WEAK SOLUTIONS FOR A GAS-LIQUID MODEL RELEVANT FOR DESCRIBING GAS-KICK IN OIL WELLS

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Abstract. The purpose of this paper is to establish a local in time existence result for a compressible gas-liquid model. The model is a drift-flux model which is composed of two continuity equations and one mixture momentum equation supplemented with a slip relation in order to take into account the possibility of flows with unequal fluid velocities. The model is highly relevant for modeling of gas kick for oil wells, which in its worst case can lead to blowout scenarios. The mathematical study of such kind of models is important for the development of simulation tools that can be employed for increased control of deep-water well operations.

The liquid phase is assumed to be incompressible whereas the gas is described by a polytropic equation of state. The model is studied in a framework previously used for investigations of the single-phase compressible Navier-Stokes model. New challenges arise due to the appearance of a *generalized* pressure term that depends on fluid masses as well as gas velocity. The local existence result is obtained by introducing a suitable transformation along the line of the works [8, 9] in a free boundary setting. This allows us to obtain sufficient pointwise control of the gas and liquid masses. The estimates are rather delicate as they must be fine enough to control a possible singular behavior associated with the pressure law as well as the slip relation. The existence result is obtained under the assumption of a sufficient small time interval combined with suitable assumptions on the regularity of the initial data, the parameters that control, respectively, the behavior of the initial masses at the boundaries of the flow domain and the decay properties of the viscosity term.

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1. Introduction

The starting point for the investigations of this work is a one-dimensional two-phase model of the drift-flux type. This model is frequently used to simulate unsteady, compressible flow of liquid and gas in pipes and wells [1, 2, 4, 6, 15, 18, 21, 26]. The model consists of two mass conservation equations corresponding to each of the two phases gas (g) and liquid (l) and one equation for the conservation of the momentum of the mixture and is given in the following form:

$$\begin{aligned} \partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] &= 0 \\ \partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] &= 0 \\ \partial_t[\alpha_g\rho_g u_g + \alpha_l\rho_l u_l] + \partial_x[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + p] &= -q + \partial_x[\varepsilon\partial_x u_{mix}], \quad u_{mix} = \alpha_g u_g + \alpha_l u_l, \end{aligned} \tag{1}$$

where $\varepsilon \geq 0$. The model is supposed under isothermal conditions. The unknowns are: ρ_l, ρ_g the liquid and gas densities; α_l, α_g volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$; u_l, u_g fluid velocities of liquid and gas; p common pressure for liquid and gas; and q representing external forces like gravity and friction. Since the momentum is given only for the mixture, we need an additional closure law, a so-called hydrodynamical closure law, which connects the two phase velocities. More generally, this law should be able to take into account the different flow regimes. In addition, we need a thermodynamical equilibrium model which specifies the fluid properties.

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More details will be given in Section 2. We refer also to [5, 6, 11, 18, 20, 21, 22, 26] for various numerical schemes which have been developed for the study of the drift-flux model.

Application. Various gas kick simulators have been developed for the purpose of studying well control aspects during exploratory and development drilling subject to high pressure and temperature bottomhole conditions. Precise predictions of wellbore pressures, liquid/gas volumes as well as flow rates at the top of the well represent central issues. Clearly, the possibility of blowout occurrences needs to be mitigated in order to avoid human casualties, financial losses (production stop and equipment losses), and finally but not least, environmental damage. We refer to [1] and references therein for more information pertaining to this subject. In particular, in [1] the simulations are based on the drift-flux model (1) equipped with density-pressure relations similar to those used in the present work as well as a slip law that is based on the formulation (32). Development of accurate and robust discretization techniques for solving the system (1) is naturally related to a good understanding of its mathematical features (long-time behavior, estimates of various quantities, compactness, etc.). In particular, it is clearly of interest to obtain existence, stability, and uniqueness result of various versions of the model (1).

Previous results. Few such results seem to exist for two-phase gas-liquid models of the form (1). In [8, 9] we studied a simplified version obtained by assuming that fluid velocities are equal, $u_g = u_l = u$, and by neglecting the external forces, i.e., $q = 0$. In addition, we neglected certain gas effects by considering a simplified momentum equation where acceleration terms depend solely on the liquid phase. This is motivated by the fact that liquid phase density typically is much higher than gas phase density. Consequently, we considered a model in the form

$$\begin{aligned} \partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u] &= 0 \\ \partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u] &= 0 \\ \partial_t[\alpha_l\rho_l u] + \partial_x[\alpha_l\rho_l u^2] + \partial_x p &= \partial_x[\varepsilon\partial_x u], \qquad p,\varepsilon \geq 0. \end{aligned} \tag{2}$$

Assuming a polytropic gas law relation $p = C\rho_g^\gamma$ with $\gamma > 1$ for the gas phase whereas the liquid phase is treated as an incompressible fluid, i.e., $\rho_l = \text{Const}$, we get a pressure law of the form

$$p(n,m) = C\Big(\frac{n}{\rho_l - m}\Big)^\gamma, \tag{3}$$

where we use the notation $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$. In particular, we see that there is a possibly singular behavior associated with pressure at transition to pure liquid phase, i.e., $\alpha_l = 1$, which yields $m = \rho_l$ and $n = 0$. In addition, we have the possibility for vacuum as in the single-phase gas model, i.e., that $\rho_g = 0$ which implies that $n = 0$ and $p = 0$. Different forms for the viscosity function ε have been considered. In [8] we used

$$\varepsilon = \varepsilon(m) = \frac{m^\beta}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0, 1/3), \tag{4}$$

whereas in [9] we considered

$$\varepsilon = \varepsilon(n,m) = \frac{n^\beta}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0, 1/3). \tag{5}$$

More recently, Yao and Zhu [29] also studied the model (2) in a flow regime where the viscosity coefficient $\varepsilon > 0$ was assumed to take the form (4). They gave a proof of the global existence and uniqueness of weak solutions when β is in $(0, 1]$ and thereby improved the result of [8]. They also gave an interesting asymptotic behavior result, and obtained the regularity of the solutions by the energy method. The same authors also presented results for the same gas-liquid model (but constant viscosity term) when the masses m, n connected continuously to a vacuum state $m = n = 0$ [30]. In a recent work we have also studied the model (2) where relevant friction and gravity terms have been included [10]. We also note that the model (2), where both fluids were assumed to be compressible and with a constant viscosity coefficient ε, was studied in [7]. A global existence result was obtained for a class of weak solutions for rather general initial data.

Why using a viscosity term that depends on volume fraction and fluid densities? Viscosity μ_m for a gas-liquid mixture may not be a well-defined quantity just in terms of fluid fractions and single phase viscosities. The mixture viscosity in fact depends strongly on dynamical processes, including bubble size, flow regime etc. Hence, motivated by lab experiments different examples of a viscosity term μ_m, where the gas-liquid mixture is considered as a single-phase fluid, have been proposed. Some of them are, see for example [27] and references therein:

$$\mu_m = \mu_l, \qquad \text{(Owen's model)}, \tag{6}$$

$$\frac{1}{\mu_m} = \frac{y}{\mu_g} + \frac{1-y}{\mu_l}, \qquad \text{(McAdams et al.'s model)}, \tag{7}$$

$$\mu_m = \frac{\mu_l \mu_g}{\mu_g + y^{1/4}(\mu_l - \mu_g)}, \qquad \text{(Lin et al's model)}, \tag{8}$$

$$\mu_m = y\mu_g + (1-y)\mu_l, \qquad \text{(Cicchitti et al.'s model)}, \tag{9}$$

$$\mu_m = \alpha_g \mu_g + \alpha_l \mu_l, \qquad \text{(Dukler et al's model)}, \tag{10}$$

$$\mu_m = \alpha_g \mu_g + \alpha_l (1 + 2.5\alpha_g)\mu_l, \qquad \text{(Beattie and Whalley's model)}. \tag{11}$$

Here y is defined as mass flux fraction

$$y = \frac{\alpha_g \rho_g u_g}{\alpha_g \rho_g u_g + \alpha_l \rho_l u_l}. \tag{12}$$

For equal fluid velocities $u_l = u_g$ this corresponds to $y = \frac{n}{n+m}$.

The above correlations for the mixture viscosity μ_m obtained from lab experiments, reflect that there is room for dependence on both volume fractions α_g, α_l, densities ρ_l, ρ_g, as well as fluid velocities u_l, u_g. The line we pursue in this work, as in [8, 9], is to consider a choice suggested by the mathematical framework that is employed. However, we now briefly describe why the coefficient used in [8] also seems relevant from a more physical point of view. In that work we studied the model (in Lagrangian coordinates)

$$\begin{aligned} \partial_t n + (nm)\partial_x u &= 0 \\ \partial_t m + m^2 \partial_x u &= 0 \\ \partial_t u + \partial_x P(n,m) &= \partial_x (E(m)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{13}$$

with

$$P(n,m) = k_1 \Big(\frac{n}{\rho_l - m}\Big)^{\gamma}, \qquad \gamma > 1, \tag{14}$$

and

$$E(m) = k_2 \Big(\frac{m}{\rho_l - m}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{15}$$

If we assume that $n << m$, then $y = \frac{n}{n+m} \approx \frac{n}{m} := c$ (according to the notation used in [8, 9]) for $0 \le y \le 1$. Moreover, typically the liquid viscosity μ_l is considerable larger than the gas viscosity μ_g, see (30). Consequently, $\mu_l >> \mu_g$ and we may approximate as follows by using the viscosity model of McAdams et al (7):

$$\frac{1}{\mu_m} = \frac{y}{\mu_g} + \frac{1-y}{\mu_l} \approx \frac{y}{\mu_g} = \frac{c}{\mu_g}. \tag{16}$$

Thus, directly motivated by the traditional single-phase viscosity term of the form $E = (\mu\rho)^{\beta+1} = C\rho^{\beta+1}$ in Lagrangian coordinates, see for example [24, 17, 19, 28, 25, 16, 3, 31], we may propose a similar viscosity coefficient $E = (\mu_m \rho_m)^{\beta+1}$ for the gas-liquid mixture model (13) where μ_m is a mixture viscosity defined by, e.g., one of the choices (6)–(11) and ρ_m is a suitable mixture density. If we define a mixture density ρ_m as

$$\rho_m = [(\alpha_g \rho_g)^{\beta+1} + (\alpha_l \rho_l)^{\beta+1}]^{\frac{1}{\beta+1}}, \tag{17}$$

and combine it with (16), then $E = (\mu_m \rho_m)^{\beta+1}$ corresponds to

$$
\begin{aligned}
E &= (\mu_m \rho_m)^{\beta+1} = \mu_m^{\beta+1}[(\alpha_g \rho_g)^{\beta+1} + (\alpha_l \rho_l)^{\beta+1}] = (\mu_m \alpha_g \rho_g)^{\beta+1} + (\mu_m \alpha_l \rho_l)^{\beta+1} \\
&= (\alpha_g \rho_l \mu_g)^{\beta+1} \Big(\frac{1}{c} \frac{n}{\rho_l - m} \Big)^{\beta+1} + (\alpha_l \mu_g)^{\beta+1} \Big(\frac{\rho_l}{c} \Big)^{\beta+1} \\
&= (\alpha_g \rho_l \mu_g)^{\beta+1} \Big(\frac{m}{\rho_l - m} \Big)^{\beta+1} + (\alpha_l \mu_g)^{\beta+1} \Big(\frac{\rho_l}{c} \Big)^{\beta+1} \\
&:= E_1 + E_2,
\end{aligned}
$$

where we have used the fact that $\rho_g = \rho_l \frac{n}{\rho_l - m}$, see also (34). Recalling that ρ_l is constant and that $c = \frac{n}{m} = c(x)$ is constant in time, the most "dynamic" part of this viscosity term is the first part

$$
E_1 = (\alpha_g \rho_l \mu_g)^{\beta+1} \Big(\frac{m}{\rho_l - m} \Big)^{\beta+1}.
$$

Comparing with (15) we see that E_1 coincides with the one that is studied in [8] except that the coefficient $(\alpha_g \mu_g)^{\beta+1}$ has been replaced by a constant.

New results and main challenges. The main novelty of this work compared to [8, 9, 29, 30, 10] is that the current model allows unequal fluid velocities, i.e., $u_g \neq u_l$. As a consequence the model, when it is rewritten in terms of Lagrangian variables, contains a generalized pressure term $\widetilde{P} = P(c,n) - h(u_g)g(cn)$ for appropriate choices of the functions h and g. In particular, the pressure now depends on the gas velocity u_g. More precisely, we consider (1) in a free boundary problem setting where the masses m and n initially occupy only a finite interval $[a,b] \subset \mathbb{R}$. That is,

$$
n(x,0) = n_0(x), \quad m(x,0) = m_0(x), \quad u_g(x,0) = u_{g,0}(x), \quad u_l(x,0) = u_{l,0}(x), \qquad x \in [a,b], \tag{18}
$$

and the following boundary conditions are imposed:

$$
n(a,t) = n(b,t) = 0, \qquad m(a,t) = m(b,t) = k^*, \qquad t \geq 0, \tag{19}
$$

where k^* is a constant to be defined later which is related to the slip law. Rewriting the model (1) in terms of Lagrangian variables (details are given in Section 2), the free boundary region $[a(t), b(t)]$ is converted into a fixed region $[0,1]$ and the variables (n, m, u_g, u_l) are replaced by (c, n, u) where $c = \frac{m - k^*}{n}$ and u corresponds to the gas velocity. The resulting model takes the form

$$
\begin{aligned}
\partial_t c &= 0 \\
\partial_t n + cn^2 \partial_x u &= 0 \\
\partial_t u + \partial_x [P(c,n) - |u| g(cn)] &= \partial_x [E(cn) \partial_x u], \quad x \in (0,1),
\end{aligned} \tag{20}
$$

where $c(x,t) = c_0(x) = \frac{m_0(x) - k^*}{n_0(x)}$. The model (20), in view of (18) and (19), is subject to the boundary conditions

$$
c(0,t) = c(1,t) = 0, \qquad n(0,t) = n(1,t) = 0, \qquad t \geq 0 \tag{21}
$$

and the initial conditions

$$
c(x,0) = c_0(x), \qquad n(x,0) = n_0(x), \qquad u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{22}
$$

Moreover, we have that

$$
P(c,n) = \Big(\frac{n}{a^* - cn} \Big)^{\gamma}, \qquad g(cn) = k^* \frac{cn}{cn + k^*}, \qquad E(cn) = \frac{[cn]^{\beta+1}}{(a^* - [cn])^{\beta+1}}. \tag{23}
$$

Here $\alpha_g^* < 1$ is an upper limit for the amount of gas that can be present, $\alpha_l^* > 0$ is the corresponding lower limit of liquid, i.e., $\alpha_g^* + \alpha_l^* = 1$. Related variables are $k^* = \rho_l \alpha_l^*$ and $a^* = \rho_l \alpha_g^*$ appearing in (23). Main challenges we have to deal with are:

- Singular behavior associated with the pressure law (23) similar to the previous works [8, 9, 29].

- Singular behavior associated with the slip law. The gas fraction α_g must not be smaller than a critical lower gas volume fraction α_g^*. This is new compared with the previous works [8, 9, 29].
- The appearance of the term $|u|g(cn)$ in (20) due to the use of a more general slip law creates new difficulties. Owing to some technical challenges we have in this work made use of the approximation that $u^2 \approx |u|$ for small velocities when we derive (20) from (1).

We obtain an existence result (Theorem 3.1) for the model (20)–(23) for a class of weak solutions and for a flow regime where the viscosity coefficient ε in (1) is of the form

$$\varepsilon = \varepsilon(m) = \frac{(m-k^*)^\beta}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0,1), \tag{24}$$

which in turn leads to the expression $E(cn)$ given in (23). Note that this viscosity coefficient (24) is a natural generalization of (4) studied in [8, 29] which corresponds to the case $k^* = 0$. In fact, if $\alpha_g^* = 1$, then $\alpha_l^* = 0$ and consequently, $a^* = \rho_l$ and $k^* = 0$. Then the model (20) reduces to the one studied in [8, 29] with the only difference that c was defined as $c = n/m$ and not $c = m/n$.

The main tool in this analysis is the introduction of a suitable variable transformation in combination with the continuation method and the pointwise estimate techniques to deal with the singularity of the solution near the free boundary. The transformation allows for application of ideas and techniques similar to those used in [24, 17, 19, 28, 25, 16, 3, 31] in previous studies of the single-phase Navier-Stokes equations.

Overview. The rest of this paper is organized as follows. In Section 2 we give more details relevant for the model (1). In particular, we derive the Lagrangian variant (20) from (1). In Section 3 we state the main theorem with its assumptions. In Section 4 we derive the basic a priori energy estimate Lemma 4.1 and state two lemmas that give pointwise control on, respectively, the masses m, n (Lemma 4.2) and fluid velocity u (Lemma 4.3). In Section 5 we derive various lemmas needed for the proof of Lemma 4.2 and Lemma 4.3. Then, in Section 6, armed with the results of Section 5, we return to these proofs. More estimates are then derived that allows us to make use of standard compactness arguments to prove existence of local (in time) weak solutions .

2. Development of the model

The purpose of this section is to give further details relevant for the drift-flux model (1). Ultimately this will lead us to the simpler model (20).

2.1. Specification of the model (1).

To close the system (1), we need to include the following additional equations: The volume fractions are related by

$$\alpha_l + \alpha_g = 1. \tag{25}$$

Thermodynamical laws specify fluid properties such as densities ρ_l, ρ_g and viscosities μ_l, μ_g. In particular we will assume that the liquid density has the following form

$$\rho_l = \rho_{l,0} + \frac{p - p_{l,0}}{a_l^2}, \tag{26}$$

where $a_l = 1000$ [m/s] is the velocity of sound in the liquid phase and $\rho_{l,0}$ and $p_{l,0}$ are given constants. Here we will assume that $\rho_{l,0} = 1000$ [kg/m^3] and $p_{l,0} = 1$ [bar]. It is often assumed that the liquid is incompressible, i.e.

$$\rho_l = \rho_{l,0}. \tag{27}$$

We assume that we consider a polytropic, isentropic ideal gas characterized by

$$p(\rho_g) = a_g^2 \rho_g^\gamma, \qquad \gamma > 1. \tag{28}$$

In other words, we have

$$\rho_g = \left(\frac{p}{a_g^2}\right)^{1/\gamma}, \qquad \gamma > 1, \tag{29}$$

where $a_g = 316$ [m/s] is the velocity of sound in the gas phase. Furthermore, the viscosity for liquid and gas are assumed to be

$$\mu_l = 5 \cdot 10^{-2} \text{ [Pa s]}, \qquad \mu_g = 5 \cdot 10^{-6} \text{ [Pa s]}. \tag{30}$$

Since we only have one momentum equation for the mixture of the two phases, the model must be supplemented with an additional hydrodynamical closure law whose purpose is to determine the fluid velocities u_l, u_g through a so-called slip relation. We may assume that the slip relation can be expressed by a general relation

$$f(\alpha_g, u_l, u_g, \rho_g, \rho_l) = 0. \tag{31}$$

A commonly used slip relation, see for example [12, 6, 1], is given by

$$f(\alpha_g, u_l, u_g, \rho_g, \rho_l) = u_g - c_0 u_{mix} - c_1 = 0, \tag{32}$$

where

$$u_{mix} = \alpha_l u_l + \alpha_g u_g,$$

and c_0, c_1 are flow dependent coefficients. c_0 is the so-called profile parameter (or distribution coefficient) whereas c_1 is the drift velocity. The gas concentration tends to be highest in the center of the pipe/well for many flow scenarios, where the local mixture velocity is also fastest. Thus, when integrated across the area of the the pipe/well, the average velocity of the gas tends to be greater than that of the liquid. This effect is represented by the c_0 parameter. c_1, on the other hand, represents the buoyancy effect. Important characteristics of the different flow patterns can be captured through appropriate choices for these two parameters. We refer to the work [10] for numerical examples that illustrate typical flow cases with unequal fluid velocities.

For the source term q we have two components

$$q = F_f + F_g,$$

where $F_g = g(\alpha_l \rho_l + \alpha_g \rho_g)\sin\theta$ represents the gravity where g is the gravitational constant and θ is the inclination. Moreover, F_f represents friction forces between the wall and the fluids.

In order to see how pressure p is related to the masses $m = \alpha_l \rho_l$ and $n = \alpha_g \rho_g$ we observe that the relation (25) can be written as

$$\frac{n}{\rho_g(p)} + \frac{m}{\rho_l(p)} = 1. \tag{33}$$

Using this, we can express the pressure p as a function P of n and m, i.e.

$$p = P(n, m).$$

In particular, assuming that liquid is incompressible, $\rho_l = \rho_{l,0}$, we get from (33) that

$$\rho_g = \rho_l \frac{n}{\rho_l - m}, \tag{34}$$

which can be plugged into (28) yielding

$$p(\rho_g) = a_g^2 \rho_l^\gamma \Big(\frac{n}{\rho_l - m}\Big)^\gamma = k_1 \Big(\frac{n}{\rho_l - m}\Big)^\gamma =: P(n, m), \qquad k_1 = a_g^2 \rho_l^\gamma. \tag{35}$$

We will use this pressure law where we for simplicity has set $k_1 = 1$ for the model we derive in the next section.

2.2. **A simplified viscous two-phase model.** As a first step, instead of working directly with the full two-phase model (1) we suggest to replace it by a simpler one. We follow along the line of [8, 9] and introduce a simplification by replacing the mixture momentum equation by the momentum equation of the liquid phase only. This is motivated by the fact that the liquid phase density is much higher than for the gas phase, typically to the order of $\rho_g/\rho_l \sim 0.001$. The liquid phase therefore plays the dominating role in the mixture momentum conservation equation, as long as the amount of gas does not become too high. We also neglect external forces like friction and gravity. To sum up, we consider the model

$$\begin{aligned} \partial_t n + \partial_x[n u_g] &= 0 \\ \partial_t m + \partial_x[m u_l] &= 0 \\ \partial_t[m u_l] + \partial_x[m u_l^2 + P(n,m)] &= \partial_x[\varepsilon(m)\partial_x u_{mix}], \qquad u_{mix} = \alpha_g u_g + \alpha_l u_l, \end{aligned} \tag{36}$$

where the pressure law $P(n,m)$ and viscosity term $\varepsilon(n,m)$ are given by

$$P(n,m) = \Big(\frac{n}{\rho_l - m}\Big)^{\gamma}, \qquad \varepsilon(m) = \frac{(m-k^*)^{\beta}}{(\rho_l - m)^{\beta+1}}, \qquad \gamma > 1, \qquad \beta > 0, \tag{37}$$

together with the constitutive relations

$$\alpha_l + \alpha_g = 1, \quad u_g - c_0 u_{mix} - c_1 = 0, \quad \rho_l = \rho_{l,0}, \quad \rho_g = \rho_g(P), \tag{38}$$

where c_0 and c_1 are assumed to be constants. As will be explained in the following the slip law $u_g - c_0 u_{mix} - c_1 = 0$ requires that the liquid mass is above a critical lower limit k^*, i.e., $m \geq k^*$. This information is taken into account in the viscosity coefficient $\varepsilon(m)$. Similarly, the upper limit for the liquid mass $m \leq \rho_l$ is also accounted for in the viscosity term (as well as the pressure term) in the same manner as for (4) and (5) employed in previous works. We note that $\varepsilon(m)$ in (37) is a natural extension of (4).

We now want to rewrite the model (36) into a form more amenable for analysis. Our approach is inspired by the work [12]. Given the slip relation

$$u_g = c_0 u_{mix} + c_1 \tag{39}$$

we introduce α_g^*, α_l^* given by

$$\alpha_g^* = \frac{1}{c_0}, \qquad \alpha_l^* = 1 - \alpha_g^*. \tag{40}$$

In the following we will assume that

$$c_0 > 1, \tag{41}$$

implying that $\alpha_g^* < 1$. This is consistent with previous applications of the slip velocity (39) in the context of gas-liquid and liquid-oil flow modeling where c_0 typically is ranging between 1.0 and 1.5. Moreover, in view of (39) it follows that

$$u_g = \frac{c_0 \alpha_l u_l + c_1}{1 - c_0 \alpha_g} = \frac{\alpha_l u_l + c_1 \alpha_g^*}{\alpha_g^* - \alpha_g} = \frac{\alpha_l u_l + c_1(1-\alpha_l^*)}{\alpha_l - \alpha_l^*} \tag{42}$$

It is implicitly assumed that $\alpha_g < \alpha_g^*$ (or equivalently, that $\alpha_l > \alpha_l^*$) for this slip law to be valid. From (42) we get

$$\alpha_l u_l = u_g(\alpha_l - \alpha_l^*) - (1-\alpha_l^*)c_1. \tag{43}$$

Next, we introduce the variable c defined by

$$c = \frac{m - \rho_l \alpha_l^*}{n} = \frac{\rho_l(\alpha_l - \alpha_l^*)}{n}. \tag{44}$$

We assume that the liquid is incompressible ρ_l=constant, i.e.,

$$c = \frac{m - k^*}{n}, \tag{45}$$

where $k^* = \rho_l \alpha_l^*$ is constant. We then apply (43), (44), and (45) and derive the following relations:

$$m = cn + k^* \tag{46}$$

$$c n u_g = \rho_l(\alpha_l - \alpha_l^*)u_g = \rho_l[\alpha_l u_l + (1-\alpha_l^*)c_1] = m u_l + \rho_l(1-\alpha_l^*)c_1 = m u_l + d, \tag{47}$$

where $d = \rho_l(1-\alpha_l^*)c_1$ is constant. In other words, $m u_l = c n u_g - d$. Employing (46) and (47) in (36) we arrive at the following form for the system in question:

$$\begin{aligned} \partial_t n + \partial_x[n u_g] &= 0 \\ \partial_t[cn] + \partial_x[c n u_g] &= 0 \\ \partial_t[c n u_g] + \partial_x[c n u_g^2] + \partial_x[m u_l^2 - c n u_g^2] + \partial_x[P(n,m)] &= \partial_x[\varepsilon(m)\partial_x u_{mix}]. \end{aligned} \tag{48}$$

We note that

$$mu_l^2 - cnu_g^2 = \alpha_l\rho_l u_l^2 - \rho_l(\alpha_l - \alpha_l^*)u_g^2 = \alpha_l\rho_l[u_l^2 - u_g^2] + k^*u_g^2. \tag{49}$$

Next, we observe in view of (43) that

$$\begin{aligned} \alpha_l(u_l - u_g) &= -u_g\alpha_l^* - (1-\alpha_l^*)c_1 &(50)\\ \alpha_l(u_l + u_g) &= u_g(2\alpha_l - \alpha_l^*) - (1-\alpha_l^*)c_1. &(51)\end{aligned}$$

Combining these two relations we get

$$\alpha_l^2(u_l^2 - u_g^2) = -u_g^2\alpha_l^*[2\alpha_l - \alpha_l^*] + c_1[\ldots]. \tag{52}$$

In the following, similar to the work [12] we restrict us to the case that

$$\alpha_l^* \neq 0 \qquad (\alpha_l^* > 0), \qquad c_1 = 0. \tag{53}$$

In other words, we neglect gas buoyancy effects represented through c_1 and relevant for vertical flow. This is also consistent with the fact that gravity effects have been neglected in the momentum equation in (36), i.e., we consider horizontal flow. Then we have

$$\alpha_l\rho_l(u_l^2 - u_g^2) = -\rho_l u_g^2\alpha_l^*[2 - \alpha_l^*/\alpha] = \rho_l u_g^2\alpha_l^*\Big[-2 + \frac{\alpha_l^*}{\alpha_l}\Big]. \tag{54}$$

In view of (49) and (54) we get

$$\begin{aligned} G(n,m,u_g) := [mu_l^2 - cnu_g^2] &= \rho_l\alpha_l^*u_g^2\Big[-2 + \frac{\alpha_l^*}{\alpha_l}\Big] + \rho_l\alpha_l^*u_g^2\\ &= \rho_l\alpha_l^*u_g^2\Big[-1 + \frac{\alpha_l^*}{\alpha_l}\Big] = \alpha_l^*u_g^2\frac{k^*-m}{\alpha_l} = -\rho_l\alpha_l^*u_g^2\frac{cn}{cn+\rho_l\alpha_l^*}\\ &= -k^*u_g^2\frac{cn}{cn+k^*} = -u_g^2 g(nc), \end{aligned} \tag{55}$$

noting from (44) that

$$\alpha_l = \frac{cn}{\rho_l} + \alpha_l^*,$$

and where we have defined the function $g(\cdot)$ as

$$g(nc) = k^*\frac{cn}{cn+k^*}. \tag{56}$$

For the pressure law we have

$$P(n,m) = \Big(\frac{n}{\rho_l - m}\Big)^\gamma = \Big(\frac{n}{[\rho_l - k^*] - cn}\Big)^\gamma = \Big(\frac{n}{a^* - cn}\Big)^\gamma := P(c,n), \tag{57}$$

where $a^* = \rho_l - k^* = \rho_l\alpha_g^*$. For the viscosity term $\varepsilon(m)$ we have

$$\varepsilon(m) = \frac{(m-k^*)^\beta}{(\rho_l - m)^{\beta+1}} = \frac{[cn]^\beta}{(a^* - [cn])^{\beta+1}} := \varepsilon(cn). \tag{58}$$

Hence, setting $u_g := u$, using (39) with the restriction (53) ($c_1 = 0$) as well as (55) in the momentum equation of (48), we obtain a gas-liquid model of the following form:

$$\begin{aligned} \partial_t n + \partial_x[nu] &= 0\\ \partial_t[cn] + \partial_x[cnu] &= 0\\ \partial_t[cnu] + \partial_x[cnu^2] + \partial_x[P(c,n) - u^2 g(cn)] &= \frac{1}{c_0}\partial_x[\varepsilon(cn)\partial_x u]. \end{aligned} \tag{59}$$

In the following we may absorb the constant $1/c_0$ into the viscosity term ε without loss of any generality.

Lagrangian coordinates. Following the approach of the works [8, 9, 29], which in turn is motivated by studies for the single-phase gas model, we suggest to study the model (59), described in terms of the variables (c, n, u), in a free boundary setting.

$$\begin{aligned} \partial_t n + \partial_x[nu] &= 0 \\ \partial_t[cn] + \partial_x[cnu] &= 0 \\ \partial_t[cnu] + \partial_x[cnu^2] + \partial_x[P(c,n) - u^2 g(cn)] &= \partial_x[\varepsilon(cn)\partial_x u], \quad \text{for } a(t) < x(t) < b(t), \end{aligned} \tag{60}$$

and $t > 0$. Initial data are

$$n(x, t=0) = n_0(x), \qquad c(x, t=0) = c_0(x) = \frac{m_0(x) - k^*}{n_0(x)}, \qquad u(x, t=0) = u_0(x), \tag{61}$$

for $x \in [a_0, b_0]$ where $a_0 = a(t=0)$ and $b_0 = b(t=0)$. Boundary conditions are set to be as follows:

$$n(a(t), t) = n(b(t), t) = 0, \qquad c(a(t), t) = c(b(t), t) = 0. \tag{62}$$

Here $a(t)$ and $b(t)$ are free boundaries, i.e., the particle path separating the gas-liquid mixture and the vacuum like state corresponding to $n = 0$ and $c = 0$, satisfying

$$\frac{da}{dt} = u(a(t), t), \qquad n(a(t), t) = c(a(t), t) = 0, \tag{63}$$

$$\frac{db}{dt} = u(b(t), t), \qquad n(b(t), t) = c(b(t), t) = 0. \tag{64}$$

Let us introduce Lagrangian coordinates by using the transformation $(x, t) \to (\xi, \tau)$ given by

$$\xi = \int_{a(t)}^{x} cn(z, t)\, dz, \qquad \tau = t, \tag{65}$$

observing that

$$\int_{a(t)}^{b(t)} cn(z, t)\, dz = \int_{a_0}^{b_0} cn(z, t=0)\, dz = \text{constant} = 1.$$

This implies that $[a(t), b(t)]$ is converted into the fixed interval $[0, 1]$ and

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = [cn]\frac{\partial}{\partial \xi}.$$

Applying this transformation in (60) gives

$$\begin{aligned} \partial_\tau n + n\partial_x u &= 0 \\ \partial_\tau[cn] + [cn]\partial_x u &= 0 \\ [cn]\partial_\tau u + \partial_x[P(c,n) - u^2 g(cn)] &= \partial_x[\varepsilon(cn)\partial_x u], \quad \text{in } 0 < \xi < 1. \end{aligned} \tag{66}$$

In other words,

$$\begin{aligned} \partial_\tau n + cn^2\partial_\xi u &= 0 \\ \partial_\tau[cn] + [cn]^2\partial_\xi u &= 0 \\ \partial_\tau u + \partial_\xi[P(c,n) - u^2 g(cn)] &= \partial_\xi[\varepsilon(cn)[cn]\partial_\xi u], \quad \text{in } 0 < \xi < 1. \end{aligned} \tag{67}$$

We now replace (τ, ξ) by (t, x). Moreover, an easy calculation shows that (67) corresponds to

$$\begin{aligned} \partial_t c &= 0 \\ \partial_t n + cn^2\partial_x u &= 0 \\ \partial_t u + \partial_x[P(c,n) - u^2 g(cn)] &= \partial_x[E(cn)\partial_x u], \quad \text{in } 0 < x < 1. \end{aligned} \tag{68}$$

with boundary conditions

$$c(0, t) = c(1, t) = 0, \qquad n(0, t) = n(1, t) = 0, \qquad t \geq 0, \tag{69}$$

and with initial conditions

$$c(x, 0) = c_0(x), \qquad n(x, 0) = n_0(x), \qquad u(x, 0) = u_0(x), \qquad x \in [0, 1], \tag{70}$$

where $c(x,t) = c_0(x) = \frac{m_0(x)-k^*}{n_0(x)}$. Moreover, we have from (57) and (58) that

$$P(c,n) = \Big(\frac{n}{a^* - cn}\Big)^{\gamma}, \qquad g(cn) = k^* \frac{cn}{cn + k^*}, \qquad E(cn) = \varepsilon(cn)[cn] = \frac{[cn]^{\beta+1}}{(a^* - [cn])^{\beta+1}}. \quad (71)$$

Finally, we shall in this work make use of the approximation that $|u| \approx u^2$, which is reasonable for small u, such that we replace $u^2 g(cn)$ by $|u|g(cn)$ in the third equation of (68). The motivation for using this approximation is to avoid some technical difficulties not yet solved. Hence, the model (68)–(71) is now consistent with the model (20)–(23).

Reformulation. For the analysis of the model (20) it will be convenient to introduce the function $Q(c,n)$ given by

$$Q(c,n) = \frac{n}{a^* - cn}, \qquad \text{which corresponds to} \qquad n = a^* \frac{Q}{1 + cQ}. \quad (72)$$

A similar approach was also used in [8, 9], however, for a different model with equal fluid velocities. The following nice relation holds for $Q(c,n)$:

$$\begin{aligned} Q(c,n)_t &= Q_c c_t + Q_n n_t = Q_n n_t \\ &= \Big(\frac{1}{a^* - cn} + \frac{cn}{(a^* - cn)^2}\Big) n_t = \frac{a^*}{(a^* - cn)^2} n_t \\ &= -\frac{a^* cn^2}{(a^* - cn)^2} u_x = -a^* c Q(c,n)^2 u_x. \end{aligned}$$

Hence, the system (20) can be replaced by

$$\begin{aligned} &\partial_t c = 0 \\ &\partial_t Q + a^* c Q^2 \partial_x u = 0 \\ &\partial_t u + \partial_x [P(Q) - |u| g(cQ)] = \partial_x [E(cQ)\partial_x u], \quad \text{in } 0 < x < 1. \end{aligned} \quad (73)$$

with

$$\begin{aligned} &P(Q) = Q(c,n)^{\gamma}, \qquad E(c,n) = \varepsilon(c,n)[cn] = [cQ]^{\beta+1} := E(cQ), \\ &g(cn) = k^* \frac{cn}{cn + k^*} = a^* k^* \frac{cQ}{k^* + (a^* + k^*)cQ} := g(cQ), \end{aligned} \quad (74)$$

since

$$cn = a^* \frac{cQ}{1 + cQ}.$$

Clearly, g as a function of cQ possesses the same features as $g(cn)$. Most importantly, it is always bounded as a function of its argument cQ. This feature of g will be crucial for the analysis that follows in Section 4. Boundary conditions for our system (73)–(74) are (in view of (72) and (69)):

$$c(0,t) = c(1,t) = 0, \qquad Q(0,t) = Q(1,t) = 0, \qquad t \ge 0. \quad (75)$$

Initial conditions are (in view of (72) and (70)):

$$c(x,0) = c_0(x), \qquad Q(x,0) = Q_0(x) = \frac{n_0}{a^* - c_0 n_0}, \qquad u(x,0) = u_0(x), \qquad x \in [0,1]. \quad (76)$$

3. A local existence result

In this section we give the main theorem of this paper, a local existence result for the model (20)–(23). Before we state the main result we describe the notation we apply throughout the paper. $W^{1,2}(I) = H^1(I)$ represents the usual Sobolev space defined over $I = (0,1)$ with norm $\|\cdot\|_{W^{1,2}}$. Moreover, $L^p(K,B)$ with norm $\|\cdot\|_{L^p(K,B)}$ denotes the space of all strongly measurable, pth-power integrable functions from K to B where K typically is subset of $\mathbb{R}$ and B is a Banach space. In addition, let $C^{\alpha}[0,1]$ for $\alpha \in (0,1)$ denotes the Banach space of functions on $[0,1]$ which are uniformly Hölder continuous with exponent α. Similarly, let $C^{\alpha,\alpha/2}(D_T)$ represent the Banach space of functions on $D_T = [0,1] \times [0,T]$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t.

Now we will give a precise description of the assumptions on the initial data (c_0, n_0, u_0) that are required for the existence result to hold. For the analysis that follows in Section 4-6 we need more precise information about how fast n_0 is decreasing towards zero and how fast m_0 will decrease towards the lower limit k^*. We now specify this more precisely as well as information about regularity properties of the initial data (c_0, n_0, u_0). In addition, information about the constraints on the parameters α, β, and γ will be given. Here $\alpha > 0$ is related to the decay towards the boundary points $x = 0, 1$, $\beta > 0$ is related to the viscosity term $E(cn)$, and $\gamma > 1$ is relevant for the pressure law $P(c, n)$, see (23).

3.1. **Assumptions.** The analysis throughout the whole paper depends on a set of assumptions.

(A1) The following assumptions are made regarding upper and lower limits on the initial masses $m_0(x), n_0(x)$:

$$A_1\phi(x)^\alpha \le m_0(x) - k^* \le A_2\phi(x)^\alpha \tag{77}$$

$$B_1\phi(x)^{3\alpha/4} \le n_0(x) \le B_2\phi(x)^{3\alpha/4}, \tag{78}$$

for some constants $A_1, A_2, B_1, B_2, \alpha > 0$ and $\phi(x) = x(1-x)$. These bounds can then be translated into upper and lower limits for the variable c which is used in our model (20) instead of m. In particular, this implies that for suitable choices of C_1 and C_2, for instance, $C_1 = A_1/B_2$ and $C_2 = A_2/B_1$

$$C_1\phi(x)^{\alpha/4} \le c_0(x) \le C_2\phi(x)^{\alpha/4}. \tag{79}$$

Clearly, $c(x,t) = c_0(x)$, hence these estimates hold for $c(x,t)$ for all times t.

(A2) We must ensure that the pressure law $P = Q^\gamma$ is well-defined at initial time. For that purpose we here assume that the initial gas phase does not vanish at some point (no transition to pure liquid flow only), i.e.,

$$\alpha_l^* \le \alpha_{l,0}(x) \le 1 - \delta, \tag{80}$$

for some $\delta > 0$. In view of (72) and (78) we may assume that there are positive constants A, B such that (see remark below)

$$\frac{AC_2}{C_1}\phi(x)^{3\alpha/4} \le Q_0 = Q(c_0, n_0) \le \frac{BC_1}{C_2}\phi(x)^{3\alpha/4}. \tag{81}$$

Thus, it follows that

$$AC_2\phi(x)^\alpha \le c_0Q_0 \le BC_1\phi(x)^\alpha. \tag{82}$$

(A3) Assumptions on $\alpha, \beta \in (0,1)$ and $\gamma > 1$ are as follows:

$$\gamma \ge \frac{4}{3}\alpha, \qquad \gamma \ge \beta + \frac{1}{3}, \qquad \gamma \ge \frac{4}{3}\beta, \qquad \frac{3}{4}\gamma > 1 + \beta - \frac{1}{\alpha},$$
$$\alpha(3\beta + 1) \le 2, \qquad \alpha\beta < \frac{3}{4}, \qquad \alpha(\beta + 1) < \frac{3}{2}.$$

(A4) Regularity assumptions on the initial data are as follows:

$$c_0, u_0 \in H^1([0,1]), \qquad [c_0Q_0]^\beta \in H^1([0,1]), \tag{83}$$

In particular, we note that $\frac{Q_0^{\gamma-1}}{c_0} \in L^1([0,1])$ due to (79) and (81) since $\int_0^1 \phi(x)^s\,dx < \infty$ for $s > -1$.

Remark 3.1. *Note that the first assumption* (77) *ensures that the initial liquid volume fraction* $\alpha_{l,0}$ *is above the lower critical limit* α_l^* *throughout the whole domain except at the boundary points* $x = 0, 1$ *where* $\alpha_{l,0} = \alpha_l^*$*, i.e.,* $\alpha_{g,0} = \alpha_g^*$.

The second equation (78) *puts an additional constraint on the pressure behavior at the end points* $x = 0, 1$ *by assuming that a vacuum state* $p = 0$ *occurs.*

Concerning the inequalities (81) *we note that* $a^* - cn_0 = \rho_l - m_0 = \rho_l(1 - \alpha_{l,0})$, *from which it follows from* (80) *that* $\rho_l \delta \leq a^* - cn_0 \leq \rho_l \alpha_g^*$. *Hence, in light of* (72) *and* (78)

$$\frac{B_1}{\rho_l \alpha_g^*}\phi(x)^{3\alpha/4} \leq \frac{n_0}{\rho_l \alpha_g^*} \leq Q_0 = \frac{n_0}{a^* - cn_0} \leq \frac{n_0}{\rho_l \delta} \leq \frac{B_2}{\rho_l \delta}\phi(x)^{3\alpha/4}.$$

From this relation we can find constants A *and* B *such that the inequalities* (81) *hold.*

3.2. **Main result.** Under the above assumptions, our main result can be stated as follows:

Theorem 3.1. *Under the assumptions (A1)–(A4), there exists a positive time constant* $T_1 > 0$ *(small time) such that the free boundary value problem* (20)–(23) *admits a weak solution* $(c, n, u)(x, t)$ *on* $[0, 1] \times [0, T_1]$ *in the sense that*

(A) *we have the following regularity:*

$$\begin{aligned} &c, n, u \in L^\infty([0,1] \times [0, T_1]) \cap C^1([0, T_1]; L^2(0,1)), \\ &E(cn)u_x \in L^\infty([0,1] \times [0, T_1]) \cap C^{1/2}([0, T_1]; L^2(0,1)). \end{aligned} \tag{84}$$

Moreover, the following pointwise estimate holds for n*:*

$$\frac{a^* A\phi(x)^{3\alpha/4}}{2 + c_0(x)A} \leq n(x,t) \leq \min\Big\{a^* \frac{3}{2} B\phi(x)^{3\alpha/4}, \frac{a^* - \mu}{c_0(x)}\Big\}, \qquad \forall t \in [0, T_1], \tag{85}$$

where $\mu = \mu(B, C_2, T_1) > 0$ *is a small constant determined in Corollary 4.1.*

(B) *Moreover, the following equations hold,*

$$\begin{aligned} &c_t = 0, \quad n_t + cn^2 u_x = 0, \\ &(c, n)(x, 0) = (c_0(x), n_0(x)), \text{ for a.e. } x \in (0,1) \text{ and any } t \geq 0, \\ &\int_0^{T_1}\int_0^1 \Big[u\phi_t + \Big(P(c,n) - |u|g(cn) - E(cn)u_x\Big)\phi_x\Big]\,dx\,dt + \int_0^1 u_0(x)\phi(x,0)\,dx = 0 \end{aligned} \tag{86}$$

for any test function $\phi(x,t) \in C_0^\infty([0,1] \times [0, T_1))$.

4. Basic estimates

Below we derive a priori estimates for (c, n, u) which are assumed to be smooth solutions of the initial boundary value problem (20)–(23). Then the method of lines can be used to construct approximate solutions of (20) and derive corresponding estimates.

In the following we will frequently take advantage of the fact that the model (20) can be rewritten in the form (73)–(76) which is more amenable for deriving various useful estimates.

Some words about the notation used for constants. We shall use C and C_i $(i = 1, \ldots)$ to denote positive constants that only depend on the initial data and other known constants as stated in the assumptions given in Section 3. Some places we also use D, E, F, K and D_i, E_i, F_i, K_i $(i = 1, \ldots)$ for the same purpose. In particular, these constants, C, D, E, F, K, are independent of the positive constant M which appears in Lemma 4.3. On the other hand, we use $\widetilde{C}$ and $\widetilde{C}_i$ $(i = 1, \ldots)$ (similarly for D, E, F, K) to represent constants that also, however, include dependence on the positive constant M.

4.1. **A priori estimates.** Now we derive a priori estimates for (c, n, u) by making use of the reformulated model (73)–(76) described in terms of (c, Q, u). We start with the standard energy estimate which is slightly more involved in our case compared to a standard single-phase model due to the appearance of the new term $|u|g(cQ)$.

Lemma 4.1 (Energy estimate)**.** *We have, under the assumption (A4), for each* $T > 0$ *the basic energy estimate*

$$\int_0^1 \Big(\frac{1}{2}u^2 + \frac{1}{a^* c(\gamma - 1)} Q(c,n)^{\gamma - 1}\Big)\,dx + \int_0^t \int_0^1 [cQ(c,n)]^{\beta+1}(u_x)^2\,dx\,ds \leq C(T), \tag{87}$$

where $0 < t < T$ *and* $C(T)$ *is a constant that depends only on the the regularity of* u_0 *and* Q_0 *as stated in Assumption (A4).*

Proof. We multiply the third equation of (73) by u and integrate over $[0,1]$ in space. Applying the boundary condition (75), the fact that $g(0)=0=E(0)$, and the equation

$$\frac{1}{a^*c(\gamma-1)}(Q^{\gamma-1})_t + Q^\gamma u_x = 0, \tag{88}$$

obtained from the second equation of (73) by multiplying with $Q^{\gamma-2}$, we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{1}{2}u^2 + \frac{1}{a^*c(\gamma-1)}Q^{\gamma-1}\Big)\,dx + \int_0^1 E(cQ)(u_x)^2\,dx = -\int_0^1 |u|g(cQ)u_x\,dx. \tag{89}$$

We may consider the splitting

$$|u|g(cQ)u_x = |u|[cQ]^{-(\beta+1)/2}g(cQ)\cdot[cQ]^{(\beta+1)/2}u_x = a\cdot b.$$

In light of the Cauchy's inequality with ε, $ab\le(1/4\varepsilon)a^2+\varepsilon b^2$, we may conclude that

$$\int_0^1 |u|g(cQ)u_x\,dx \le (1/4\varepsilon)\int_0^1 u^2[cQ]^{-(\beta+1)}g(cQ)^2\,dx + \varepsilon\int_0^1 [cQ]^{\beta+1}(u_x)^2\,dx. \tag{90}$$

For the first term on the right hand side of (90) we observe that

$$[cQ]^{-(\beta+1)}g(cQ)^2 = (a^*k^*)^2\frac{[cQ]^{1-\beta}}{(k^*+(a^*+k^*)cQ)^2} \le C_1, \tag{91}$$

since the function $g(y)=y^{1-\beta}/(k^*+(a^*+k^*)y)^2$ clearly is bounded for all $y>0$ where $0<\beta<1$. Hence,

$$\int_0^1 |u|g(cQ)u_x\,dx \le (1/4\varepsilon)C_1\int_0^1 u^2\,dx + \varepsilon\int_0^1 E(cQ)(u_x)^2\,dx. \tag{92}$$

We plug (92) into the right hand side of (89) and observe that the first term on the right hand side of (92) is handled by application of Gronwall's lemma. Moreover, by an appropriate choice of ε the second term can be adsorbed in the corresponding term appearing in the left hand side of (89). From this, (87) follows. □

Remark 4.1. *Note that the special properties of $g(\cdot)$, it is bounded and goes through zero, plays a crucial role in the above proof of the fundamental energy estimate* (87).

The next lemma deals with upper and lower estimates for $Q(c,n)$. In view of the fact that $n=a^*\frac{Q}{1+cQ}$ these estimates can directly be translated into corresponding estimates for n. Along the line of [3, 31] we use the continuation method, in combination with semidiscrete versions of the various lemmas derived below, to obtain pointwise control on Q (and n).

Lemma 4.2. *Under the assumptions of Theorem 3.1 there exists a time $T_1=T_1(A,B,M,\|u_0\|_{H^1})$ such that if*

$$\frac{1}{3}A\phi(x)^{3\alpha/4} \le Q(c,n) \le 2B\phi(x)^{3\alpha/4}, \qquad \forall t\in[0,T], \tag{93}$$

where $T\in(0,T_1]$ is any fixed positive constant, then we have the following estimate

$$\frac{1}{2}A\phi(x)^{3\alpha/4} \le Q(c,n) \le \frac{3}{2}B\phi(x)^{3\alpha/4}, \qquad \forall t\in[0,T]. \tag{94}$$

The proof will be given in Section 6 after a set of useful lemmas has been derived. We also note here that T_1 is defined as

$$T_1 = \min\{\widetilde{T}_1,\widetilde{T}_2,\widetilde{T}_3,\widetilde{T}_4,1\}, \tag{95}$$

where $\widetilde{T}_1$ is given by (144), $\widetilde{T}_2$ by (148), $\widetilde{T}_3$ by (154), and $\widetilde{T}_4$ by (126). We use the 'tilde' notation to indicate that these constants depend on the positive constant M appearing in Lemma 4.3.

Corollary 4.1. *If $Q(c,n)$ satisfies the lower and upper bounds of* (94)*, then the following bounds hold for n:*

$$\frac{a^*A\phi(x)^{3\alpha/4}}{2+c_0(x)A} \le n(x,t) \le \min\Big\{a^*\frac{3}{2}B\phi(x)^{3\alpha/4}, \frac{a^*-\mu}{c_0(x)}\Big\}, \qquad \forall t\in[0,T], \tag{96}$$

where $\mu=\mu(B,C_2,T_1)>0$ is a small positive constant.

Proof. We observe that $Q(c,n)$ is a strictly increasing function with respect to n for a fixed c and its inverse is given by $Q^{-1}(c,y) = a^* \frac{y}{1+cy}$, which also is an increasing function. Consequently,

$$n \leq Q^{-1}(\frac{3}{2}B\phi(x)^{3\alpha/4}) = a^* \frac{\frac{3}{2}B\phi(x)^{3\alpha/4}}{1+\frac{3}{2}c(x)B\phi(x)^{3\alpha/4}} \leq \frac{3}{2}a^* B\phi(x)^{3\alpha/4}.$$

At the same time it is clear from the upper bound of Q given by (94) and the expression (72) for Q, that

$$cQ = \frac{cn}{a^* - cn} < C(B, C_2, T_1).$$

In other words, since $\frac{cn}{a^*-cn}$ tends to infinity as $cn \to a^*_-$, cn must be a certain distance $\mu > 0$ below of a^*, and μ only depends on B, C_2, and T_1, i.e., $cn < a^* - \mu(B, C_2, T_1)$. Similarly,

$$n \geq Q^{-1}(\frac{1}{2}A\phi(x)^{3\alpha/4}) = a^* \frac{\frac{1}{2}A\phi(x)^{3\alpha/4}}{1+\frac{1}{2}c(x)A\phi(x)^{3\alpha/4}} \geq \frac{\frac{1}{2}a^* A\phi(x)^{3\alpha/4}}{1+\frac{1}{2}c(x)A}.$$

□

The next lemma concerns the pointwise control of the fluid velocity u. It involves a positive constant M such that $M > \{1, \|u_0\|_\infty\}$ and is determined by the inequality (132). The proof of Lemma 4.3 will also be given in Section 6.

Lemma 4.3. *Under the assumptions of Theorem 3.1 and* (93)*, if*

$$\|u\|_{L^\infty} \leq M, \qquad \forall t \in [0,T], \tag{97}$$

then we get

$$\|u\|_{L^\infty} \leq \frac{1}{2}M, \qquad \forall t \in [0,T]. \tag{98}$$

5. Proof of some lemmas

Before we return to Lemmas 4.2 and 4.3 and give proofs we shall derive some more regularity estimates. These results will be derived under assumptions (93) and (97). In other words, we assume the following estimates:

$$\frac{1}{3}A\phi(x)^{\frac{3}{4}\alpha} \leq Q(x,t) \leq 2B\phi(x)^{\frac{3}{4}\alpha}, \qquad \forall t \in [0,T], \tag{99}$$

$$\|u(\cdot,t)\|_\infty \leq M, \qquad \forall t \in [0,T], \tag{100}$$

for an appropriate time T to be specified later. The results in this section are obtained for constants C_3, C_5, and C_6 that are independent of M when

$$t \leq T \leq \min\{1, \frac{1}{M}\}. \tag{101}$$

However, the constant $\widetilde{C}_4$ appearing in Lemma 5.2 depends on M. Consistent with what we have said before we shall in the following calculations use the convention that $\widetilde{C}_i$ represents a constant that depends on M, whereas C_i only depends on known constants and constants appearing in the Assumptions (A1)–(A4).

Lemma 5.1 (Additional regularity)**.** *Under the assumptions of Theorem 3.1 as well as* (99) *and* (100) *we have, for* $t \in [0,T]$*, the estimate*

$$\int_0^1 \Big(\partial_x([cQ]^\beta)\Big)^2 dx \leq C_3, \tag{102}$$

for a constant $C_3 = C_3(\|[c_0Q_0]^\beta\|_{H^1}, \|c_0\|_{H^1}, \|u_0\|_{L^2}, B, C, C_1, C_2, T)$.

Proof. From the second equation of (73) we derive the equation

$$\frac{1}{\beta a^*}([cQ]^\beta)_t + [cQ]^{\beta+1}u_x = 0. \tag{103}$$

Using (103) in the third equation of (73) and integrating in time over $[0,t]$ we arrive at

$$u(x,t) - u_0(x) + \int_0^t \partial_x\Big(P(Q) - |u|g(cQ)\Big)\,ds = -\frac{1}{\beta a^*}\Big(\partial_x([cQ]^\beta) - \partial_x([c_0Q_0]^\beta)\Big). \tag{104}$$

Multiplying (104) by $\beta a^*(\partial_x[cQ]^\beta)$ and integrating over $[0,1]$ in x, we get

$$\begin{aligned}
&\int_0^1 (\partial_x[cQ]^\beta)^2\,dx \\
&\qquad = \int_0^1 (\partial_x[cQ]^\beta)(\partial_x[c_0Q_0]^\beta)\,dx \\
&\qquad\quad - \beta a^* \int_0^1 (\partial_x[cQ]^\beta)\Big[(u-u_0) + \int_0^t \partial_x\Big(P(Q) - |u|g(cQ)\Big)\,ds\Big]\,dx \\
&\qquad \le \varepsilon_1 \int_0^1 (\partial_x[cQ]^\beta)^2\,dx + K_1 \int_0^1 (\partial_x[c_0Q_0]^\beta)^2\,dx \\
&\qquad\quad + \varepsilon_2 \int_0^1 (\partial_x[cQ]^\beta)^2\,dx + K_2 \int_0^1 (u-u_0)^2\,dx \\
&\qquad\quad + \varepsilon_3 \int_0^1 (\partial_x[cQ]^\beta)^2\,dx + K_3 \int_0^1 \Big(\int_0^t \partial_x\Big(P(Q) - |u|g(cQ)\Big)\,ds\Big)^2\,dx,
\end{aligned} \tag{105}$$

where we have used Cauchy's inequality $\varepsilon > 0$, $ab \le (1/4\varepsilon)a^2 + \varepsilon b^2$. For the last term we can apply Hölder's inequality for the term $\int_0^t (P(Q) - |u|g(cQ))_x\,ds$ and estimate as follows:

$$\int_0^1 \Big(\int_0^t \partial_x\Big(P(Q) - |u|g(cQ)\Big)\,ds\Big)^2\,dx \le t\int_0^t \int_0^1 \Big(\partial_x[P(Q) - |u|g(cQ)]\Big)^2\,dx\,ds \tag{106}$$

Moreover, we have for $P(Q) = Q^\gamma$:

$$\begin{aligned}
\int_0^1 ([Q^\gamma]_x)^2\,dx &= \int_0^1 [\gamma Q^{\gamma-1}Q_x]^2\,dx = \int_0^1 [\gamma Q^{\gamma-1}(cQ\cdot\frac{1}{c})_x]^2\,dx \\
&= \int_0^1 \Big[\gamma\frac{Q^{\gamma-1}}{c}(cQ)_x - \gamma\frac{cQ^\gamma}{c^2}c_x\Big]^2\,dx \\
&= \int_0^1 \Big[\gamma\frac{Q^{\gamma-1}}{\beta c}[cQ]^{1-\beta}([cQ]^\beta)_x - \gamma\frac{Q^\gamma}{c}c_x\Big]^2\,dx \\
&= \int_0^1 \Big[\gamma\frac{Q^{\gamma-\beta}}{\beta c^\beta}([cQ]^\beta)_x - \gamma\frac{Q^\gamma}{c}c_x\Big]^2\,dx \\
&\le K_4\int_0^1 \Big[([cQ]^\beta)_x\Big]^2\,dx + K_5\int_0^1 [c_x]^2\,dx.
\end{aligned} \tag{107}$$

The first term on the right hand side of (107) is controlled by use of Gronwall's lemma, the second by assuming $c \in H^1$,i.e., Assumption (A4). Here we have also used Assumption (A1) which ensures that

$$C_2^{-1}\phi(x)^{-\alpha/4} \le \frac{1}{c} \le C_1^{-1}\phi(x)^{-\alpha/4},$$

and in view of (99) we conclude that

$$\frac{Q^{\gamma-\beta}}{c^\beta} \le K_4(B,C_1)\phi(x)^{\frac{3}{4}\alpha(\gamma-\beta)-\frac{1}{4}\alpha\beta} = K_4(B,C_1)\phi(x)^{\alpha(\frac{3}{4}\gamma-\beta)} \le K_4(B,C_1),$$

for $\gamma \ge \frac{4}{3}\beta$. Moreover,

$$\frac{Q^\gamma}{c} \le K_5(B,C_1)\phi(x)^{\frac{3}{4}\alpha\gamma-\frac{1}{4}\alpha} = K_5(B,C_1)\phi(x)^{\alpha(\frac{3}{4}\gamma-\frac{1}{4})} \le K_5(B,C_1),$$

if $\gamma \geq 1/3$, which clearly is satisfied. Similarly, we have

$$\begin{aligned}\int_0^1 ([|u|g(cQ)]_x)^2\,dx &\leq 2\int_0^1 (|u|_x)^2 g(cQ)^2\,dx + 2\int_0^1 |u|^2 g'(cQ)^2([cQ]_x)^2\,dx \\ &= 2\int_0^1 [cQ]^{\beta+1}(u_x)^2 g(cQ)^2[cQ]^{-(\beta+1)}\,dx + \frac{2}{\beta^2}\int_0^1 |u|^2 g'(cQ)^2[cQ]^{2(1-\beta)}\Big[([cQ]^\beta)_x\Big]^2\,dx \\ &\leq K_6\int_0^1 E(cQ)(u_x)^2\,dx + K_7M^2\int_0^1 \Big[([cQ]^\beta)_x\Big]^2\,dx,\end{aligned} \tag{108}$$

where we have used that $g(cQ)^2[cQ]^{-(\beta+1)}$ and $g'(cQ)^2[cQ]^{2(1-\beta)}$ are bounded even independent of the bound (99), in view of the special form of the function $g(\cdot)$. We also note that we have used the estimate (100).

The conclusion of combining (105)–(108), where we also apply (87) for estimating the first term of the right hand side of (108) as well as the fourth term on the right hand side of (105), is that

$$\begin{aligned}\int_0^1 (\partial_x[cQ]^\beta)^2\,dx &\leq K_8 + \int_0^t \Big(K_8 t(1+M^2)\int_0^1 (\partial_x[cQ]^\beta)^2\,dx\Big)ds \\ &\leq K_8 + \int_0^t \Big(K_8(1+M)\int_0^1 (\partial_x[cQ]^\beta)^2\,dx\Big)ds,\end{aligned}$$

where $K_8 = K_8(B, C, C_1, \|[c_0Q_0]^\beta\|_{H^1}, \|c_0\|_{H^1}, \|u_0\|_{L^2}, T)$ and we have employed (101). Thus, application of Gronwall's inequality gives the estimate

$$\int_0^1 (\partial_x[cQ]^\beta)^2\,dx \leq K_8\exp\{(1+M)t\} \leq C_3,$$

where we again have used (101). By this, (102) has been proved. □

Lemma 5.2 (Additional regularity)**.** *Under the assumptions of Theorem 3.1,* (99) *and* (100) *we have, for $t \in [0,T]$, the estimate*

$$\int_0^1 [cQ]^{\beta+3}(\partial_x u)^4\,dx \leq \widetilde{C}_4 + C_5\Big(\int_0^1 (\partial_t u)^2\,dx\Big)^2 \tag{109}$$

for constants $\widetilde{C}_4 = \widetilde{C}_4(C_1, A, B, M, T)$ and $C_5 = C_5(C_1, A, B, T)$ determined by (113)*.*

Proof. Integrating the third equation of (73) over the interval $[x,1]$ and using the boundary condition (75) we get

$$\int_x^1 u_t\,dy - (Q^\gamma - |u|g(cQ)) = -E(cQ)u_x.$$

This corresponds to

$$\begin{aligned}u_x &= -\frac{1}{E(cQ)}\int_x^1 u_t\,dy + \frac{Q^\gamma}{E(cQ)} - \frac{|u|g(cQ)}{E(cQ)} \\ &= -[cQ]^{-\beta-1}\int_x^1 u_t\,dy + c^{-\beta-1}Q^{\gamma-\beta-1} - |u|[cQ]^{-\beta-1}g(cQ).\end{aligned}$$

Consequently, we can estimate as follows:

$$\begin{aligned}(u_x)^4[cQ]^{\beta+3} &\leq K_9[cQ]^{\beta+3}\Big([cQ]^{-4\beta-4}\Big(\int_x^1 u_t\,dy\Big)^4 + c^{-4\beta-4}Q^{4\gamma-4\beta-4} + u^4[cQ]^{-4\beta-4}g(cQ)^4\Big) \\ &\leq K_9\Big([cQ]^{-3\beta-1}\Big(\int_x^1 u_t\,dy\Big)^4 + c^{-3\beta-1}Q^{4\gamma-3\beta-1} + u^4[cQ]^{-3\beta-1}g(cQ)^4\Big).\end{aligned}$$

Hence, integrating over $[0,1]$ gives us

$$\begin{aligned}&\int_0^1 (u_x)^4[cQ]^{\beta+3}\,dx \\ &\leq K_9\int_0^1 \Big([cQ]^{-3\beta-1}\Big(\int_x^1 u_t\,dy\Big)^4 + c^{-3\beta-1}Q^{4\gamma-3\beta-1} + u^4[cQ]^{-3\beta-1}g(cQ)^4\Big)\,dx.\end{aligned} \tag{110}$$

Clearly, the last term on the right hand side is bounded in view of (100), the properties of $g(\cdot)$, and $0 < \beta < 1$. Similarly, the middle term is bounded, in view of Assumption (A1) and (99) since

$$c^{-3\beta-1}Q^{4\gamma-3\beta-1} \le C_1^{-1}B\phi(x)^{-\frac{1}{4}\alpha(3\beta+1)}\phi(x)^{\frac{3}{4}\alpha(4\gamma-3\beta-1)} = BC_1^{-1}\phi(x)^{\alpha(3\gamma-1-3\beta)} \le BC_1^{-1}, \tag{111}$$

if

$$\alpha(3\gamma - 1 - 3\beta) \ge 0,$$

that is, $\gamma \ge \beta + \frac{1}{3}$. For the first term on the right hand side of (110) we have by Young's inequality:

$$\begin{aligned} \int_0^1 [cQ]^{-3\beta-1}\Big(\int_x^1 u_t\,dy\Big)^4 dx &\le K_{10}\int_0^1 [cQ]^{-3\beta-1}\Big(\int_0^1 (u_t)^2\,dy\Big)^2 \phi(x)^2\,dx \\ &\le K_{10}\Big(\int_0^1 (u_t)^2\,dy\Big)^2, \end{aligned} \tag{112}$$

for a suitable choice of K_{10}. For the last choice of $K_{10} = K_{10}(C_1, A)$ we have used that

$$[cQ]^{-(3\beta+1)}\phi(x)^2 \le [C_1^{-1}A^{-1}]^{3\beta+1}\phi(x)^{-\alpha(3\beta+1)+2} \le [C_1^{-1}A^{-1}]^{3\beta+1},$$

if the following relation holds

$$-\alpha(3\beta+1) + 2 \ge 0, \qquad \text{i.e. } 2 \ge \alpha(3\beta+1).$$

If $0 < \beta < 1$ it follows that α must satisfy that $0 < \alpha < 1/2$. Thus, from (110)–(112) it follows

$$\int_0^1 (u_x)^4[cQ]^{\beta+3}\,dx \le K_{11}(1+M^4) + K_{11}\Big(\int_0^1 (\partial_t u)^2\,dx\Big)^2, \tag{113}$$

where $K_{11} = K_{11}(C_1, A, B, T)$.

□

Lemma 5.3 (Additional regularity)**.** *Under the assumptions of Theorem 3.1,* (99) *and* (100) *we have, for $t \in [0,T]$, the estimate*

$$\int_0^1 (\partial_t u)^2\,dx + \int_0^t\int_0^1 E(cQ)(u_{tx})^2\,dx\,ds \le C_6, \tag{114}$$

for a constant $C_6 = C_6(C, C_1, A, B, T)$.

Proof. We have from (73) that

$$u_t + (P(Q) - |u|g(cQ))_x = (E(cQ)u_x)_x,$$

which gives

$$\begin{aligned} u_{tt} &= \Big(E(cQ)u_x - P(Q) + |u|g(cQ)\Big)_{xt} \\ &= \Big(E(cQ)_t u_x + E(cQ)u_{xt} - P(Q)_t\Big)_x + \Big(|u|_t g(cQ) + |u|g(cQ)_t\Big)_x, \end{aligned}$$

Multiplying by u_t, integrating over $[0,1]$ and performing integration by parts, where we use the boundary condition (75), we get

$$\begin{aligned} \int_0^1 \partial_t\Big(\frac{1}{2}u_t^2\Big)\,dx = &-\int_0^1 \Big(E(cQ)_t u_x + E(cQ)u_{xt} - P(Q)_t\Big)u_{tx}\,dx \\ &- \int_0^1 \Big(|u|_t g(cQ) + |u|g(cQ)_t\Big)u_{tx}\,dx. \end{aligned} \tag{115}$$

The following equations can be obtained from the second equation of (73):

$$P(Q)_t = -\gamma a^* c Q^{\gamma+1}u_x, \tag{116}$$

and

$$g(cQ)_t = g'(cQ)cQ_t = -a^* g'(cQ)[cQ]^2 u_x, \tag{117}$$

and

$$E(cQ)_t = c^{\beta+1}(Q^{\beta+1})_t = (\beta+1)c^{\beta+1}Q^\beta Q_t = -(\beta+1)a^*[cQ]^{\beta+2}u_x. \tag{118}$$

Note that we in fact have used the equations (116)–(118) when we conclude that the contribution from boundary terms appearing in (115) is zero. Thus, it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \Big(\frac{1}{2}u_t^2\Big)\,dx + \int_0^1 E(cQ)(u_{xt})^2\,dx \\
&= -\int_0^1 E(cQ)_t u_x u_{tx}\,dx + \int_0^1 P(Q)_t u_{tx}\,dx - \int_0^1 |u|_t g(cQ)u_{tx}\,dx - \int_0^1 |u|g(cQ)_t u_{tx}\,dx \\
&= (\beta+1)a^*\int_0^1 [cQ]^{\beta+2}(u_x)^2 u_{tx}\,dx - \gamma a^*\int_0^1 cQ^{\gamma+1}u_x u_{tx}\,dx \\
&\qquad\qquad - \int_0^1 g(cQ)\mathrm{sgn}(u)u_t u_{tx}\,dx + a^*\int_0^1 |u|g'(cQ)[cQ]^2 u_x u_{tx}\,dx.
\end{aligned} \tag{119}$$

For the first term on the right hand side of (119) we use the splitting

$$[cQ]^{\beta+2}(u_x)^2 u_{tx} = \Big([cQ]^{(\beta+3)/2}(u_x)^2\Big)\Big([cQ]^{(\beta+1)/2}u_{tx}\Big).$$

Hence,

$$\int_0^1 [cQ]^{\beta+2}(u_x)^2 u_{tx}\,dx \le \frac{1}{4\varepsilon_1}\int_0^1 [cQ]^{(\beta+3)}(u_x)^4\,dx + \varepsilon_1\int_0^1 E(cQ)(u_{tx})^2\,dx, \tag{120}$$

by application of Cauchy's inequality $\varepsilon > 0$, $ab \le (1/4\varepsilon)a^2 + \varepsilon b^2$. The last term on the right hand side of (120) can be adsorbed on the left hand side of (119), whereas the first is bounded by (109).

Similarly, for the second term on the right hand side of (119) we have

$$cQ^{\gamma+1}u_x u_{tx} = c^{1-\frac{3\beta+5}{4}}Q^{\gamma+1-\frac{3\beta+5}{4}}\Big([cQ]^{(\beta+3)/4}u_x\Big)\Big([cQ]^{(\beta+1)/2}u_{tx}\Big). \tag{121}$$

Two times applications of Cauchy's inequality $\varepsilon > 0$, $ab \le (1/4\varepsilon)a^2 + \varepsilon b^2$ then give us

$$\begin{aligned}
&\int_0^1 cQ^{\gamma+1}u_x u_{tx}\,dx \\
&\quad\le \frac{1}{4\varepsilon_2}\int_0^1 c^{2(1-\frac{3\beta+5}{4})}Q^{2(\gamma+1-\frac{3\beta+5}{4})}[cQ]^{(\beta+3)/2}(u_x)^2\,dx + \varepsilon_2\int_0^1 E(cQ)(u_{tx})^2\,dx \\
&\quad\le \frac{1}{8\varepsilon_2}\int_0^1 c^{4(1-\frac{3\beta+5}{4})}Q^{4(\gamma+1-\frac{3\beta+5}{4})}\,dx + \frac{1}{8\varepsilon_2}\int_0^1 [cQ]^{(\beta+3)}(u_x)^4\,dx + \varepsilon_2\int_0^1 E(cQ)(u_{tx})^2\,dx \\
&\quad\le K_{13} + \frac{1}{8\varepsilon_2}\int_0^1 [cQ]^{(\beta+3)}(u_x)^4\,dx + \varepsilon_2\int_0^1 E(cQ)(u_{tx})^2\,dx,
\end{aligned} \tag{122}$$

where we have used assumption (A1) and (99) to estimate as follows:

$$\int_0^1 c^{4(1-\frac{3\beta+5}{4})}Q^{4(\gamma+1-\frac{3\beta+5}{4})}\,dx \le K_{12}\int_0^1 \phi(x)^{\alpha(1-\frac{3\beta+5}{4})}\cdot\phi(x)^{3\alpha(\gamma+1-\frac{3\beta+5}{4})}\,dx \le K_{13},$$

if

$$\alpha\Big[1 - \frac{3\beta+5}{4} + 3(\gamma+1-\frac{3\beta+5}{4})\Big] > -1,$$

that is,

$$3(\gamma-\beta) > 1 - \frac{1}{\alpha},$$

which clearly holds for $\alpha \in (0,1)$ and $\gamma > \beta$. As far as the third term on the right hand side of (119) is concerned, we note that

$$g(cQ)u_t u_{tx} = a^*k^*\Big([cQ]^{(\beta+1)/2}u_{tx}\Big)\Big(\frac{[cQ]^{(1-\beta)/2}}{k^* + (a^*+k^*)[cQ]}u_t\Big). \tag{123}$$

Hence, we conclude again from Cauchy's inequality and the fact that $\beta \in (0,1)$ that

$$\begin{aligned}
&\int_0^1 g(cQ)\mathrm{sgn}(u)u_t u_{tx}\,dx \\
&\quad\le K_{14}(\varepsilon_3)\int_0^1 \frac{[cQ]^{(1-\beta)}}{(k^*+(a^*+k^*)[cQ])^2}(u_t)^2\,dx + \varepsilon_3\int_0^1 E(cQ)(u_{tx})^2\,dx \\
&\quad\le K_{14}(\varepsilon_3)\int_0^1 (u_t)^2\,dx + \varepsilon_3\int_0^1 E(cQ)(u_{tx})^2\,dx \\
&\quad\le \frac{K_{14}(\varepsilon_3)}{2} + \frac{K_{14}(\varepsilon_3)}{2}\Big(\int_0^1 (u_t)^2\,dx\Big)^2 + \varepsilon_3\int_0^1 E(cQ)(u_{tx})^2\,dx.
\end{aligned} \tag{124}$$

Finally, for the fourth term on the right hand side of (119) we have

$$|u|g'(cQ)[cQ]^2 u_x u_{tx} = \Big(|u|g'(cQ)[cQ]^{2-\frac{\beta+1}{2}}u_x\Big)\Big([cQ]^{\frac{\beta+1}{2}}u_{tx}\Big).$$

We can repeat the arguments similar to those of the estimate of (121) leading to (122).

$$\begin{aligned}
&\int_0^1 |u|g'(cQ)[cQ]^2 u_x u_{tx}\,dx \\
&\le \frac{1}{4\varepsilon_4}\int_0^1 |u|^2 g'(cQ)^2[cQ]^{3-\beta}(u_x)^2\,dx + \varepsilon_4\int_0^1 E(cQ)(u_{tx})^2\,dx \\
&\le \frac{1}{8\varepsilon_4}\int_0^1 |u|^4 g'(cQ)^4[cQ]^{3+\beta-4\beta}\,dx + \frac{1}{8\varepsilon_4}\int_0^1 [cQ]^{\beta+3}(u_x)^4\,dx + \varepsilon_4\int_0^1 E(cQ)(u_{tx})^2\,dx \\
&\le \frac{M^4}{8\varepsilon_4}\int_0^1 g'(cQ)^4[cQ]^{3(1-\beta)}\,dx + \frac{1}{8\varepsilon_4}\int_0^1 [cQ]^{\beta+3}(u_x)^4\,dx + \varepsilon_4\int_0^1 E(cQ)(u_{tx})^2\,dx \\
&\le K_{15}M^4 + \frac{1}{8\varepsilon_4}\int_0^1 [cQ]^{\beta+3}(u_x)^4\,dx + \varepsilon_4\int_0^1 E(cQ)(u_{tx})^2\,dx,
\end{aligned} \tag{125}$$

where we have used the splitting

$$[cQ]^{3-\beta} = [cQ]^{\frac{3+\beta}{2}}\cdot[cQ]^{\frac{3+\beta}{2}-2\beta}$$

and the properties of $g'(\cdot)$. Clearly, $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ can be chosen such that the associated term $\int_0^1 E(cQ)(u_{xt})^2\,dx$ on the right hand side of (120), (122), (124), and (125) can be incorporated in the corresponding term on the left hand side of (119).

Consequently, in view of (119), (120), (122), (124), and (125) in combination with (109) (Lemma 5.2), see also (113), we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 (u_t)^2\,dx + \int_0^1 E(cQ)(u_{xt})^2\,dx \\
&\qquad\le K_{16}(1+M^4) + K_{17}\Big(\int_0^1 (u_t)^2\,dx\Big)^2,
\end{aligned}$$

for appropriate constants K_{16} and K_{17}. That is,

$$\begin{aligned}
&\int_0^1 (u_t)^2\,dx + \int_0^t\int_0^1 E(cQ)(u_{xt})^2\,dx\,ds \\
&\qquad\le K_{16}(1+M^4)t + K_{17}\int_0^t\Big(\int_0^1 (u_t)^2\,dx\Big)^2\,ds.
\end{aligned}$$

Following the arguments of [3, 31] we may conclude that by choosing $\widetilde{T}_4(M) > 0$ sufficiently small such that for all $t \in [0,\widetilde{T}_4]$ we have

$$\frac{1}{\frac{1}{K_{16}(1+M^4)\widetilde{T}_4} - K_{17}\widetilde{T}_4} \le C_6, \tag{126}$$

where C_6 is independent of M. Hence, for $t \in [0,T]$, recall that $T \in (0,T_1]$ and by (95) it follows that $T_1 \le \widetilde{T}_4$, we then have

$$\int_0^1 (u_t)^2\,dx + \int_0^t \int_0^1 E(cQ)(u_{xt})^2\,dx\,ds \\ \le K_{16}(1+M^4)\widetilde{T}_4 + K_{17}\int_0^t \Big(\int_0^1 (u_t)^2\,dx\Big)^2\,ds.$$

Application of (a nonlinear version of) Gronwall's inequality then gives

$$\int_0^1 (u_t)^2\,dx + \int_0^t \int_0^1 E(cQ)(u_{xt})^2\,dx\,ds \le \frac{1}{\frac{1}{K_{16}(1+M^4)\widetilde{T}_4} - K_{17}t} \le \frac{1}{\frac{1}{K_{16}(1+M^4)\widetilde{T}_4} - K_{17}\widetilde{T}_4} \le C_6,$$

where C_6 has been defined by (126). □

6. Proof of Lemma 4.2 and Lemma 4.3

Equipped with the results of Section 5, we shall return to the proof of Lemma 4.2 and Lemma 4.3. First we give a proof of Lemma 4.3 where we rely on the result of Lemma 5.3. Then we shall derive Lemma 6.1 and 6.2, the first one provides control over the L^2 continuity in time of u, i.e., estimate of $\|u(\cdot,t)-u_0(\cdot)\|_2$, whereas the second one gives control over the quantity $|\int_0^t\int_0^x u_t\,dx\,dt|$. Finally, armed with these estimates we can give a proof of Lemma 4.2. We end this section by Lemma 6.3 which provides more L^2 continuity type of estimates with respect to time for, respectively, Q, u, and Eu_x. Then, Lemma 6.4 gives an upper pointwise bound for Eu_x as well as BV estimates for Eu_x, u, and Q.

6.1. **Proof of Lemma 4.3.**

Proof. We obtain from the third equation of (73) by integrating over $[x,1]$ and employing the boundary condition (75)

$$u_x = -\frac{1}{E(cQ)}\int_x^1 u_t\,dy + \frac{Q^\gamma}{E(cQ)} - \frac{|u|g(cQ)}{E(cQ)}.$$

This implies that

$$\int_0^1 |u_x|\,dx \le \int_0^1 \frac{1}{E(cQ)}\Big|\int_x^1 u_t\,dy\Big|\,dx + \int_0^1 \frac{Q^\gamma}{E(cQ)}\,dx + \int_0^1 \frac{|u|g(cQ)}{E(cQ)}\,dx := A_1 + A_2 + A_3. \quad (127)$$

Then we estimate as follows:

$$\begin{aligned} A_2 &= \int_0^1 Q^{\gamma-\beta-1}c^{-\beta-1}\,dx \le C_1^{-(\beta+1)}\int_0^1 \phi(x)^{-\frac14\alpha(\beta+1)}Q^{\gamma-\beta-1}\,dx \\ &\le D_2\int_0^1 \phi(x)^{-\frac14\alpha(\beta+1)}\cdot\phi(x)^{\frac34\alpha(\gamma-\beta-1)}\,dx = D_2\int_0^1 \phi(x)^{-\frac14\alpha(\beta+1)+\frac34\alpha(\gamma-\beta-1)}\,dx \\ &= D_2\int_0^1 \phi(x)^{\frac14\alpha[-\beta-1+3\gamma-3\beta-3)]}\,dx = D_2\int_0^1 \phi(x)^{\alpha[\frac34\gamma-\beta-1]}\,dx \le D_2(C_1,A,B), \end{aligned} \quad (128)$$

for some constant D_2 which is independent of M and where we have used assumption (A1) which implies that

$$C_2^{-1}\phi(x)^{-\alpha/4} \le c^{-1} \le C_1^{-1}\phi(x)^{-\alpha/4}, \qquad \text{i.e.} \qquad c^{-(\beta+1)} \le C_1^{-(\beta+1)}\phi(x)^{-\frac{\alpha}{4}(\beta+1)},$$

together with estimate (93). Here we must also assume that $\alpha[\frac34\gamma-\beta-1] > -1$, that is,

$$\frac34\gamma > 1+\beta-\frac{1}{\alpha},$$

which follows from Assumption (A3). Furthermore, by combining Assumption (A1) and (93) we obtain

$$\frac{1}{2BC_2}\phi(x)^{-\alpha\beta} \le [cQ]^{-\beta} \le \frac{3}{C_1A}\phi(x)^{-\alpha\beta}. \quad (129)$$

Using this estimate together with (97) we can estimate as follows:

$$
\begin{aligned}
A_3 &= \int_0^1 \frac{|u|g(cQ)}{E(cQ)}\,dx = \int_0^1 |u| \frac{a^* k^* [cQ]^{-\beta}}{k^* + (a^* + k^*)cQ}\,dx \le D_3 \int_0^1 |u|[cQ]^{-\beta}\,dx \\
&\le D_3 \int_0^1 |u|\phi(x)^{-\alpha\beta}\,dx \le D_3 M^{1/2} \int_0^1 \phi(x)^{-\alpha\beta} \cdot |u|^{1/2}\,dx \\
&\le D_3 M^{1/2} \Big(\int_0^1 \phi(x)^{-\alpha\beta q}\,dx\Big)^{1/q} \Big(\int_0^1 |u|^{\frac{1}{2}p}\,dx\Big)^{1/p}, \qquad (p,q) = (4, \frac{4}{3}), \\
&\le D_3 M^{1/2} \Big(\int_0^1 |u|^2\,dx\Big)^{1/4} \le D_3(A, C_1, C) M^{1/2},
\end{aligned}
\tag{130}
$$

where we have used Hölder's inequality with $p = 4$ and $q = \frac{4}{3}$, estimate (87) of Lemma 4.1, and the fact that $-\alpha\beta q = -\alpha\beta\frac{4}{3} > -1$ or $\alpha\beta < \frac{3}{4}$. As above D_3 is a suitable constant that is independent of M but may depend on C_1, C_2, A, and B. Finally, we see that

$$
\begin{aligned}
A_1 &= \int_0^1 \frac{1}{E(cQ)} \Big| \int_x^1 u_t\,dy \Big|\,dx \\
&\le D_1 \int_0^1 [cQ]^{-\beta-1}\phi(x)^{1/2} \Big(\int_0^1 |u_t|^2\,dy\Big)^{1/2} dx \le D_1 \int_0^1 [cQ]^{-(\beta+1)}\phi(x)^{1/2}\,dx \\
&\le D_1 \int_0^1 \phi(x)^{1/2-\alpha(\beta+1)}\,dx \le D_1(C_6, A, C_1),
\end{aligned}
\tag{131}
$$

for a suitable constant $D_1 = D_1(C_6, A, C_1)$ (note dependence on C_6 since we have used Lemma 5.3), if $1/2 - \alpha(\beta+1) > -1$, that is, $\alpha(\beta+1) < \frac{3}{2}$. Hence, in view of (127), (128), (130), and (131) the conclusion is that

$$
\int_0^1 |u_x|\,dx \le D + DM^{1/2}
$$

for a constant D which is independent of M. In other words, in view of Sobolev's embedding theorem that $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$ and Lemma 4.1

$$
\|u\|_\infty \le \int_0^1 |u|\,dx + \int_0^1 |u_x|\,dx \le \sqrt{2C} + D + DM^{1/2}.
$$

Clearly, for positive constants $D_4 = \sqrt{2C} + D$ and $D_5 = D$ then

$$
\frac{1}{M}(D_4 + D_5 M^{1/2}) \le \frac{1}{2}, \tag{132}
$$

for a suitable large M. Hence, $\|u\|_\infty \le \frac{1}{2}M$ for a suitable choice of M, and the lemma has been proved. □

6.2. **More lemmas under the assumption of** (93).

Lemma 6.1. *Under the assumptions of Theorem 3.1 and* (93)*, there exists a positive constant* $D_1 = D_1(C, C_1, C_2, A, B, \|u_0\|_{H^1}, T)$ *such that, for all* $t \in [0,T]$

$$
\int_0^1 |u(x,t) - u_0(x)|^2\,dx \le D_1 t. \tag{133}
$$

Proof. Clearly, from the third equation of (73) we get

$$
(u - u_0)_t + (Q^\gamma - |u|g(cQ))_x = (E(cQ)u_x)_x.
$$

Multiplying by $(u - u_0)$ and integrating over the spatial domain $[0,1]$ together with application of integration by parts and use of the boundary conditions (75) and the fact that $g(0) = 0$, we arrive at the following form

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{1}{2}(u-u_0)^2\,dx + \int_0^1 E(cQ)(u_x)^2\,dx \\
&= \int_0^1 E(cQ)u_x u_{0,x}\,dx + \int_0^1 Q^\gamma u_x\,dx - \int_0^1 |u|g(cQ)u_x\,dx \\
&\qquad - \int_0^1 Q^\gamma u_{0,x}\,dx + \int_0^1 |u|g(cQ)u_{0,x}\,dx.
\end{aligned} \tag{134}
$$

We can estimate as follows:

$$
E(cQ)u_x u_{0,x} = E(cQ)^{1/2}u_{0,x}\cdot E(cQ)^{1/2}u_x,
$$

hence, by Cauchy's inequality and use of (93) we get

$$
\begin{aligned}
\int_0^1 E(cQ)u_x u_{0,x}\,dx &\le \frac{1}{4\varepsilon}\int_0^1 E(cQ)u_{0,x}^2\,dx + \varepsilon\int_0^1 E(cQ)u_x^2\,dx \\
&\le \frac{1}{4\varepsilon}\max(E(cQ))\int_0^1 u_{0,x}^2\,dx + \varepsilon\int_0^1 E(cQ)u_x^2\,dx \\
&\le D_\varepsilon + \varepsilon\int_0^1 E(cQ)u_x^2\,dx.
\end{aligned} \tag{135}
$$

The first term on the right hand side of (135) is bounded due to (93), Assumption (A1), and Assumption (A4). Moreover, for the second term on the right hand side of (134) we have

$$
Q^\gamma u_x = Q^\gamma [cQ]^{-(\beta+1)/2}\cdot [cQ]^{(\beta+1)/2}u_x.
$$

Hence,

$$
\begin{aligned}
\int_0^1 Q^\gamma u_x\,dx &\le \frac{1}{4\varepsilon}\int_0^1 Q^{2\gamma}[cQ]^{-(\beta+1)}\,dx + \varepsilon\int_0^1 E(cQ)u_x^2\,dx \\
&\le D_\varepsilon \int_0^1 \phi(x)^{\alpha(\frac{3}{2}\gamma-\beta-1)}\,dx + \varepsilon\int_0^1 E(cQ)u_x^2\,dx.
\end{aligned} \tag{136}
$$

For the first term on the right hand side of (136) we have used an estimate similar to (129) which implies that

$$
Q^{2\gamma}[cQ]^{-(\beta+1)} \le D(C_1,A)\phi(x)^{\frac{3}{2}\gamma\alpha}\phi(x)^{-\alpha(\beta+1)} \le D(C_1,A)\phi(x)^{\alpha(\frac{3}{2}\gamma-\beta-1)}.
$$

Consequently, the first term on the right hand side of (136) is bounded if $\alpha(\frac{3}{2}\gamma-\beta-1) > -\alpha > -1$, i.e., $\gamma > \frac{2}{3}\beta$. For the third term on the right hand side of (134) we have

$$
\int_0^1 |u|g(cQ)u_x\,dx \le \int_0^1 |u-u_0|g(cQ)u_x\,dx + \int_0^1 |u_0|g(cQ)u_x\,dx \tag{137}
$$

The first term on the right hand side of (137) can be treated by the same technique as for Lemma 4.1, the second term can be handled similarly by employing that $u_0 \in L^2([0,1])$. The two last terms on the right hand side of (134) can obviously be estimated in the same manner. Using these estimates in combination with (134) implies (133). □

Lemma 6.2. *Under the assumptions of Theorem 3.1 and* (93)*, there exists a positive constant* $\widetilde{E}_1 = \widetilde{E}_1(M, D_1, T)$ *and a positive integer* m *with* $m > \frac{1}{1-\alpha\beta} > 1$ *such that, for all* $t \in [0,T]$

$$
\Big|\int_0^t \int_0^x \partial_t u\,dy\,ds\Big| \le \widetilde{E}_1\phi(x)^{\frac{m-1}{m}}t^{1/m}, \qquad 0 \le x \le \frac{1}{2}. \tag{138}
$$

Proof. We can estimate as follows by using Hölder's inequality:

$$
\begin{aligned}
&\Big|\int_0^t\int_0^x \partial_t u\,dy\,ds\Big| = \Big|\int_0^x [u(y,t)-u_0(y)]\,dy\Big| \\
&\le \int_0^x |u(y,t)-u_0(y)|\,dy \le \int_0^1 \chi_{[0,x]}(y)|u(y,t)-u_0(y)|\,dy \\
&\le 2^{\frac{m-1}{m}}\phi(x)^{\frac{m-1}{m}}\Big(\int_0^1 |u(y,t)-u_0(y)|^m\,dy\Big)^{\frac{1}{m}}, \qquad x\in[0,\frac{1}{2}], \\
&\le 2^{\frac{m-2}{m}}\cdot 2^{\frac{m-1}{m}}\phi(x)^{\frac{m-1}{m}}M^{\frac{m-2}{m}}\Big(\int_0^1 |u(y,t)-u_0(y)|^2\,dy\Big)^{\frac{1}{m}} \\
&\le 2^{\frac{2m-3}{m}}D_1^{\frac{1}{m}}\phi(x)^{\frac{m-1}{m}}M^{\frac{m-2}{m}}t^{\frac{1}{m}} = \widetilde{E}_1(M,D_1,T)\phi(x)^{\frac{m-1}{m}}t^{\frac{1}{m}},
\end{aligned}
$$

where we use $p=m$ and $q=m/(m-1)$. Moreover, we have used that $\|u_0\|_\infty, \|u\|_\infty \le M$ from Lemma 4.3 as well as the estimate of Lemma 6.1.

□

Remark 6.1. *In a similar manner we can obtain the estimate*

$$
\Big|\int_0^t\int_x^1 \partial_t u\,dy\,ds\Big| \le \widetilde{E}_1\phi(x)^{\frac{m-1}{m}}t^{1/m}, \qquad \frac{1}{2}\le x\le 1. \tag{139}
$$

Note that the M dependence of $\widetilde{E_1}$ (as well as other constants) will be controlled in the following proof of Lemma 4.2 by the "small" time $T>0$ where $T<T_1$ and T_1 is defined by (95). In particular, suitable small times $\widetilde{T}_1$, $\widetilde{T}_2$, and $\widetilde{T}_3$ will be introduced in the proof below to eliminate dependence on M.

6.3. **Proof of Lemma 4.2.** Now we are in a position where we can give a proof of Lemma 4.2.

Proof. We start with the following equality which follows directly by combing the second and third equation of (73)

$$
[cQ]^\beta + a^*\beta\int_0^t Q^\gamma\,ds = [cQ]_0^\beta - a^*\beta\int_0^t\int_0^x u_t\,dy\,ds + a^*\beta\int_0^t |u|g(cQ)\,ds. \tag{140}
$$

Here we have combined

$$
\frac{1}{a^*\beta}([cQ]^\beta)_t + E(cQ)u_x = 0,
$$

and

$$
\int_0^x u_t\,dy + [Q^\gamma - |u|g(cQ)] = E(cQ)u_x.
$$

First, we observe that from Lemma 6.2 we have

$$
\Big|\int_0^t\int_0^x u_t\,dy\,ds\Big| \le \widetilde{E}_1(M)\phi(x)^{\frac{m-1}{m}}t^{1/m}. \tag{141}
$$

Moreover,

$$
\Big|\int_0^t |u|g(cQ)\,ds\Big| \le a^*M\int_0^t [cQ]\,ds \le 2a^*BC_2M\phi(x)^\alpha t = \widetilde{E}_2(M)\phi(x)^\alpha t. \tag{142}
$$

Consequently, in view of (140) where we set $a^*\beta=1$ without loss of generality, we can employ (82) from Assumption (A2) together with (141) and (142) and estimate as follows:

$$
\begin{aligned}
[cQ]^\beta + \int_0^t Q^\gamma\,ds &\ge [C_2A]^\beta\phi(x)^{\beta\alpha} - \widetilde{E}_1\phi(x)^{\frac{m-1}{m}}t^{1/m} - \widetilde{E}_2 t\phi(x)^\alpha \\
&\ge [C_2A]^\beta\phi(x)^{\beta\alpha} - \widetilde{E}_1\phi(x)^{\alpha\beta}t^{1/m} - \widetilde{E}_2 t^{1/m}\phi(x)^{\alpha\beta} \\
&= \Big([C_2A]^\beta - [\widetilde{E}_1+\widetilde{E}_2]t^{1/m}\Big)\phi(x)^{\beta\alpha},
\end{aligned} \tag{143}
$$

since $\phi(x)^\alpha \leq \phi(x)^{\alpha\beta}$ and $\phi(x)^{(m-1)/m} \leq \phi(x)^{\alpha\beta}$ since $m > 1/(1-\alpha\beta)$, that is, $(m-1)/m > \alpha\beta$ as assumed in Lemma 6.2. We have also used that $t^{1/m} > t$ for $t < 1$ and $m > 1$. Now we can define

$$\widetilde{T}_1 = \Big(\frac{[C_2A]^\beta - (\frac{2}{3}[C_2A])^\beta}{\widetilde{E}_1 + \widetilde{E}_2}\Big)^m > 0. \tag{144}$$

For $0 \leq t \leq \widetilde{T}_1$ we get from (143)

$$[cQ]^\beta + \int_0^t Q^\gamma\, ds \geq \Big(\frac{2}{3}[C_2A]\Big)^\beta \phi(x)^{\alpha\beta}. \tag{145}$$

To ensure the lower limit of $[cQ]$ from (145) we must determine an upper limit for $\int_0^t Q^\gamma\, ds$ for sufficient small t. For this purpose we set

$$Z(t) = \int_0^t Q^\gamma(x,s)\, ds. \tag{146}$$

Then an inequality equation for $Z(t)$ can be derived. However, first we note that similar to the proof of (143) we get

$$[cQ]^\beta + \int_0^t Q^\gamma\, ds \leq \Big([C_1B]^\beta + [\widetilde{E}_1 + \widetilde{E}_2]t^{1/m}\Big)\phi(x)^{\beta\alpha}, \tag{147}$$

Now we can define

$$\widetilde{T}_2 = \Big(\frac{(\frac{3}{2}[C_1B])^\beta - [C_1B]^\beta}{\widetilde{E}_1 + \widetilde{E}_2}\Big)^m > 0. \tag{148}$$

For $0 \leq t \leq \widetilde{T}_2$ we get from (147)

$$[cQ]^\beta \leq \Big(\frac{3}{2}[C_1B]\Big)^\beta \phi(x)^{\beta\alpha} \tag{149}$$

Now we return to the task of deriving a lower limit for $[cQ]$. We again follow along the line of [3, 31]. From (147) and (146) we get

$$c^\beta Z'(t)^{\frac{\beta}{\gamma}} + Z(t) \leq \widetilde{K}(t)\phi(x)^{\beta\alpha},$$

that is,

$$Z'(t)^{\frac{\beta}{\gamma}} + c^{-\beta}Z(t) \leq \widetilde{K}(t)\phi(x)^{\beta\alpha}c^{-\beta} \leq C_1^{-\beta}\widetilde{K}(t)\phi(x)^{\beta\alpha}\phi(x)^{-\alpha\beta/4} \leq C_1^{-\beta}\widetilde{K}(t)\phi(x)^{\frac{3}{4}\beta\alpha} := \widetilde{K}_2(t),$$

for $\widetilde{K}(t) = ([C_1B]^\beta + [\widetilde{E}_1 + \widetilde{E}_2]t^{1/m})$. Clearly, $Z(0) = 0$ and then we can deduce, by assuming that $0 < \beta < \gamma$ that

$$Z'(t)^{\frac{\beta}{\gamma}} + K_1 Z(t) \leq \widetilde{K}_2(t) \leq \widetilde{K}_2(\widetilde{T}_2), \qquad K_1 = c^{-\beta}, \qquad \widetilde{K}_2 = C_1^{-\beta}\widetilde{K}(t)\phi(x)^{\frac{3}{4}\beta\alpha}, \tag{150}$$

for $t \in [0, \widetilde{T}_2]$. That is, we have an ODE inequality of the form

$$Z'(t)^{1/p} + AZ(t) \leq B, \qquad p = \frac{\gamma}{\beta} > 1. \tag{151}$$

The solution of $Z'(t)^{1/p} + AZ(t) = B$ can be found by writing this ODE in the form

$$Z'(t) = (B - AZ(t))^p.$$

Introducing $U(t) = B - AZ(t)$ we get

$$U'(t) = -AU^p, \qquad U(t=0) = B.$$

This gives the solution

$$\frac{1}{1-p}U^{1-p}\Big|_B^U = -At,$$

that is,

$$B\Big(1 - \frac{1}{[1 + (p-1)AtB^{p-1}]^{\frac{1}{p-1}}}\Big) = AZ.$$

From this it follows that the following inequality holds

$$AZ \le B\Big(1 - \frac{1}{[1+(p-1)AtB^{p-1}]^{\frac{1}{p-1}}}\Big), \qquad t \in [0, \widetilde{T}_2]. \tag{152}$$

In view of (150), (151) and (152) we get the inequality

$$\begin{aligned} Z(t) &\le c^{\beta}[C_1^{-\beta}\widetilde{K}(\widetilde{T}_2)\phi(x)^{\frac{3}{4}\alpha\beta}]\Big(1 - \frac{1}{(1+(p-1)[c^{-\beta}][C_1^{-\beta}\widetilde{K}(\widetilde{T}_2)\phi(x)^{\frac{3}{4}\alpha\beta}]^{p-1}t)^{\frac{1}{p-1}}}\Big) \\ &\le C_2^{\beta}C_1^{-\beta}\widetilde{K}(\widetilde{T}_2)\phi(x)^{\alpha\beta}\Big(1 - \frac{1}{(1+(p-1)[c^{-\beta}][C_1^{-\beta}\widetilde{K}(\widetilde{T}_2)\phi(x)^{\frac{3}{4}\alpha\beta}]^{p-1}t)^{\frac{1}{p-1}}}\Big). \end{aligned} \tag{153}$$

Now we observe that $p-1 = \frac{\gamma-\beta}{\beta} > 0$, hence

$$[c^{-\beta}][\phi(x)^{\frac{3}{4}\alpha\beta}]^{\frac{\gamma-\beta}{\beta}} \le C_1^{-\beta}\phi(x)^{-\frac{\alpha\beta}{4}}\phi(x)^{\frac{3}{4}\alpha(\gamma-\beta)} \le C_1^{-\beta}\phi(x)^{\alpha(\frac{3}{4}\gamma-\beta)} \le C_1^{-\beta},$$

for $\gamma \ge \frac{4}{3}\beta$. From this estimate it's clear that we can get the expression

$$(1+(p-1)[c^{-\beta}][C_1^{-\beta}\widetilde{K}(\widetilde{T}_2)\phi(x)^{\frac{3}{4}\alpha\beta}]^{p-1}t)^{\frac{1}{p-1}}$$

as close to 1 (from above) as desirable for a small enough time interval (that depends on M due to the appearance of $\widetilde{K}$). Thus, is is also clear that for small enough times, let's say, $t \in [0, \widetilde{T}_3]$ for $\widetilde{T}_3 = \widetilde{T}_3(M) > 0$ we have for the right hand side of (153)

$$\begin{aligned} &C_2^{\beta}C_1^{-\beta}\widetilde{K}(\widetilde{T}_2)\Big(1 - \frac{1}{(1+(p-1)[c^{-\beta}][C_1^{-\beta}\widetilde{K}(\widetilde{T}_2)\phi(x)^{\frac{3}{4}\alpha\beta}]^{p-1}t)^{\frac{1}{p-1}}}\Big) \\ &\qquad \le \Big(\frac{2}{3}[C_2A]\Big)^{\beta} - \Big(\frac{1}{2}[C_2A]\Big)^{\beta}. \end{aligned} \tag{154}$$

Inserting this in (153) gives that

$$-Z(t) \ge -\phi(x)^{\alpha\beta}\Big[\Big(\frac{2}{3}[C_2A]\Big)^{\beta} - \Big(\frac{1}{2}[C_2A]\Big)^{\beta}\Big].$$

Employing this in (145) we get

$$[cQ]^{\beta} \ge -Z(t) + \Big(\frac{2}{3}[C_2A]\Big)^{\beta}\phi(x)^{\alpha\beta} \ge \Big(\frac{1}{2}[C_2A]\Big)^{\beta}\phi(x)^{\alpha\beta}. \tag{155}$$

In view of (155) and (149) we have

$$\Big(\frac{1}{2}[AC_2]\Big)^{\beta}\phi(x)^{\alpha\beta} \le [cQ]^{\beta} \le \Big(\frac{3}{2}[BC_1]\Big)^{\beta}\phi(x)^{\beta\alpha}.$$

In other words,

$$\Big(\frac{1}{2}[AC_2]\Big)\phi(x)^{\alpha} \le [cQ] \le \Big(\frac{3}{2}[BC_1]\Big)\phi(x)^{\alpha},$$

which clearly, in view of the assumption (79), gives us

$$\Big(\frac{1}{2}A\Big)\phi(x)^{\frac{3}{4}\alpha} \le Q \le \Big(\frac{3}{2}B\Big)\phi(x)^{\frac{3}{4}\alpha}.$$

Hence, the desired estimate (94) has been obtained. □

6.4. **More regularity results in space and time under the assumptions of Theorem 3.1.** In the next lemma we obtain L^2-continuity in time estimates for the quantities (Q, u, E).

Lemma 6.3. *Under the assumptions of Theorem 3.1, we have for* $0 < s < t < T_1$ *and appropriate constants* $\widetilde{F}_1$, F_2, *and* $\widetilde{F}_3$ *the following estimates:*

$$\int_0^1 |Q(x,t) - Q(x,s)|^2\,dx \le \widetilde{F}_1|t-s|^2, \tag{156}$$

$$\int_0^1 |u(x,t) - u(x,s)|^2\,dx \le F_2|t-s|^2, \tag{157}$$

$$\int_0^1 |E(cQ)u_x(x,t) - E(cQ)u_x(x,s)|^2\,dx \le \widetilde{F}_3|t-s|. \tag{158}$$

Proof. We have, by using the second equation of (73) and Hölder's inequality where we tactically have assumed $s < t$,

$$\begin{aligned}
\int_0^1 |Q(x,t) - Q(x,s)|^2\,dx &= \int_0^1 \Big|\int_s^t Q_\xi(x,\xi)\,d\xi\Big|^2\,dx \\
&= a^* \int_0^1 \Big|\int_0^{T_1} \chi_{[s,t]}(\xi)(cQ^2u_x)(x,\xi)\,d\xi\Big|^2\,dx \\
&\le a^* \int_0^1 (t-s)\Big(\int_s^t |(cQ^2u_x)(x,\xi)|^2\,d\xi\Big)\,dx = a^*|t-s|\int_s^t\int_0^1 c^2Q^4u_x^2\,dx\,d\xi \\
&= a^*|t-s|\int_s^t\int_0^1 (c^{\frac{1-\beta}{2}}Q^{\frac{5-\beta}{2}})[cQ]^{\frac{\beta+3}{2}}(u_x)^2\,dx\,d\xi \\
&\le \frac{a^*}{2}|t-s|\Big[\int_s^t\Big(\int_0^1 c^{1-\beta}Q^{5-\beta}\,dx + \int_0^1 [cQ]^{\beta+3}(u_x)^4\,dx\Big)\,d\xi\Big] \\
&\le F_1(B,C_2)|t-s|\int_s^t\Big(\int_0^1 \phi(x)^{\alpha(4-\beta)}\,dx + \int_0^1 [cQ]^{\beta+3}(u_x)^4\,dx\Big)\,d\xi \\
&\le \widetilde{F}_1(B,C_2,\widetilde{C}_4,C_5,C_6)|t-s|^2.
\end{aligned} \tag{159}$$

Here we have employed Young's inequality, Lemma 5.2, and Lemma 5.3 as well as Assumption (A2) and the pointwise estimate of Q given by Lemma 4.2. Next, we consider (157). We get by Hölder's inequality

$$\begin{aligned}
\int_0^1 |u(x,t) - u(x,s)|^2\,dx &= \int_0^1 \Big|\int_s^t u_\xi(x,\xi)\,d\xi\Big|^2\,dx \\
&\le |t-s|\int_s^t\int_0^1 u_\xi^2\,dxd\xi \le C_6|t-s|^2,
\end{aligned}$$

where we have used Lemma 5.3 again. Finally, for (158) we estimate as follows by using Young's inequality:

$$\begin{aligned}
\int_0^1 |E(cQ)u_x(x,t) - E(cQ)u_x(x,s)|^2\,dx &= \int_0^1 \Big|\int_s^t \partial_\xi(E(cQ)u_x)(x,\xi)\,d\xi\Big|^2\,dx \\
&\le |t-s|\int_0^1 \Big|\int_s^t \Big(E(cQ)_\xi u_x + E(cQ)u_{x\xi}\Big)(x,\xi)d\xi\Big|\,dx \\
&= |t-s|\int_0^1 \Big|\int_s^t \Big(-(\beta+1)a^*[cQ]^{\beta+2}(u_x)^2 + E(cQ)u_{x\xi}\Big)(x,\xi)d\xi\Big|\,dx \\
&\le F_3|t-s|\int_0^{T_1}\int_0^1 \Big[[cQ]^{\beta+1}[cQ]^{\beta+3}(u_x)^4 + E(cQ)E(cQ)(u_{xt})^2\Big]\,dx\,dt \\
&\le \widetilde{F}_3(B,C_2,\widetilde{C}_4,C_5,C_6)|t-s|,
\end{aligned}$$

by using Lemma 5.2, and Lemma 5.3 and the pointwise upper bound for $[cQ]$ to obtain the last inequality. □

Corollary 6.1. *Under the assumptions of Theorem 3.1, we get for $0 < s < t < T_1$ that*

$$\int_0^1 |n(x,t) - n(x,s)|^2\,dx \le (a^*)^4\widetilde{F}_1|t-s|^2, \tag{160}$$

where $\widetilde{F}_1$ is the constant from (156).

Proof. First we recall from (68) that n satisfies the equation $n_t + cn^2u_x = 0$ and that n and Q are related by

$$\frac{n}{Q} = a^* - cn, \tag{161}$$

see (72), and that Corollary 4.1 implies that

$$cn \le a^*. \tag{162}$$

Hence, we can follow along the same line as for the estimate (156) and calculate as follows:

$$\begin{aligned}
\int_0^1 |n(x,t)-n(x,s)|^2\,dx &= \int_0^1 \Big|\int_s^t n_\xi(x,\xi)\,d\xi\Big|^2\,dx \\
&= \int_0^1 \Big|\int_0^{T_1} \chi_{[s,t]}(\xi)(cn^2u_x)(x,\xi)\,d\xi\Big|^2\,dx \\
&\le \int_0^1 (t-s)\Big(\int_s^t |(cn^2u_x)(x,\xi)|^2\,d\xi\Big)\,dx \le |t-s|\max\Big(\frac{n}{Q}\Big)^4 \int_s^t\int_0^1 c^2Q^4u_x^2\,dx\,d\xi \\
&\le (a^*)^4F_1(B,C_2)|t-s|\int_s^t\Big(\int_0^1 \phi(x)^{\alpha(4-\beta)}\,dx + \int_0^1 [cQ]^{\beta+3}(u_x)^4\,dx\Big)\,d\xi \\
&\le (a^*)^4\widetilde{F}_1(B,C_2,\widetilde{C}_4,C_5,C_6)|t-s|^2,
\end{aligned}$$

where we have used (161) and (162) and the calculations of the last part of (159). □

Lemma 6.4. *Under the assumptions of Theorem 3.1 and for $t \in [0,T_1]$, we get*

$$\|E(cQ)u_x(x,t)\|_{L^\infty([0,1]\times[0,T_1])} \le \widetilde{F}_4, \tag{163}$$

$$\int_0^1 |\partial_x(E(cQ)\partial_x u)|\,dx \le \widetilde{F}_5, \tag{164}$$

$$\int_0^1 |\partial_x u|\,dx \le \widetilde{F}_6, \tag{165}$$

$$\int_0^1 |\partial_x Q|\,dx \le F_7. \tag{166}$$

Proof. The estimate (165) follows as a byproduct of the proof of Lemma 4.3. The estimate (163) follows from (164) since

$$|[E(cQ)u_x](x,t)| = |\int_0^x (E(cQ)u_y)_y\,dy| \le \int_0^1 |(E(cQ)u_y)_y|\,dy \le \widetilde{F}_5, \qquad t\in[0,T_1].$$

From the momentum equation we have

$$(E(cQ)u_x)_x = u_t - (Q^\gamma)_x + (|u|g(cQ))_x, \tag{167}$$

The estimate (164) can be obtained by observing:

$$\begin{aligned}
&\int_0^1 |(E(cQ)u_x)_x|\,dx \le \int_0^1 |u_t|\,dx + \int_0^1 |(Q^\gamma)_x|\,dx + \int_0^1 |u|_x g(cQ)\,dx + \int_0^1 |u||g(cQ)_x|\,dx \\
&\le \sqrt{C_6} + \gamma\int_0^1 |Q^{\gamma-1}Q_x|\,dx + \max g(cQ)\int_0^1 |u_x|\,dx + \int_0^1 |u|g'(cQ)[cQ]_x\,dx \\
&\le \sqrt{C_6} + \frac{\gamma}{\beta}\int_0^1 |Q^{\gamma-\beta}([cQ]^\beta c^{-\beta})_x|\,dx + \widetilde{F}_6 + \frac{1}{\beta}\int_0^1 |u|g'(cQ)[cQ]^{1-\beta}([cQ]^\beta)_x\,dx \\
&\le \sqrt{C_6} + \frac{\gamma}{\beta}\int_0^1 |Q^{\gamma-\beta}c^{-\beta}([cQ]^\beta)_x|\,dx + \gamma\int_0^1 |Q^\gamma c^{-1}c_x|\,dx \\
&\qquad + \widetilde{F}_6 + \frac{1}{2\beta}\int_0^1 |u|^2g'(cQ)^2[cQ]^{2-2\beta}\,dx + \frac{1}{2\beta}\int_0^1 ([cQ]^\beta)_x^2\,dx \\
&\le \sqrt{C_6} + F_9(B,C,C_1,C_3) + \widetilde{F}_6 := \widetilde{F}_5.
\end{aligned}$$

Here Cauchy's inequality has been used repeatedly, Lemma 4.1, Lemma 4.2, Lemma 5.1, Lemma 5.3, and Assumptions (A1)–(A4). Finally, the estimate (166) follows since

$$\begin{aligned}\int_0^1 |Q_x|\,dx &\le \int_0^1 |[cQ]_x \frac{1}{c}|\,dx + \int_0^1 |[cQ](c^{-1})_x|\,dx\\ &\le \frac{1}{\beta}\int_0^1 |c^{-\beta}Q^{1-\beta}([cQ]^\beta)_x|\,dx + \int_0^1 c^{-1}Q|c_x|\,dx\\ &\le \frac{1}{2\beta}\int_0^1 ([cQ]^\beta)_x^2\,dx + \frac{1}{2\beta}\int_0^1 c^{-2\beta}Q^{2-2\beta}\,dx + \int_0^1 c^{-1}Q|c_x|\,dx \le F_7,\end{aligned}$$

where we have used Lemma 4.2, Lemma 5.1, and Assumptions (A1)–(A4). □

Corollary 6.2. *Under the assumptions of Theorem 3.1 and for* $t \in [0,T_1]$, *we get*

$$\int_0^1 |\partial_x n|\,dx \le F_8. \tag{168}$$

Proof. We have that $cQ = cn/(a^* - cn)$, consequently,

$$\partial_x([cQ]^\beta) = \beta\Big(\frac{cn}{a^*-cn}\Big)^{\beta-1}\frac{a^*}{(a^*-cn)^2}\partial_x[cn], \tag{169}$$

from which we get

$$\partial_x n = \frac{1}{a^*\beta}(a^*-cn)^{\beta+1}\frac{n^{1-\beta}}{c^\beta}\partial_x([cQ]^\beta) - \frac{n}{c}\partial_x c.$$

Hence, it follows easily that

$$\begin{aligned}\int_0^1 |\partial_x n|\,dx &\le F_{81}\int_0^1 \phi(x)^{\frac{1}{2}\alpha}|\partial_x c|\,dx + F_{82}\int_0^1 |\partial_x([cQ]^\beta)|\phi(x)^{\alpha(\frac{3}{4}-\beta)}\,dx\\ &\le F_{81} + F_{82}\Big(\int_0^1 (\partial_x([cQ]^\beta))^2\,dx\Big)^{1/2}\Big(\int_0^1 \phi(x)^{2\alpha(\frac{3}{4}-\beta)}\,dx\Big)^{1/2} \le F_8,\end{aligned} \tag{170}$$

since $\beta,\alpha \in (0,1)$ and by application of Lemma 5.1, Corollary 4.1, and Assumptions (A1)–(A4). □

6.5. **The proof of Theorem 3.1.** In order to construct a weak solution to the initial boundary value problem we can directly adopt the standard approach and apply the line method as described in [13, 14, 3, 31, 9], see also the important references within these papers for more details. Since this will not introduce new elements to what is already found in these papers we leave the details to the readers. Here we just note that having formulated a semidiscrete version of the model (68)–(71), the basic theory of ordinary differential equations then guarantees the local existence of smooth solutions (c_i, n_i, u_i), for $i = 0,\dots,N$ such that

$$0 < c_i(t) < \infty, \qquad 0 < n_i(t) < \infty, \qquad |u_i(t)| < \infty.$$

Let $[0, T^h]$ be the maximal time interval on which the smooth solution exist. For the analysis below we must show that the solutions are actually locally defined on $[0,T_1]$ where T_1 is defined by (95). In particular, it can be ensured that $T^h > T_1$, see [3, 31], and (c_i, n_i, u_i) are well defined for $i = 0,\dots,N$, for all $t \in [0,T_1]$. Based on the work of Section 4, 5, and 6 we can obtain semidiscrete versions of the various lemmas. By defining appropriate approximate solutions $(c^N, n^N, u^N)(x,t)$ and using Helly's theorem, we can prove Theorem 3.1.

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