(m^k_{j+1} - m^k_j). \tag{5.17}$$

and $\kappa^k_{j+1/2}$ given by (5.10).

REMARK 13. *The above scheme is semi-implicit in the sense that we solve implicitly for the mass flux (momentum) m^{k+1} while we solve explicitly for the density ρ^{k+1} along the following two steps:*

- *First, we solve simultaneously for m^{k+1}_j and $P^{k+1}_{j+1/2}$ from the two equations given by* (5.16)*. This corresponds to solving a linear problem $Ax = b$ where the matrix A is tridiagonal.*
- *Second, we solve for ρ^{k+1}_j from* (5.14) *by making use of the updated momentum m^{k+1}_j obtained from the first step.*

In the next section we will do some simple numerical experiments whose purpose is to give some insight into basic accuracy and stability properties.

5.2. An X-FORCE scheme for the isothermal two-fluid model. We now describe in more detail an explicit and semi-implicit X-FORCE scheme for the 4-equations isothermal two-fluid model given by

$$\begin{aligned} \partial_t m_{\mathrm{k}} + \partial_x I_{\mathrm{k}} &= 0, & m_{\mathrm{k}} &= \alpha_{\mathrm{k}}\rho_{\mathrm{k}}, \qquad I_{\mathrm{k}} = m_{\mathrm{k}} v_{\mathrm{k}}, \\ \partial_t I_{\mathrm{k}} + \partial_x J_{\mathrm{k}} + \alpha_{\mathrm{k}}\partial_x p &= -(\Delta p)\partial_x \alpha_{\mathrm{k}}, & J_{\mathrm{k}} &= I_{\mathrm{k}} v_{\mathrm{k}}, \end{aligned} \tag{5.18}$$

where $\rho_{\mathrm{k}} = \rho_{\mathrm{k}}(p)$ is density, v_{k} fluid velocity, α_{k} volume fraction, and $p = p(m_{\mathrm{l}}, m_{\mathrm{g}})$ is pressure common to both phases, and the index $\mathrm{k} = \mathrm{l}, \mathrm{g}$ represents the two phases. Moreover, Δp is a given function which depends on the primitive variables and its purpose is to ensure that the model becomes hyperbolic (real eigenvalues). For more details regarding the structure of the two-fluid model we refer to [9] and references therein. Here it suffices to observe that for α_k we have the relation

$$\alpha_{\mathrm{g}} + \alpha_{\mathrm{l}} = 1. \tag{5.19}$$

This relation can be rewritten as

$$\frac{m_{\mathrm{g}}}{\rho_{\mathrm{g}}(p)} + \frac{m_{\mathrm{l}}}{\rho_{\mathrm{l}}(p)} = 1, \qquad m_{\mathrm{k}} = \alpha_{\mathrm{k}}\rho_{\mathrm{k}}, \tag{5.20}$$

from which we obtain a relation yielding the pressure $p = p(m_{\mathrm{l}}, m_{\mathrm{g}})$ as a nonlinear function which depends on the masses m_{k}. By using the relation (5.20), we can derive the differential

$$dp = \kappa(\rho_{\mathrm{l}} dm_{\mathrm{g}} + \rho_{\mathrm{g}} dm_{\mathrm{l}}) \tag{5.21}$$

where

$$\kappa = \frac{1}{\frac{\partial \rho_{\mathrm{l}}}{\partial p}\alpha_{\mathrm{l}}\rho_{\mathrm{g}} + \frac{\partial \rho_{\mathrm{g}}}{\partial p}\alpha_{\mathrm{g}}\rho_{\mathrm{l}}}. \tag{5.22}$$

The purpose of this section is to demonstrate, within the general framework of Section 2, how we can reproduce the central-type scheme which was used in the previous works [6, 7, 8]. First, in view of section 5.1, we propose to associate the convective terms with the g component, whereas the nonlinear pressure function is associated with the h-component. Hence, we may write the model in the form

$$\partial_t u + \partial_x g(u) + \alpha(u)\partial_x h(u) = q(u), \tag{5.23}$$

where $u = (m_{\mathrm{g}}, m_{\mathrm{g}}, I_{\mathrm{g}}, I_{\mathrm{l}})^T$, $g = (I_{\mathrm{g}}, I_{\mathrm{l}}, J_{\mathrm{g}}, J_{\mathrm{l}})^T$, $\alpha = (0, 0, \alpha_{\mathrm{g}}, \alpha_{\mathrm{l}})$, $h = (0, 0, p, p)^T$, and $q = -(\Delta p)(0, 0, \partial_x \alpha_{\mathrm{g}}, \partial_x \alpha_{\mathrm{l}})^T$. The q term, which contains spatial derivatives, is here handled as a source term since the impact is weak and its main purpose is to ensure that the model becomes hyperbolic, not to represent any real physical effect.

We note that the model (5.23) is slightly different from the original model equation (1.1) due to the occurrence of the non-conservative term $\alpha(u)\partial_x h(u)$. However, the framework described in Section 2.2 naturally allows us to treat this case. In particular, the extended model corresponding to (2.18) now is in the form

$$\begin{aligned} v_t + g(v+w)_x &= 0, & v(\cdot, 0) &= u^k(\cdot), \\ w_t + \alpha(v+w)\widetilde{h}_x &= 0, & w(\cdot, 0) &= 0, \\ \widetilde{h}_t + a(x,t)g(v+w)_x &= 0, & \widetilde{h}(\cdot, 0) &= h(u^k)(\cdot), \end{aligned} \tag{5.24}$$

where $a(x,t)$ represents the Jacobian $D_u h$

$$a = D_u h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \kappa\rho_{\mathrm{l}} & \kappa\rho_{\mathrm{g}} & 0 & 0 \\ \kappa\rho_{\mathrm{l}} & \kappa\rho_{\mathrm{g}} & 0 & 0 \end{pmatrix}, \tag{5.25}$$

where κ is given by (5.22) by using (5.21). Thus, the last set of equations of (5.24) corresponds to the *pressure evolution equation*

$$\frac{\partial p}{\partial t} + \kappa\Big(\rho_{\rm l}\frac{\partial}{\partial x}(\alpha_{\rm g}\rho_{\rm g}v_{\rm g}) + \rho_{\rm g}\frac{\partial}{\partial x}(\alpha_{\rm l}\rho_{\rm l}v_{\rm l})\Big) = 0. \tag{5.26}$$

In light of (2.29), we see that the X-FORCE scheme takes the general form (here we have neglected the "source term" q appearing on the right hand side of (5.23))

$$u_j^{k+1} = u_j^k - \lambda\Big(G_{j+1/2}^k - G_{j-1/2}^k\Big) - \lambda\alpha_j^k\Big(H_{j+1/2}^{k+1} - H_{j-1/2}^{k+1}\Big), \qquad \alpha_j^k = \alpha(u_j^k), \tag{5.27}$$

where $G_{j+1/2}^k$ and $H_{j+1/2}^{k+1}$ are given by (2.30) when we apply a forward Euler discretization of the semi-discrete system corresponding to (2.22) and (2.23).

An explicit X-FORCE scheme. To sum up, the resulting numerical fluxes $G_{j+1/2}^k$ and $H_{j+1/2}^{k+1}$ are then given by

$$G_{j+1/2}^k = \frac{1}{2}\begin{pmatrix} I_{{\rm g},j}^k + I_{{\rm g},j+1}^k \\ I_{{\rm l},j}^k + I_{{\rm l},j+1}^k \\ J_{{\rm g},j}^k + J_{{\rm g},j+1}^k \\ J_{{\rm l},j}^k + J_{{\rm l},j+1}^k \end{pmatrix} - \frac{1}{4\lambda}\begin{pmatrix} m_{{\rm g},j+1}^k - m_{{\rm g},j}^k \\ m_{{\rm l},j+1}^k - m_{{\rm l},j}^k \\ I_{{\rm g},j+1}^k - I_{{\rm g},j}^k \\ I_{{\rm l},j+1}^k - I_{{\rm l},j}^k \end{pmatrix}, \tag{5.28}$$

and

$$H_{j+1/2}^{k+1} = \begin{pmatrix} 0 \\ 0 \\ P_{j+1/2}^{k+1} \\ P_{j+1/2}^{k+1} \end{pmatrix}, \tag{5.29}$$

where

$$P_{j+1/2}^{k+1} = \frac{1}{2}(p_j^k + p_{j+1}^k) - \lambda[\kappa\rho_{\rm l}]_{j+1/2}^k(I_{{\rm g},j+1}^k - I_{{\rm g},j}^k) - \lambda[\kappa\rho_{\rm g}]_{j+1/2}^k(I_{{\rm l},j+1}^k - I_{{\rm l},j}^k). \tag{5.30}$$

We may follow (2.26) and define $[\kappa\rho_{\rm g}]_{j+1/2}^k$ as the average

$$[\kappa\rho_{\rm g}]_{j+1/2}^k = \frac{1}{2}([\kappa\rho_{\rm g}]_j^k + [\kappa\rho_{\rm g}]_{j+1}^k), \tag{5.31}$$

where κ now is given by (5.22). A similar definition can used for $[\kappa\rho_{\rm l}]_{j+1/2}^k$.

An implicit X-FORCE scheme. We closely follow the strategy we used for the isothermal Euler model in the previous subsection and now propose a semi-implicit variant of the above two-fluid explicit X-FORCE scheme. More precisely,

- we consider a backward Euler discretization of the semi-discrete system corresponding to (2.22) and (2.23),
- in order to obtain a linearly implicit scheme, we introduce a linearization of the convective terms of the momentum equations by considering

$$J_{{\rm g},j}^{k+1} \approx I_{{\rm g},j}^{k+1}v_{{\rm g},j}^k = J_{{\rm g},j}^{k+1/2}, \qquad J_{{\rm l},j}^{k+1} \approx I_{{\rm l},j}^{k+1}v_{{\rm l},j}^k = J_{{\rm l},j}^{k+1/2}. \tag{5.32}$$

This results in the following numerical fluxes.

$$G_{j+1/2}^{k+1/2} = \frac{1}{2}\begin{pmatrix} I_{{\rm g},j}^{k+1} + I_{{\rm g},j+1}^{k+1} \\ I_{{\rm l},j}^{k+1} + I_{{\rm l},j+1}^{k+1} \\ J_{{\rm g},j}^{k+1/2} + J_{{\rm g},j+1}^{k+1/2} \\ J_{{\rm l},j}^{k+1/2} + J_{{\rm l},j+1}^{k+1/2} \end{pmatrix} - \frac{1}{4\lambda}\begin{pmatrix} m_{{\rm g},j+1}^k - m_{{\rm g},j}^k \\ m_{{\rm l},j+1}^k - m_{{\rm l},j}^k \\ I_{{\rm g},j+1}^k - I_{{\rm g},j}^k \\ I_{{\rm l},j+1}^k - I_{{\rm l},j}^k \end{pmatrix}, \tag{5.33}$$

and

$$P^{k+1}_{j+1/2} + \lambda[\kappa\rho_{\rm l}]^k_{j+1/2}(I^{k+1}_{{\rm g},j+1} - I^{k+1}_{{\rm g},j}) + \lambda[\kappa\rho_{\rm g}]^k_{j+1/2}(I^{k+1}_{{\rm l},j+1} - I^{k+1}_{{\rm l},j}) = \frac{1}{2}(p^k_j + p^k_{j+1}). \tag{5.34}$$

The corresponding scheme then takes the form

- Mass equations

$$\begin{aligned} m^{k+1}_{{\rm g},j} &= m^k_{{\rm g},j} - \lambda\Big(I^{k+1/2}_{{\rm g},j+1/2} - I^{k+1/2}_{{\rm g},j-1/2}\Big), \\ m^{k+1}_{{\rm l},j} &= m^k_{{\rm l},j} - \lambda\Big(I^{k+1/2}_{{\rm l},j+1/2} - I^{k+1/2}_{{\rm l},j-1/2}\Big) \end{aligned} \tag{5.35}$$

  with

$$\begin{aligned} I^{k+1/2}_{{\rm g},j+1/2} &= \frac{1}{2}(I^{k+1}_{{\rm g},j} + I^{k+1}_{{\rm g},j+1}) - \frac{1}{4\lambda}(m^k_{{\rm g},j+1} - m^k_{{\rm g},j}), \\ I^{k+1/2}_{{\rm l},j+1/2} &= \frac{1}{2}(I^{k+1}_{{\rm l},j} + I^{k+1}_{{\rm l},j+1}) - \frac{1}{4\lambda}(m^k_{{\rm l},j+1} - m^k_{{\rm l},j}). \end{aligned} \tag{5.36}$$

- Momentum equations

$$\begin{aligned} I^{k+1}_{{\rm g},j} + \lambda\alpha^k_{{\rm g},j}\Big(P^{k+1}_{j+1/2} - P^{k+1}_{j-1/2}\Big) + \lambda\Big(J^{k+1/2}_{{\rm g},j+1/2} - J^{k+1/2}_{{\rm g},j-1/2}\Big) &= I^k_{{\rm g},j} + \Delta t q^k_{{\rm g},j} \\ I^{k+1}_{{\rm l},j} + \lambda\alpha^k_{{\rm l},j}\Big(P^{k+1}_{j+1/2} - P^{k+1}_{j-1/2}\Big) + \lambda\Big(J^{k+1/2}_{{\rm l},j+1/2} - J^{k+1/2}_{{\rm l},j-1/2}\Big) &= I^k_{{\rm l},j} + \Delta t q^k_{{\rm l},j} \\ P^{k+1}_{j+1/2} + [\kappa\rho_{\rm l}]^k_{j+1/2}\lambda\Big(I^{k+1}_{{\rm g},j+1} - I^{k+1}_{{\rm g},j}\Big) + [\kappa\rho_{\rm g}]^k_{j+1/2}\lambda\Big(I^{k+1}_{{\rm l},j+1} - I^{k+1}_{{\rm l},j}\Big) & \\ = \frac{1}{2}(p^k_j + p^k_{j+1}), & \end{aligned} \tag{5.37}$$

  with

$$\begin{aligned} J^{k+1/2}_{{\rm g},j+1/2} &= \frac{1}{2}(J^{k+1/2}_{{\rm g},j} + J^{k+1/2}_{{\rm g},j+1}) - \frac{1}{4\lambda}(I^k_{{\rm g},j+1} - I^k_{{\rm g},j}), \\ J^{k+1/2}_{{\rm l},j+1/2} &= \frac{1}{2}(J^{k+1/2}_{{\rm l},j} + J^{k+1/2}_{{\rm l},j+1}) - \frac{1}{4\lambda}(I^k_{{\rm l},j+1} - I^k_{{\rm l},j}), \end{aligned} \tag{5.38}$$

where $J^{k+1/2}_{{\rm g},j}$ and $J^{k+1/2}_{{\rm l},j}$ are given by (5.32).

Remark 14. *The above scheme is semi-implicit in the sense that we solve implicitly for the mass flux (momentum) $I^{k+1}_{\rm g}$ and $I^{k+1}_{\rm l}$ while we solve explicitly for the masses $m^{k+1}_{\rm g}$ and $m^{k+1}_{\rm l}$ along the following two steps:*

- *First, we solve simultaneously for $I^{k+1}_{{\rm g},j}$, $I^{k+1}_{{\rm l},j}$, and $P^{k+1}_{j+1/2}$ from the three equations given by* (5.37)*. This corresponds to solving a linear problem $Ax = b$ where the matrix A possesses a band structure.*
- *Second, we solve for $m^{k+1}_{{\rm g},j}$ and $m^{k+1}_{{\rm l},j}$ from* (5.35) *by making use of the updated momentum $I^{k+1}_{{\rm g},j}$ and $I^{k+1}_{{\rm l},j}$ obtained from the first step.*

Explicit and semi-implicit X-FORCE schemes similar to those described above have been explored extensively, respectively, in [7] and [6, 8]. In particular, it is demonstrated that the semi-implicit X-FORCE scheme possesses a weak CFL condition which relates the CFL number to the maximal fluid velocity and not the speed of the sonic waves. Another feature of the X-FORCE schemes presented in these works is that an upwind component is incorporated in the convective fluxes. As a consequence, they are able to give an accurate resolution of the important mass waves comparable to that of a Roe scheme.

6. Some numerical experiments. First we study basic properties of second-order X-FORCE schemes for nonlinear scalar conservation laws. Then we demonstrate characteristic behavior of the three X-FORCE schemes described in Section 5.1 for the isothermal Euler model.

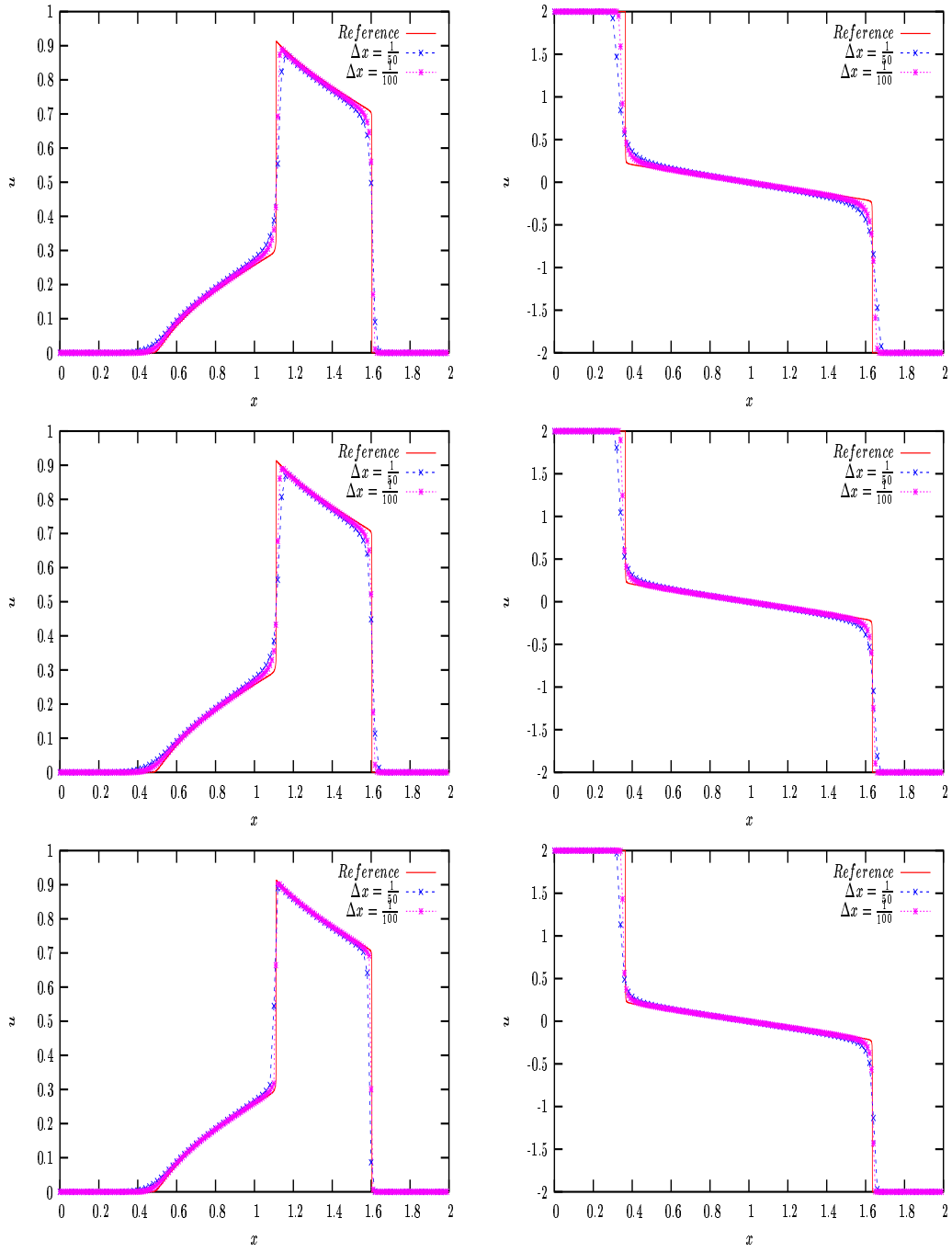

FIG. 6.1. **Left column:** *from top to bottom: XF1, XF2, and XF3 with* f_a *flux.* **Right column:** *from top to bottom: XF1, XF2, and XF3 with* f_b *flux.*

6.1. Scalar conservation law. The purpose of this test is to explore the behavior of three different X-FORCE schemes for a nonlinear scalar conservation law $u_t + f(u)_x = 0$. We consider second-order type of schemes as described in Section 4.4. In particular, we want to check whether the non-oscillatory behavior ensured by Corollary 4.2 and 4.8 for the first-order X-FORCE class

seems to carry over to the second-order X-FORCE class. First we consider an example with

$$f_a(u) = \frac{u^2}{u^2 + (1-u)^2}. \tag{6.1}$$

Then an example with

$$f_b(u) = \frac{(u^2-1)(u^2-4)}{4}. \tag{6.2}$$

For both cases we consider three X-FORCE schemes corresponding to the following three different splittings:

$$\begin{aligned}
&\text{XF1:} \quad g(u) = \begin{cases} u & \text{for } f_a, \\ \frac{1}{2}(2-|u|) & \text{for } f_b. \end{cases}, \qquad h(u) = f(u) - g(u), \\
&\text{XF2:} \quad g(u) = f(u), \qquad h(u) = 0, \\
&\text{XF3:} \quad g(u) = h(u) = \frac{1}{2} f(u).
\end{aligned}$$

Note that XF2 is nothing but the nonstaggered NT scheme presented in [14]. XF3 represents a second-order FORCE scheme. In particular, this second order FORCE flux represents an alternative to the second order FORCE scheme presented in [2]. Finally, XF-1 is an interesting candidate since the second-order interpolation is applied only to a linear part of the nonlinear flux $f(u)$. As initial data we use

$$u_a^0(x) = \begin{cases} 1 & 0.5 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}, \qquad u_b^0(x) = \begin{cases} 2 & x < 1 \\ -2 & \text{otherwise,} \end{cases}$$

and solve on the domain $[0, 2]$. We note that $\max_{u\in[0,1]} |f_a'(u)| = 2$ and $\max_{u\in[-2,+2]} |f_b'(u)| = 3$. For the case with the f_a flux we have used the following CFL numbers

$$\text{CFL}_{\text{XF1}} = 0.5, \qquad \text{CFL}_{\text{XF2}} = 0.5, \qquad \text{CFL}_{\text{XF3}} = 0.95.$$

The choice of CFL numbers have been guided by the stability conditions as described in Section 4 for the respective first-order schemes. We have computed solutions for two different grid sizes $\Delta x = 1/50$ and $\Delta x = 1/100$ where we have used $\theta = 1$ in (4.36) and (4.41). Results after a time $T = 0.5$ are shown in the left column of Fig. 6.1. Reference solutions have been generated by applying the FORCE scheme on a very fine grid.

Similarly, for the case with the f_b flux we have used

$$\text{CFL}_{\text{XF1}} = 0.25, \qquad \text{CFL}_{\text{XF2}} = 0.5, \qquad \text{CFL}_{\text{XF3}} = 0.75.$$

Solutions are shown in the right column of Fig. 6.1 at time $T = 1.2$. Particularly, the numerical results indicate that the second order X-FORCE class, that might be seen as a generalization of the second order non-staggered NT scheme presented in [14], is TVD. More precisely, the following observations can be made from the above numerical examples:

(i) We can apply the interpolation technique used in [14] to only a linear part $g(u)$ of the flux $f(u)$ (XF1) and obtain results similar to those produced by the non-staggered NT scheme (XF2).

(ii) The second-order FORCE scheme (XF3), which applies linear interpolants to the flux $f(u)$ similar to the non-staggered NT scheme (XF2), clearly seems to be TVD under a weaker CFL condition than the non-staggered NT scheme.

Thus, the above simulations indicate that it can be potentially fruitful to explore various candidates in the X-FORCE class. In particular, for systems there should be room for exploiting this freedom to explore interesting X-FORCE schemes besides the non-staggered NT scheme and the higher order FORCE scheme. However, more investigations are necessary before any decisive conclusion can be drawn.

6.2. Isothermal Euler model. Now we want to pay some attention to the relation between explicit and implicit time discretization. For that purpose we test the different first order X-FORCE schemes corresponding to **scheme 1** (explicit), **scheme 2** (explicit), and **scheme 3** (implicit) of Section 5.1. More precisely, we consider the first order schemes for the model (1.2) together with the pressure law

$$p(\rho) = K\rho^\gamma, \qquad \gamma > 1. \tag{6.3}$$

The sound velocity σ is given by

$$\sigma^2 = \frac{\partial p}{\partial \rho} = K\gamma\rho^{\gamma-1}, \qquad \gamma > 1.$$

We note that the model possesses two characteristic sonic wave velocities given by

$$\lambda = u \pm \sigma.$$

We use $\gamma = 1.4$ and $K = 1$ for the numerical calculations. We consider the Riemann problem with $(\rho_L, u_L) = (1, 0)$ and $(\rho_R, u_R) = (2, 0)$.

In order to shed more light on the convergence properties of the proposed X-FORCE schemes we have also included another well known central scheme, the relaxed scheme proposed by Jin and Xin [15], in our example. We also have used the second-order relaxed scheme to calculate reference solutions. The relaxed scheme has been tested extensively for various conservation laws. For the relaxed scheme it is known (we refer to [15] for details) that the so-called subcharacteristic condition given by

$$A - F'(v)^2 \geq 0, \qquad \text{for all } v, \tag{6.4}$$

must be satisfied for a diagonal matrix $A = \text{diag}\{a_{11}, a_{22}\}$ and the Jacobian F' associated with the conservation law. The matrix A determines the numerical dissipation associated with the relaxed scheme. In many cases we can choose that A has the simpler form

$$A = aI, \qquad a > 0.$$

In this case (and if the model is one-dimensional), the condition (6.4) is satisfied if

$$\max\{|u - \sigma|, |u + \sigma|\} < \sqrt{a}.$$

For the above Riemann problem we apply the following rough estimate

$$\sigma^2 = \gamma\rho^{\gamma-1} \leq 1.4 \max(\rho)^{0.4} = 1.4 \cdot 2^{0.4} \approx 1.85.$$

Consequently, we may estimate as follows (assuming $|u| \leq 0.5$)

$$\max(|u - \sigma|, |u + \sigma|) \leq 1.4 + 0.5 < 2,$$

we may choose $a = 4$. We define the CFL number as

$$\text{CFL} = \sqrt{a}\frac{\Delta t}{\Delta x} = 2\frac{\Delta t}{\Delta x}, \tag{6.5}$$

which relates the time step Δt and the spatial discretization parameter Δx to the maximal eigenvalue.

Test 1. We compare **scheme 1** (the FORCE scheme given by (5.2)–(5.4)), **scheme 2** (the X-FORCE scheme given by (5.5)–(5.7)), and the first order relaxed scheme of Jin. We compute solutions after time $T = 1.5$ on a grid of 100 cells. Results for two different CFL numbers, CFL=0.5 and CFL=1.0, are shown, respectively, in Fig. 7.1 and Fig. 7.2. The CFL number is inserted in the relation (6.5), from which the corresponding time step is determined.

We observe that the performance of the various central schemes is very similar with respect to accuracy. However, for CFL=0.5 the X-FORCE scheme gives the best resolution. This difference is clearly reduced by increasing the time step Δt, that is, setting CFL=1 in (6.5).

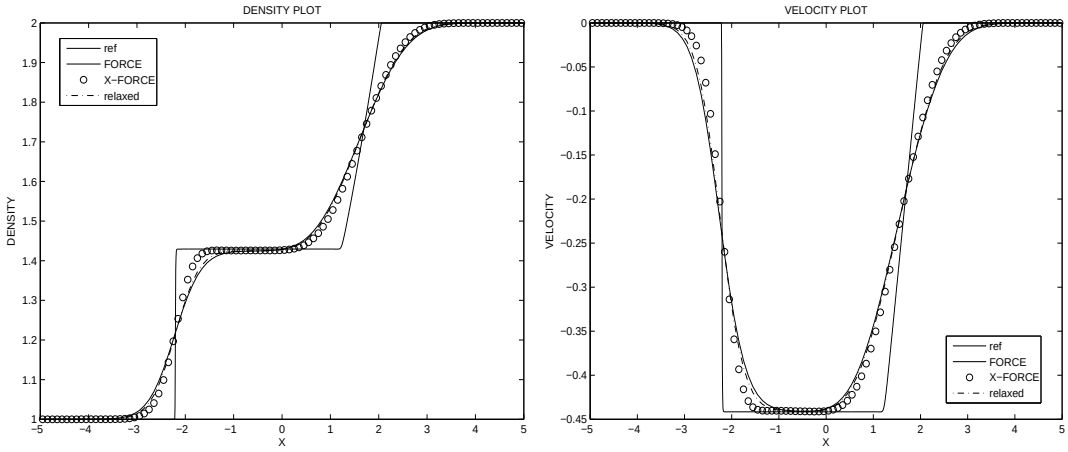

FIG. 7.1. *We consider the performance of the first order FORCE, X-FORCE, and relaxed scheme with CFL=0.5 and* $N = 100$ *grid cells.* $T = 1.5$.

Test 2. In Fig. 7.3 we consider the implicit variant of the X-FORCE scheme (**scheme 3**) defined by (5.14)–(5.17). We have shown results produced by this scheme for a grid of $N = 100$ cells. The behavior is close to that of the explicit variant for CFL numbers of the same order. As expected, higher CFL numbers introduce a smearing effect.

Finally, we want to check if solutions remain stable when we use very high CFL numbers. For that purpose we consider a finer grid of $N = 2000$ cells so that Δx in (6.5) becomes small. This allows us to still compute solutions at time $T = 1.5$ at the same time as we use high CFL numbers. Results are shown in Fig. 7.4 for CFL$= 10, 20, 40, 80$ and we observe that the implicit X-FORCE scheme remains stable. In fact, apparently the implicit scheme is stable for any CFL number, however at the cost of introducing a stronger smearing of the fronts.

7. Some concluding remarks. In [6]–[8] we proposed to use a centred type three-point scheme as a building block in the construction of some central-upwind type schemes for an isothermal two-fluid model commonly used within the petroleum industry. A main ingredient in this construction was the splitting of the flux into a convective term and a pressure term. The convective term was discretized by using the modified Lax-Friedrichs scheme whereas an appropriate numerical flux for the pressure term was obtained by developing a pressure evolution equation.

The purpose of this work has been to see how some of these ideas can be taken over to a *general* nonlinear system of conservation laws. We have implemented these ideas in a systematic and consistent manner by approximating the original model by an extended model obtained via (i) a decomposition of the original model into two sub-systems corresponding to a general splitting $f = g + h$ of the flux f, (ii) development of flux evolution equations for the h flux.

In particular, we have observed that a consistent discretization of this extended model gives rise to a class of centred type schemes which contains the FORCE scheme [20] as a special case. Thus, we have denoted the class as extended FORCE (X-FORCE). A higher order X-FORCE class is proposed by following along the line of [14]. We have verified that the X-FORCE schemes for scalar conservation laws are TVD, monotone, and entropy consistent under suitable CFL conditions. We have also illustrated how we naturally can construct an implicit X-FORCE scheme for the isothermal Euler equations and an isothermal 4-equations two-fluid model. These examples demonstrate potential merits in replacing the flux splittings that reproduce the FORCE scheme or the non-staggered NT scheme with splittings more naturally tailored to the structure of the model under consideration.

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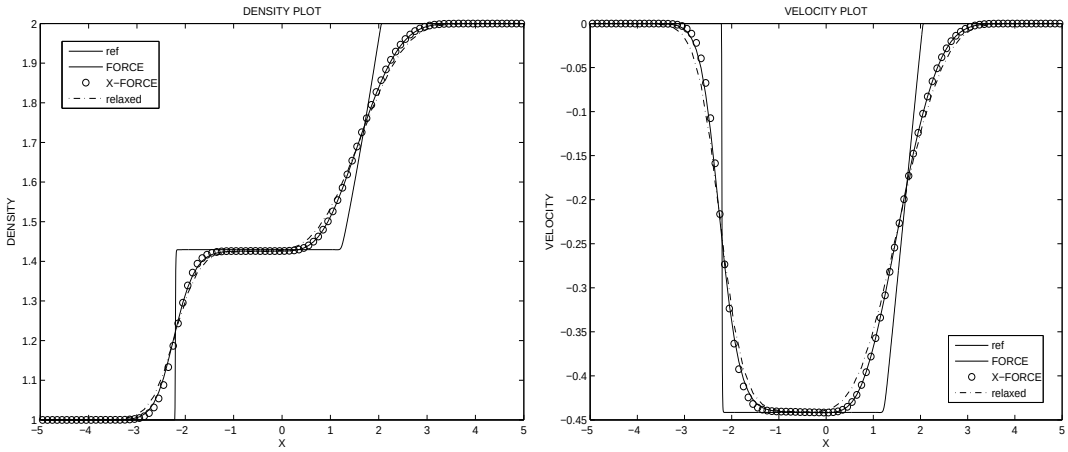

FIG. 7.2. *We consider the performance of the first order FORCE, X-FORCE, and relaxed scheme with CFL=1.0 and* $N = 100$ *grid cells.* $T = 1.5$.

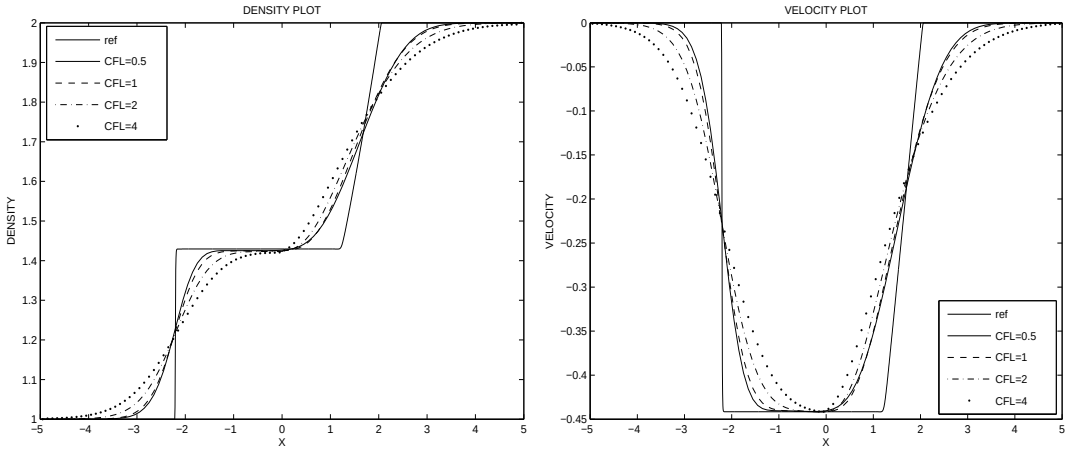

FIG. 7.3. *We consider the performance of the implicit X-FORCE scheme for different CFL numbers with* $N = 100$ *grid cells.* $T = 1.5$.

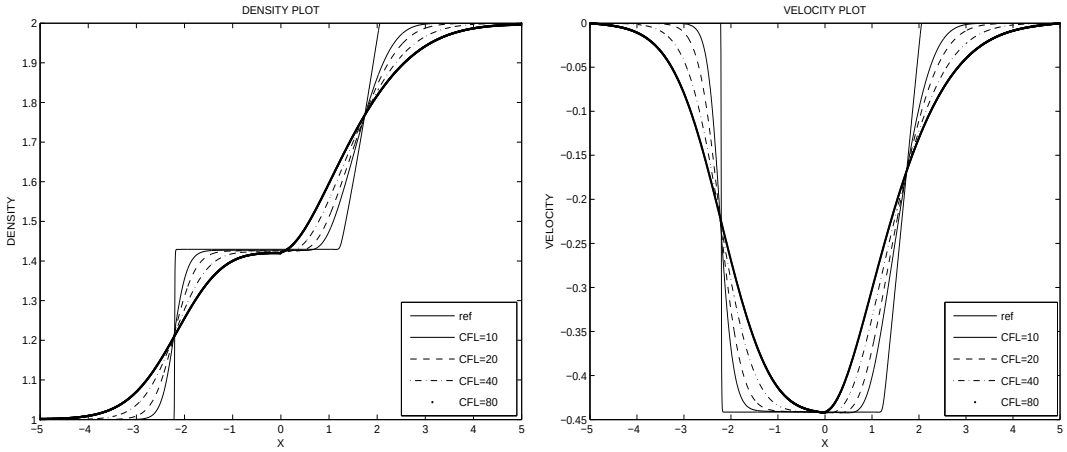

FIG. 7.4. *We consider the performance of the implicit X-FORCE scheme for different large CFL numbers with* $N = 2000$ *grid cells.* $T = 1.5$.

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