Relative Permeability From Thermodynamics

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Summary

A new theory is reviewed for single-component, two-phase flow in porous media. It integrates thermodynamics with fluid mechanics and includes wettability in the thermodynamic description. It does not make use of the relative permeability concept. Wettability is included through a function that is part of the input to the model, together with an equation of state. A dialogue between this new model and the existing one has been established with the purpose of obtaining some of the input functions of both models. First, we have used information available in the existing model so as to calculate the function necessary for the description of wettability in the new model. The new model then being complete, we use it to find the two central functions necessary to the existing model, the relative permeabilities. We show that these are well defined when flow is characterized as a traveling wave, and we give a formula for their calculation. The formula assumes that a capillary pressure correlation and an equation of state are known, in addition to a constant related to the thickness of the transition region between the two phases.

Introduction

A new model for two-phase flow in porous media recently has been developed and presented in a paper[1] referred to from this point on as P[1]. Reviews of the theory and some recent developments are given in the next section.

The model assumes a fluid consisting of one chemical component and capable of changing phase. The temperature is assumed to be constant and uniform. The model contains a number of parameters and assumes that a number of functions are known. Among the parameters one finds the porosity ϕ, the absolute permeability K, and one additional parameter, denoted by Λ (see the section "Theoretical Background" below). Among the functions one finds the fluid viscosity as a function of the fluid density and an equation of state. The latter has to be of the van der Waals type to ensure that phase transitions take place under changing conditions of density.

The model includes one additional function, I, which describes the wetting properties of the rock, and which is assumed to be known. More importantly, the model does not include relative permeabilities.

In view of the last two characteristics, we have set up a "dialogue" between this new model and the existing one, with a double purpose. First, we use information available in the existing model so as to calculate the I function. The new model then being complete, we use it to throw some light on the relative permeability concept: Because the new model does not use relative permeabilities it should, if properly used, be capable of calculating them.

The paper is organized as follows. The section entitled "Theoretical Background" contains a review of the new model and some new theoretical developments. Among other things, it is shown that the previously-mentioned I function can be calculated by using known results about capillary pressure (the first part of the dialogue). The section entitled "Application to Relative Permeabilities" contains the second part of the dialogue. We use the model to produce numerical results concerning a stabilized flow type in a vapor-wet rock. We then infer, from these results and from the structure of the model, formulas giving the relative permeabilities vs. saturation in terms of the parameters describing the rock and the fluid, especially the thermodynamic functions and the capillary pressure. To our knowledge, formulas of this type (i.e., not based on correlations) have not appeared previously in the literature on the subject. In addition, we have found that there exists a functional relationship between the entry pressure, the pore distribution index, and the residual vapor saturation of a vapor-wet rock. It should be emphasized that we have, in this paper, restricted considerations to stabilized flow—that is, a flow type that can be characterized as a traveling wave.

The qualitative behavior of the new permeability formulas is investigated in the "Discussion" section.

Concerning wider applications of the new model, at least in its present form, it should be kept in mind that it is characterized as a "one-component, two phases" model. As such, it is directly applicable to the study of cases referred to in the steam-water literature.[2,3] It can, however, be applied to two-phase flows of systems such as gas-oil, which are not one-component, but whose thermodynamic behavior can, in certain well-defined circumstances, be accounted for by an equation of state of the van der Waals type.

Theoretical Background

At the pore (or Navier-Stokes) level, P[1] assumes a fluid consisting of a single chemical component. This fluid is capable of existing in either a one-phase state, which may be called "liquid" or "vapor," or in a two-phase state, which would then be a coexistence of the liquid and the vapor. The diffuse-interface theory (shortened to DI in the sequel) of two-phase flow[4,5] is then adopted. The central postulate of DI is that when two phases coexist, a transition region of nonzero width exists between them where the density varies continuously. One can then consider the two phases as manifestations of one and the same fluid. This brings about a simplification relative to the traditional model of two-phase flow. Indeed, this model assumes that the two phases are separated by a surface across which the density jumps discontinuously and considers each phase as a separate fluid. The reduction to *one* fluid leads to just one set of conservation equations, those of mass, momentum, and energy. These equations must of course be coupled to an equation of state describing a fluid capable of existing in two phases—in other words, an equation of state of the van der Waals type.

The DI model has an additional attractive characteristic: It allows the description of the wetting properties of the solid that is in contact with the fluid. The basic theory was developed by Cahn,[6] who assumed a fluid-solid interaction of a special kind. The application of the Cahn theory to fluids is attributable to Seppecher.[7] This author's essential result is based on two facts. First, the momentum conservation equation that is a consequence of DI is a Navier-Stokes equation modified by the addition of a term proportional to $\nabla\nabla^2\rho$, where ρ is the fluid density. Second, the wetting properties of the solid are taken care of by the inclusion of an additional boundary condition of the type

$$\mathbf{n} \cdot \boldsymbol{\nabla}\rho = f(\theta), \quad \text{(1)}$$

where $f(\theta)$ is a known function[1] of the wetting angle θ. The problem of solving the modified Navier Stokes equation is then well posed.

The physical effect of the boundary condition given by Eq. 1 is to sharply increase (for $\theta<\pi/2$) or decrease (for $\theta>\pi/2$) the density at a vicinity of the surface characterized by a distance of the order

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of a molecular diameter. The extreme thinness of this layer of modified density is essential to the upscaling process, which is described in the next paragraph.

The upscaling from pore level equations to macroscopic, Darcy-level equations has been done in P[1] by the Marle[8] averaging technique. The only difficulty is to upscale the boundary condition, Eq. 1, in a meaningful way. The problem has been solved by ignoring Eq. 1 itself, but including its effects through the introduction of a surface fluid obeying its own flow equations. This is called the Σ-fluid (its density is denoted ρ_Σ), and it is supposed to interact with what we may now call the *regular fluid.* The flow equations of the regular fluid and of the Σ-fluid can now easily be upscaled with the Marle technique.

The fluid-solid interaction responsible for the wetting at the Navier-Stokes level must be replaced, at the upscaled level, by a postulated interaction energy (called I from this point on) between the regular fluid and the Σ-fluid. This interaction energy caused by wetting, I, has the same status as the equation of state or the phenomenological parameters such as permeability, porosity, and so on (i.e., it is a part of the data that the model expects to be given). Some of the properties of I are known, as will be shown below, and these properties will be used to calculate I in terms of the capillary pressure.

The transition region between phases that is postulated in DI is thin, and the gradient of density is large in that region. Accordingly, DI incorporates, in the thermodynamical description of the fluid, an additional energy proportional to the squared gradient of the density, $|\nabla\rho|^2$. This has two effects: First, the Navier-Stokes equations are modified in the manner already mentioned above; second, pressure and chemical potential have the traditional meanings *in the bulk fluid* (i.e., far from the transition region), where the squared gradient of the density can be neglected. As a notational consequence of the second effect, pressure and chemical potential of the bulk fluid carry the superscript b.

The averaging of the pore-level equations over many pores inside a representative volume is done in P[1] by assuming that the following parameters are uniform throughout the porous medium: Porosity ϕ, pore surface per unit volume A_Σ, and temperature. This averaging produces equations describing two-phase flow in porous media where the dependent variables are R and R_Σ, the macroscopic averages of ρ and ρ_Σ. (In P[1], ρ and R carry a subscript f and are said to be quantities of the f-fluid, as a necessary distinction from the Σ-fluid. The f-fluid is identical to the what we above have called the regular fluid, and the subscript f is suppressed here.) The central equation is a mass balance equation for R:

$$\frac{\partial R}{\partial t} + \nabla \cdot (R\mathbf{V}) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $\mathbf{V}$ = the macroscopic (Darcy-like) velocity:

$$\mathbf{V} = -\frac{KR}{\phi\eta}\nabla(M^b + M^c - \Lambda\nabla^2 R + G). \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In this expression, K = the absolute permeability, ϕ = the porosity, and G = the gravitational potential giving rise to an acceleration equal to $-\nabla G$; η = the fluid viscosity, assumed to be a known function of R (see the end of this section); M^b = the bulk chemical potential and is a function of R; M^c = the chemical potential caused by wetting and is a function of R and R_Σ. Λ = an additional phenomenological parameter related to the width of the transition region between the phases; it has been shown in P[1] (section 6.3) that a measure for this width is $2\sqrt{\tilde{\Lambda}}/(\tilde{R}_l - \tilde{R}_v)$ (where the tildes indicate dimensionless quantities; see the Nomenclature).

Actually, upscaling gives rise to *two* mass-balance equations, one for R, the other for R_Σ, each of them with a source term. In addition, the two source terms are equal but of opposite signs, indicating a transfer of mass from the regular fluid to the Σ-fluid and vice versa. It has, however, been shown in P[1] that the source terms are negligibly small, so that Eq. 2 is the correct mass-balance equation for R. Also, the mass-balance equation for R_Σ has not been written in anticipation of a new result derived below, namely that R_Σ can be neglected for the type of wetting we shall be considering (incomplete wetting).

Chemical potentials replace pressures as the naturally occurring quantities in the Darcy-like velocity, but we shall see that pressures can be reinstated in certain circumstances. The information necessary to determine these chemical potentials is contained in the Helmholtz free energy of the bulk fluids (fluid proper and Σ-fluid), which has the form

$$\phi\Psi(R,R_\Sigma) = \phi\Psi^b(R) + A_\Sigma F_\Sigma(R_\Sigma) + \phi I(R,R_\Sigma). \quad \ldots\ldots\ldots\ldots (4)$$

Here Ψ^b = the free energy of the averaged fluid proper, F_Σ = the free energy of the averaged surface fluid, and I = the energy of interaction between the two fluids. M^b and M^c are defined below, together with the corresponding chemical potentials of the Σ-fluid:

$$M^b = \frac{d\Psi^b}{dR}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

$$M^c = \frac{\partial I}{\partial R}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

$$M_\Sigma = \frac{dF_\Sigma}{dR_\Sigma}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5c)$$

$$M_\Sigma^c = \frac{\phi}{A_\Sigma}\frac{\partial I}{\partial R_\Sigma}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5d)$$

It has been shown in P[1] that, for neutral wetting at the pore level (90° wetting angle), density ρ_Σ vanishes. Consequently, R_Σ vanishes, and the interaction energy I between the regular fluid and the Σ-fluid can be set equal to zero. The problem is completely defined by Eqs. 2 and 3 (with $M^c = 0$) and 5a.

For nonneutral wetting $I(R,R_\Sigma)$ determines M^c as a function of R and R_Σ. There is a need for an additional equation which, according to P[1], is

$$M_\Sigma = M, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where the left side is defined by Eq. 5c, while the right side is

$$M = M^b - \Lambda\nabla^2 R - \frac{1}{2}|\mathbf{V}|^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

P[1] concludes that the equations for two-phase flow in porous media resulting from DI are Eqs. 2, 3, and 6. As additional input, all thermodynamic functions appearing on the right side of Eq. 4 must be known, as well as the fluid viscosity η in Eq. 3.

It is also pointed out in P[1] that the chemical potentials can be replaced, in Eq. 3, by the more familiar pressures, through the following formulas:

$$\frac{1}{R}\frac{dP^b}{dR} = \frac{dM^b}{dR}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8a)$$

$$R\nabla M^c = \nabla P^c. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8b)$$

Here, P^b = the bulk pressure, related to the density R through an equation of state, while P^c = the capillary pressure. Eq. 8a is a thermodynamical relation valid at constant temperature. Eq. 8b, on the other hand, is merely a formal definition that is not compatible with P^c being a function of R and R_Σ. Indeed, the assumption $P^c = P^c\ (R,R_\Sigma)$ leads directly to

$$\frac{\partial}{\partial R_\Sigma}\frac{\partial P^c}{\partial R} - \frac{\partial}{\partial R}\frac{\partial P^c}{\partial R_\Sigma} = \frac{\partial M^c}{\partial R_\Sigma}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

so that P^c is not a function of state. We shall return to this presently.

It is of some interest to point out that the combination of Eqs. 2 and 3, and of Eqs. 4, 5a, and 5b, gives

$$\frac{\partial R}{\partial t} = \nabla \cdot \left[\frac{KR^2}{\phi\eta}\nabla\left(\frac{\partial\Psi}{\partial R} - \Lambda\nabla^2 R + G\right)\right]. \quad \ldots\ldots\ldots\ldots (10)$$

When G is identically zero, this reduces to the Cahn-Hilliard equation.

This completes the review of P[1]. What follows are further developments to the model.

The Incomplete-Wetting Approximation. It has been shown (see Eqs. 35 in P[1]) that, at equilibrium,

$$M^b + M^c + G - \Lambda\nabla^2 R = \bar{M}, \qquad (11a)$$

$$M_\Sigma + M^c_\Sigma + G = \bar{M}, \qquad (11b)$$

where $\bar{M}$ = a constant (see below, Eq. 17). Assuming equilibrium with zero macroscopic velocity everywhere, we get from Eq. 7 that $M = M^b - \Lambda\nabla^2$ so that Eqs. 11 above can be written $M + M^c + G = \bar{M}$, $M_\Sigma + M^c_\Sigma + G = \bar{M}$.

These equations, together with Eq. 6, now give $M^c = M^c_\Sigma$, which, with Eqs. 5b and 5d, yields

$$\frac{\partial I}{\partial(\phi R)} = \frac{\partial I}{\partial(A_\Sigma R_\Sigma)}. \qquad (12)$$

It is easy to show that this is a necessary and sufficient condition for I to be a function of $\phi R + A_\Sigma R_\Sigma$.

It is shown in P[1] that $\phi R/(A_\Sigma R_\Sigma)$ (the ratio of averaged fluid proper to averaged Σ-fluid per unit volume) is on the same order of magnitude as $q = \phi/(\ell A_\Sigma)$. Using the Karman-Kozeny equation[9] $[K = \phi^3/(5A^2_\Sigma)]$, one gets

$$q = \sqrt{5K/\phi}/\ell. \qquad (13)$$

In this paper we shall say that we have *complete wetting* when the wetting angle is either 0 or 180°; otherwise, we shall say that we have *incomplete wetting*.

It is known[1] that for incomplete wetting, $\ell \approx 10^{-10}$ m so that q is quite large and the dependence of I on $A_\Sigma R_\Sigma$ can be neglected. For complete wetting, on the other hand, experiments[10] indicate that ℓ may increase by a factor of many hundreds; the dependence of I on $A_\Sigma R_\Sigma$ can then become important for permeabilities on the order of the millidarcy.

Accordingly, we call $I = I(R)$ the incomplete-wetting approximation. We assume, for the rest of this paper, that this approximation is valid.

A direct consequence of the incomplete-wetting approximation is that

$$M^c = \frac{dI}{dR}. \qquad (14)$$

The equations for two-phase flow in porous media resulting from DI are now Eqs. 2 and 3, where the chemical potentials are defined by Eqs. 5a and 14. In addition, the F_Σ contribution to the free energy Ψ, Eq. 4, can be neglected so that one can write

$$\Psi(R) = \Psi^b(R) + I(R). \qquad (15)$$

The two functions on the right side, Ψ^b and I, must be known because M^b and M^c are their derivatives. They are determined below. The fluid viscosity η appearing in Eq. 3 must also be known.

Another consequence of the incomplete wetting approximation is that the capillary pressure P^c can be defined unambiguously as a function of R only. Indeed, Eqs. 8b and 14 show that, in this approximation, P^c is an equation of state that depends on the one variable R and which is related to the chemical potential M^c by

$$\frac{dP^c}{dR} = R\frac{dM^c}{dR}. \qquad (16)$$

We shall show below that this equation allows us to obtain I, the interaction energy caused by wetting.

The Helmholtz Free Energy of the Bulk Fluid. The incomplete-wetting approximation has no bearing on the calculations of this section.

Usually, the relevant thermodynamic properties of the fluid are known through an equation of state $P^b = P^b(R)$. It would then be easy to get the needed M^b-function by using Eq. 8a. It will, however, be made clear in a later section that knowledge of the free energy $\Psi^b(R)$ is important, so this section concentrates on a method to calculate it. It has been shown[11] that, for van der Waals type $P^b(R)$ curves, Ψ^b has the form

$$\Psi^b(R) = W(R) + \bar{M}R - P, \qquad (17)$$

where $W(R)$ has two minima where it is zero; $\bar{M}$ and $\bar{P}$ (the chemical potential and the pressure of the fluid at equilibrium) are functions of the temperature alone, so that they can be considered constant for calculations at uniform temperature. **Fig. 1** shows a typical equation of state of the van der Waals type and its resulting W-function. It is stated here for later reference that a continuous P^b vs. R curve implies a smooth W function. However, solutions to the Cahn-Hilliard equation, Eq. 10, are known[12] where, for physical reasons, W is not smooth at the minima. The only important feature of W is to have two global minima of value zero. Function W is then said to support two phases,[12] the densities of which are the values of the density at which the minima of W occur.

The densities of the phases, say vapor and liquid, are denoted by R_v and R_l, and are such that the Maxwell construction (also called *the Maxwell equal area rule*[13] when referring to the pressure-vs.-volume diagram) applies:

$$\int_{R_v}^{R_l} P^b \frac{dR}{R^2} = P^b(R_v)\left(\frac{1}{R_v} - \frac{1}{R_l}\right). \qquad (18)$$

Note that the $\bar{P}$ of Eq. 17 is either $P^b(R_v)$ or $P^b(R_l)$.

By definition,

$$P^b = R^2 \frac{d}{dR}\left(\frac{\Psi^b}{R}\right). \qquad (19)$$

Using Eq. 17 on the right side, one easily gets

$$W(R) = P^b(R_v)\left(1 - \frac{R}{R_v}\right) + R\int_{R_v}^{R} P^b(R')\frac{dR'}{R'^2}. \qquad (20)$$

Note that $W(R_v) = 0$ by choice of integration constants, and that $W(R_l) = 0$ because of the Maxwell construction. It is also easily checked that the derivative $W'(R)$ vanishes at R_v and R_l.

The expression for M^b now follows from its definition, as given by Eq. 5a:

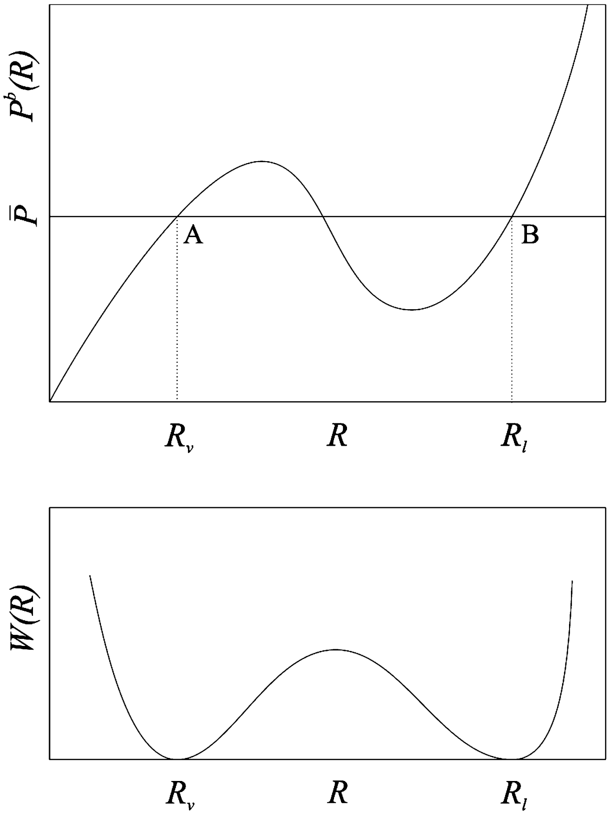

Fig. 1—A van der Waals type equation of state (top) and the *W*-function (bottom).

$$M^b = \int_{R_v}^{R} \frac{1}{R'}\frac{dP^b}{dR'}\,dR' + \bar{M}. \quad \text{(21)}$$

Because only the gradient of M^b appears in the the flow equation, the value of $\bar{M}$ is irrelevant, and one can set

$$\bar{M} = 0. \quad \text{(22)}$$

Note that we must have $M^b(R_v) = M^b(R_l) = \bar{M} = 0$. This is easily checked by using partial integration in Eq. 21 and referring to Eq. 18. The determination of M^b through the equation of state is thus completed. The determination of M^c follows.

The Interaction Energy Caused by Wetting. The incomplete-wetting approximation is now assumed (i.e., we assume that the interaction energy I depends exclusively on R). An unambiguous definition of the capillary pressure P^c as a function of R is then possible by Eq. 16.

We now look at this equation as a means to determine M^c through known correlations for the capillary pressure. However, we shall see that it is essential to determine I first. To get I, we use Eq. 5b, now rewritten as

$$M^c = \frac{dI}{dR}, \quad \text{(23)}$$

in Eq. 16:

$$\frac{d^2 I}{dR^2} = \frac{1}{R}\frac{dP^c}{dR}. \quad \text{(24)}$$

The formal integration of this equation gives

$$I(R) = R\int \frac{P^c}{R^2}\,dR - \alpha + \beta R. \quad \text{(25)}$$

The bounds of integration and the constants α and β depend on the wetting properties, as will be shown below. It is convenient to start by defining saturation.

The averaged fluid is at equilibrium in a two-phase state if its density R is between R_v and R_l (see Fig. 1). The volume fractions of the two phases can be taken as the definitions of the vapor and liquid saturations, S_v and S_l. This implies that

$$S_v = \frac{R_l - R}{R_l - R_v}, \quad \text{(26a)}$$

$$S_l = \frac{R_l - R_v^*}{R_l - R_v}. \quad \text{(26b)}$$

If the rock is vapor-wet, there will be a residual vapor saturation, S_{vr}, attributable to the fact that the density of what is now the liquid phase is $R_l^* < R_l$. According to Eq. 26a,

$$S_{vr} = \frac{R_l - R_l^*}{R_l - R_v}. \quad \text{(27)}$$

If the rock is liquid-wet, there will, correspondingly, be a residual liquid saturation

$$S_{lr} = \frac{R_v^* - R_v}{R_l - R_v}, \quad \text{(28)}$$

where $R_v^* > R_v$ is now the density of the vapor phase (see **Fig. 2**). From a different point of view, one can say that the shifted densities are inferred from measured residual saturations by

$$R_v^* = R_v + (R_l - R_v)S_{lr}, \quad \text{(29a)}$$

$$R_l^* = R_l - (R_l - R_v)S_{vr}. \quad \text{(29b)}$$

The free energy Ψ, Eq. 15, reduces to Ψ^b, Eq. 17, for neutral wetting ($I = 0$). The phase transition mechanism is then regulated by W. For nonneutral wetting, this mechanism is regulated by W+I. One then expects W+I to be a modified W-function, the modification consisting of a shift of the minima along the R-axis.

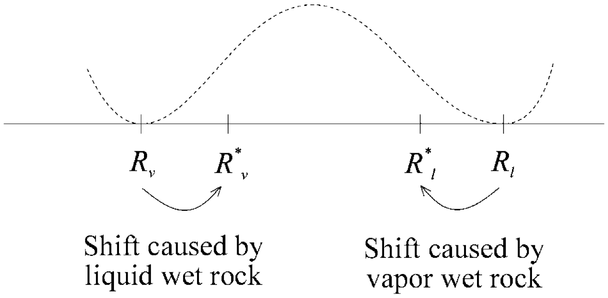

Fig. 2—Shifts in the densities of the liquid and vapor phases caused by the wetting properties. The *W* function is shown in broken line.

We now turn to the determination of I. We shall in this paper, for the sake of simplicity, limit ourselves to the case of a vapor-wet medium.

The Interaction Energy for Vapor-Wet Rock. The rock being vapor-wet, one expects a residual vapor saturation in a process where, say, the porous medium is filled initially with vapor, which is then displaced by liquid. The equilibrium value of the averaged vapor saturation remains unchanged, equal to R_v, but the averaged liquid density will be $R_l^* < R_l$. We then want $I(R)$ to be such that W+I has the form shown in **Fig. 3.** Obviously, there are three necessary conditions on I:

$$I(R_v) = 0, \quad \text{(30a)}$$

$$I'(R_v) = 0, \quad \text{(30b)}$$

$$I(R_l^*) = -W(R_l^*), \quad \text{(30c)}$$

where I' is the derivative of I. The two first conditions imply that W+I resembles W near $R = R_v$, and the third implies that W+I vanishes at $R = R_l^*$. Because, for vapor-wet medium, $P^c(R_v)$ is a finite constant (usually called the entry pressure), we can write Eq. 25 as

$$I(R) = R\int_{R_v}^{R} \frac{P^c(R')}{R'^2}\,dR' + \beta R - \alpha. \quad \text{(31)}$$

β and α are determined by Eqs. 30a and 30b. One finds

$$I(R) = P^c(R_v)\left(1 - \frac{R}{R_v}\right) + R\int_{R_v}^{R} \frac{P^c(R')}{R'^2}\,dR'. \quad \text{(32)}$$

We now use the Brooks-Corey[14] capillary pressure vs. saturation correlation,

$$P^c = C\left(\frac{1 - S_{vr}}{S_v - S_{vr}}\right)^a,\ 0 < a < 1, \quad \text{(33)}$$

where C = the entry pressure and $1/a$ the pore-size distribution index ($a = 0$ corresponding to uniform pore size). Using Eqs. 26a and 27, one finds that

$$I(R) = CJ(R), \quad \text{(34a)}$$

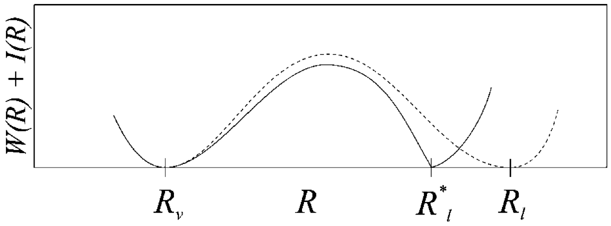

Fig. 3—Expected form of the *W*+*I* function for vapor-wet rock. Note that it is not necessarily smooth at R_l^*. The *W*-function is shown in broken line.

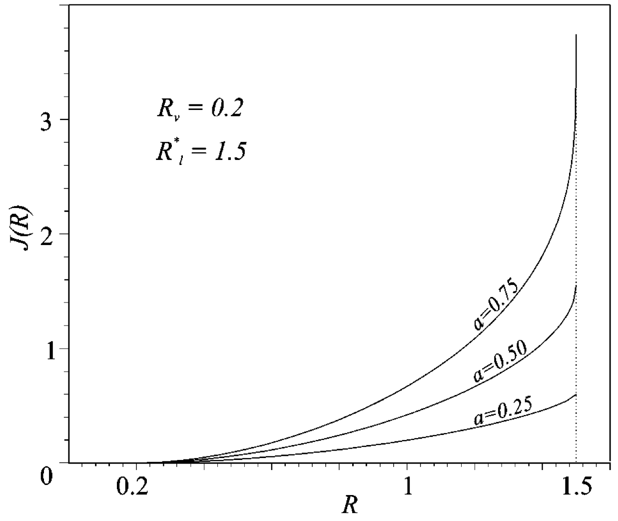

Fig. 4—Function *J*(*R*) for vapor-wet rock (Eq. 34b), plotted vs. *R* for the numerical values indicated.

$$J(R) = 1 - \frac{R}{R_v} + R \int_{R_v}^{R} \frac{1}{R'^2}\left(\frac{R_l^* - R_v}{R_l^* - R'}\right)^a dR'. \qquad (34b)$$

Plots of $J(R)$ are shown in **Fig. 4.**

Eq. 30c now turns out to be a condition on *C:*

$$C = -\frac{W(R_l^*)}{J(R_l^*)}. \qquad (35)$$

We see that $C<0$. Eq. 35 is an equation between R_v, R_l^*, C, and a, involving the thermodynamic function W of the fluid. Referring to Eq. 29b, we see that we can look upon Eq. 35 as an equation containing the pure liquid and vapor densities R_v and R_l as parameters and linking the two constants C and a to the residual vapor saturation S_{vr}. Referring to **Fig. 5,** we see that C goes to zero when S_{vr} goes to zero, because R_l^* goes then to R_l.

The I function is now determined for the vapor-wet case, and $W+I$ has the form shown in Fig. 5. Note that the determination of the constant C in the capillary pressure correlation, Eq. 33, is essential to make sure that $W+I$ has two minima, thus supporting the existence of two phases.

The function $M^c(R)$ for the vapor-wet case is now known through its definition (Eq. 14).

The capillary pressure function being defined for $S_v>S_{vr}$ only, I and consequently M^c are only known for $R_v<R<R_l^*$ (see Eqs. 34). For most numerical applications, one must supply a function M^c which is valid outside this interval. Obviously, it seems natural to use Eq. 34b for $R<R_v$ also (this was done in the plots of Figs. 4 and 5). For $R>R_l^*$, however, a continuation must be found that is best adapted to the physics of the problem. An example is given elsewhere in this paper.

The Fluid Viscosity. We shall use a formula for viscosity proposed by Arrhenius and used in P[1], namely

$$\eta(R) = \eta_l^{S_l}\eta_v^{S_v}, \qquad (36)$$

where S_l and S_v are the liquid and vapor saturations defined by Eqs. 26.

Application to Relative Permeabilities

The numerical experiments described here are restricted to 1D situations. The coordinate axis (or x-axis) points in the direction of the gravitational force. The solution $R(x,t)$ to the following equation is sought (see Eqs. 2 and 3):

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial x}\left[\frac{KR^2}{\phi\eta}\frac{\partial}{\partial x}\left(\frac{d\Psi}{dR} - \Lambda\frac{\partial^2 R}{\partial x^2} + G\right)\right]. \qquad (37)$$

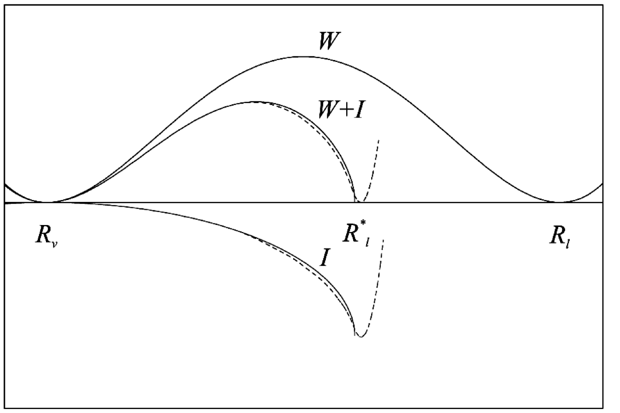

Fig. 5—Vapor-wet case: Function *I*(*R*) resulting from P^c vs. saturation given by Eq. 33, plotted vs. *R*. Functions *W* and *W+I* are also shown. Concerning the broken lines, see the text following Eq. 49.

In this equation, $G=-gx$ ($g>0$ is the acceleration caused by gravity); η is the fluid viscosity given by Eq. 36; Ψ is given by Eqs. 15, 17, and 22. Thus,

$$\frac{d\Psi}{dR} = \frac{dW}{dR} + \frac{dI}{dR} = M^b + M^c. \qquad (38)$$

Following P[1], we now introduce dimensionless variables. The dimensionless counterpart of any quantity q is denoted $\tilde q$. The definitions of the dimensionless variables are given in the Nomenclature. The dimensionless version of Eq. 37 is

$$\frac{\partial \tilde R}{\partial \tilde t} = \frac{\partial}{\partial \tilde x}\left[\frac{\tilde R^2}{\tilde\eta}\frac{\partial}{\partial \tilde x}\left(\tilde M^b + \tilde M^c - \tilde\Lambda\frac{\partial^2 \tilde R}{\partial \tilde x^2} - \tilde g \tilde x\right)\right]. \qquad (39)$$

We shall also need the dimensionless version of the velocity, Eq. 3:

$$\tilde V = -\frac{\tilde R}{\tilde\eta(\tilde R)}\frac{\partial}{\partial \tilde x}\left(\tilde M^b + \tilde M^c - \tilde\Lambda\frac{\partial^2}{\partial \tilde x^2} - \tilde g \tilde x\right). \qquad (40)$$

The I appearing in Eq. 38 is given by Eqs. 34 and 35, with the modifications described in the second paragraph following Eq. 49.

As for W, because we shall not try to replicate the behavior of any specific fluid with a known equation of state, we shall not follow the method outlined by Eqs. 17 through 20. Instead, we shall do the opposite (i.e., postulate the simplest W-function with the behavior shown in Fig. 1). We have chosen a $\tilde W$ that is very easy to use in the sense that it allows us to choose $\tilde R_v$ and $\tilde R_l$ at will; otherwise, using van der Waals or more realistic equations of state (Peng-Robinson and others), these constants must be obtained through calculations involving the Maxwell rule. We use

$$\tilde W(\tilde R) = (\tilde R - \tilde R_v)^2(\tilde R - \tilde R_l)^2, \qquad (41)$$

(called the pseudo-van der Waals form in P[1]). It follows from Eqs. 5a, 17, and 22 that

$$\tilde M^b = 2(\tilde R - \tilde R_v)(\tilde R - \tilde R_l)(2\tilde R - \tilde R_v - \tilde R_l). \qquad (42)$$

The equation of state $\tilde P^b = \tilde P^b(\tilde R)$ follows from Eqs. 19, 17, and the condition $\tilde P^b(0)=0$, which determines $\bar P$. It is easily found that

$$\tilde P^b = 3\tilde R^4 - 4(\tilde R_v + \tilde R_l)\tilde R^3 + (\tilde R_v^2 + \tilde R_l^2 + 4\tilde R_v\tilde R_l)\tilde R^2, \qquad (43)$$

$$\bar P = \tilde R_v^2\tilde R_l^2. \qquad (44)$$

The $\tilde P^b(R)$–function has the expected behavior shown in Fig. 1 when we choose $\tilde R_v<\tilde R_l$.

It is known experimentally that R_l and R_v are respectively larger and smaller than the critical value R_c. This translates to $\tilde R_l>1$ and $\tilde R_v<1$. To increase the qualitative correctness of the model while still keeping it simple, we put

$$\tilde{R}_l = 1 + r_0 \qquad (45a)$$

$$\tilde{R}_v = 1 - r_0, \qquad (45b)$$

where r_0 is chosen between zero and 1. The model, as used here, assumes that temperature is constant, but it is useful to keep in mind that $r_0 = 0$ at the critical temperature, then increases with decreasing temperature. (See Appendix A in P[1] where it is shown that, for the pseudo-van der Waals equation of state, r_0 is proportional to $\sqrt{1-T/T_c}$.)

Vapor-Wet Rock, Gravity Drainage by Liquid Phase. We consider a 1D porous medium where flow occurs vertically. It is initially filled with vapor, and it is assumed that liquid is provided at the top, at a constant pressure, and that the vapor can flow out at the bottom, also at constant pressure.

We solve Eq. 39 with the following assumptions.

- The $\tilde{x}$-axis points downward, and $0 \le \tilde{x} \le 1$.
- The initial condition is

$$\tilde{R}(\tilde{x},0) = \tilde{R}_0(\tilde{x}), \qquad (46)$$

where $\tilde{R}_0 \approx \tilde{R}_l^*$ for small values of $\tilde{x}$, and $\tilde{R}_0 \approx \tilde{R}_v$ otherwise.

- The boundary conditions are

$$\tilde{R}(0,\tilde{t}) = \tilde{R}_l^*, \qquad (47a)$$

$$\tilde{R}(1,\tilde{t}) = \tilde{R}_v, \qquad (47b)$$

$$\frac{\partial \tilde{R}}{\partial \tilde{x}}(0,\tilde{t}) = 0, \qquad (47c)$$

$$\frac{\partial \tilde{R}}{\partial \tilde{x}}(1,\tilde{t}) = 0. \qquad (47d)$$

- The values of the parameters are

$$\tilde{R}_v = 0.4,\ \tilde{R}_l = 1.6,\ a = 0.5,\ S_{vr} = 0.4, \qquad (48)$$

implying that $\tilde{R}_l^* = 1.12$. In addition, we have chosen

$$\eta_v/\eta_l = 0.1,\ \tilde{g} = 0.5,\ \tilde{\Lambda} = 0.01. \qquad (49)$$

The solution is expected to show a gradual sinking of the liquid displacing the gas so that there should be, after some time has elapsed, a layer of liquid with density $\tilde{R}_l^*$ at the top, a two-phase transition region, and a layer of gas with density $\tilde{R}_v$ at the bottom. The above value of $\tilde{\Lambda}$ has been chosen so that the three regions have approximately the same width. (Refer to the paragraph following Eq. 3. With the notation introduced by Eqs. 45, a measure for this width is $\sqrt{\tilde{\Lambda}/r_0}$, as referred to the unit of length.)

We have used a commercially available program using finite elements (FEMLAB, operating under MATLAB).[1] FEMLAB is a registered trademark of COMSOL AB; MATLAB is a registered trademark of TheMathWorks Inc. The program requires continuous and smooth functions as input, implying that, first, $\tilde{R}_0(\tilde{x})$ must be smooth (infinitely differentiable is best) and, second, I must be continued smoothly, to the right of $\tilde{R}_l^*$, as indicated on Fig. 5 by the broken lines. The most important consequence, related to the second smoothing above, implies removing the infinity from the P^c curve.

The result of the calculation of $\tilde{R}(\tilde{x},\tilde{t})$ is shown in **Fig. 6.** The following facts emerge from this figure:

- The transition from the liquid ($\tilde{R} \approx \tilde{R}_l^*$) plateau to the steep-slope region is fast when compared to the transition from steep-slope region to the vapor ($\tilde{R} \approx \tilde{R}_v$) plateau. This is obviously because of the medium being vapor-wet.
- $\tilde{R}(\tilde{x},\tilde{t})$ behaves like a wave, travelling at constant velocity and without distortion, so that $\tilde{R}(\tilde{x},\tilde{t}_1)$ is (up to calculational errors) a translated version of $\tilde{R}(\tilde{x},\tilde{t}_2)$.

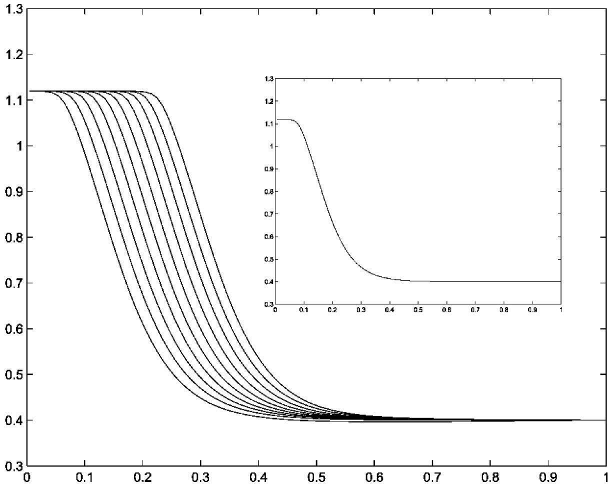

Fig. 6—Vapor-wet rock. Gravity drainage by liquid phase. Plot of $\tilde{R}(\tilde{x},\tilde{t})$ vs. $\tilde{x}$ for (left to right) $\tilde{t}$ = 0.05 to 0.5 in steps of 0.05. Eqs. solved: 39, 46, and 47; numerical values: Eqs. 48 and 49. Inset: Long-time solution of the same equations with the same numerical values as above, except that $\tilde{g}$ = 0.

- Each $\tilde{R}(\tilde{x},\tilde{t})$ is a translated version of the curve shown in the inset. The latter is the long-time solution of Eqs. 39, 46, and 47 with the numerical values given in Eqs. 48 and 49, except that $\tilde{g} = 0$. In other words, the curve in the inset is the static equilibrium solution without gravity. As such, it must (and does) satisfy Eq. 11a with $G = 0$. Keeping in mind Eq. 22 and using dimensionless quantities, we see that the curve in the inset obeys

$$\tilde{M}^b + \tilde{M}^c = \tilde{\Lambda}\frac{d^2\tilde{R}}{d\tilde{x}^2} = 0. \qquad (50)$$

We mention here for future reference that, when one takes into account Eqs. 5a and 14, the following first integral can be found:

$$\frac{d\tilde{R}}{d\tilde{x}} = -\sqrt{\frac{2[\tilde{W}(\tilde{R}) + \tilde{I}(\tilde{R})]}{\tilde{\Lambda}}}, \qquad (51)$$

obeying conditions of Eq. 47.

We shall return to these properties of the solution later.

We now present a method for using the previous solution to calculate relative permeabilities to vapor and liquid, k_{rv} and k_{rl}. These do not exist in the model described in this paper, so that their calculation presupposes that they can be defined with quantities that can be calculated in the model. The definitions, in terms of dimensionless quantities, are

$$\tilde{V}_v = \frac{k_{rv}}{\tilde{\eta}(\tilde{R}_v)}\left(-\frac{\partial \tilde{p}_v}{\partial \tilde{x}} + \tilde{g}\tilde{R}_v\right), \qquad (52a)$$

$$\tilde{V}_l^* = \frac{k_{rl}}{\tilde{\eta}(\tilde{R}_l^*)}\left(-\frac{\partial \tilde{p}_1^*}{\partial \tilde{x}} + \tilde{g}\tilde{R}_l^*\right), \qquad (52b)$$

One here assumes that, at a point in the two-phase region, the fluid consists of a vapor phase with density $\tilde{R}_v$, viscosity $\tilde{\eta}(\tilde{R}_v)$, velocity $\tilde{V}_v$, pressure $\tilde{p}_v$, and a liquid phase where the corresponding quantities are starred and have subscript l (the star as used here reminds that the medium is vapor-wet so that the liquid phase is a mixture of liquid and residual vapor).

We now consider a time $\tilde{t}_0$ at which the solution has, as closely as possible, the characteristics of an infinite 1D system consisting of liquid (and residual vapor) to the left, a transitional two-phase region in the middle, and vapor to the right, the whole flowing from left to right as a standing wave. Referring to Fig. 6, we see that a number of solutions with values of $\tilde{t}_0$ around 0.5 have the required characteristics. The considerations which now follow are valid for solutions at such $\tilde{t}_0$ s.

The phase densities and viscosities to use in Eqs. 52 are already well defined, and the main problem is to derive phase velocities and phase pressures. We have not found a rigorous derivation, so that the results presented are based on guesses based in turn on the physics of the situation.

Basing the definition of phase velocities on the concept of momentum, one can say that the local momenta carried by each phase must add to the the fluid momentum $\tilde{R}\tilde{V}$:

$$\tilde{R}_v\tilde{V}_v + \tilde{R}_l^*\tilde{V}_l^* = \tilde{R}\tilde{V}. \quad \text{(53)}$$

This equation does not allow the calculation of the two-phase velocities, but we can use it as a guide; any formulas we derive for $\tilde{V}_v$ and $\tilde{V}_l^*$ should verify it.

We introduce the following notation:

$$\Gamma_v^*(\tilde{x},\tilde{t}_0) = \frac{\tilde{R}(0,\tilde{t}_0)\tilde{V}(0,\tilde{t}_o) - \tilde{R}(\tilde{x},\tilde{t}_0)\tilde{V}(\tilde{x},\tilde{t}_0)}{\tilde{R}(0,\tilde{t}_0)\tilde{V}(0,\tilde{t}_0) - \tilde{R}(1,\tilde{t}_0)\tilde{V}(1,\tilde{t}_0)}. \quad \text{(54)}$$

Note that Γ_v^* is a normalized vapor momentum, constructed on the same lines as the normalized vapor saturation

$$S_v^*(\tilde{x},\tilde{t}_0) = \frac{\tilde{R}(0,\tilde{t}_0) - \tilde{R}(\tilde{x},\tilde{t}_0)}{\tilde{R}(0,\tilde{t}_0) - \tilde{R}(1,\tilde{t}_0)} = \frac{\tilde{R}_l^* - \tilde{R}}{\tilde{R}_l^* - \tilde{R}_v}, \quad \text{(55)}$$

where the second equality follows from Eqs. 47a and 47b.

The simplest expressions for the local phase velocities, satisfying Eq. 53, are

$$\tilde{V}_v(\tilde{x},\tilde{t}_0) = \tilde{V}(1,\tilde{t}_0)\Gamma_v^*(\tilde{x},\tilde{t}_0), \quad \text{(56a)}$$

$$\tilde{V}_l^*(\tilde{x},\tilde{t}_0) = \tilde{V}(0,\tilde{t}_0)[1 - \Gamma_v^*(\tilde{x},\tilde{t}_0)]. \quad \text{(56b)}$$

A confirmation of the fact that these expressions are the correct choices comes from a comparison of the plot of Γ_v^* vs. $\tilde{x}$ with that of S_v^* vs. $\tilde{x}$; see **Fig. 7.** The two curves are practically indistinguishable, meaning that $\tilde{V}_v$ and $\tilde{V}_l^*$ are proportional to, respectively, S_v^* and $1 - S_v^*$. We shall see later that this leads to relative permeability curves with the desired straight-line behavior when the capillary pressure vanishes.

The numerical solution thus leads to

$$\tilde{V}_v(\tilde{x},\tilde{t}_0) = \tilde{V}(1,\tilde{t}_0)S_v^*, \quad \text{(57a)}$$

$$\tilde{V}_l(\tilde{x},\tilde{t}_0) = \tilde{V}(0,\tilde{t}_0)[1 - S_v^*]. \quad \text{(57b)}$$

We can go one step further and express the boundary velocities in terms of densities by using Eq. 40. Indeed, because the density at $\tilde{t}_0$ has flat plateaus near the left and right boundaries (Fig. 6), and the first, second, and third derivatives of $\tilde{R}$ with respect to $\tilde{x}$ vanish at $\tilde{x} = 0$ and $\tilde{x} = 1$ (this is verified to be the case, up to numerical errors), we find that

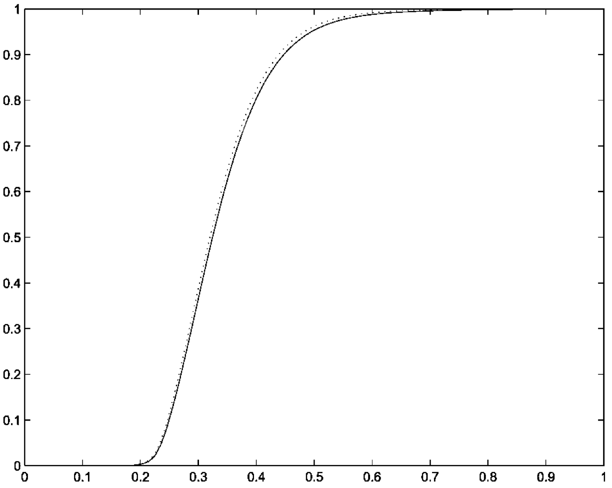

Fig. 7—Plot of Γ_v^* vs. $\tilde{x}$ (solid line) and of S_v^* vs. $\tilde{x}$ (dotted line) at $\tilde{t}$ = 0.5. See the caption of Fig. 6.

$$\tilde{V}(0,\tilde{t}) = \frac{\tilde{g}\tilde{R}_l^*}{\tilde{\eta}(\tilde{R}_l^*)}, \quad \text{(58a)}$$

$$\tilde{V}(1,\tilde{t}) = \frac{\tilde{g}\tilde{R}_v}{\tilde{\eta}(\tilde{R}_v)}, \quad \text{(58b)}$$

so that

$$\tilde{V}_v(\tilde{x},\tilde{t}_0) = \frac{\tilde{g}\tilde{R}_v}{\tilde{\eta}(\tilde{R}_v)} S_v^*, \quad \text{(59a)}$$

$$\tilde{V}_l^*(\tilde{x},\tilde{t}_0) = \frac{\tilde{g}\tilde{R}_l^*}{\tilde{\eta}(\tilde{R}_l^*)}(1 - S_v^*). \quad \text{(59b)}$$

We now turn to the definitions of the phase pressures, or rather of the pressure gradients, needed in Eqs. 52.

The required definitions are based first on the observation that, because the phase-separating mechanism is now based on the *W*+*I*-function shown in Fig. 5 (broken curve), then the gradient of $P^b + P^c$ at *equilibrium* is zero, in analogy with the fact that the gradient of P^b at equilibrium is zero when only *W* is present; see Fig. 1 where the horizontal line between A and B is the *equilibrium* P^b, and compare with **Fig. 8.** This means that

$$\left[\frac{\partial \tilde{P}^b}{\partial \tilde{x}}\right]_{\text{eq}} = -\frac{\partial \tilde{P}^c}{\partial \tilde{x}}. \quad \text{(60)}$$

The right side of this equation is easily obtained in terms of $\partial\tilde{R}/\partial\tilde{x}$, keeping in mind that $\tilde{P}^c$ is a function of S_v^* only and that S_v^* is defined in Eq. 55. We know, however, that because of the properties of the solution, $\partial\tilde{R}/\partial\tilde{x}$ is given by Eq. 51. Thus, $\partial\tilde{P}^b/\partial\tilde{x}$ at equilibrium can be written in terms of the derivative of the capillary pressure with respect to the normalized vapor saturation, and of the thermodynamic functions *W* and *I.* We introduce the following notation:

$$\gamma(S_v^*) = \frac{[\tilde{g}\tilde{R}_v(\tilde{R}_l - \tilde{R}_v)]^{-1}}{(1 - S_{vr})} \frac{d\tilde{P}^c}{dS_v} \sqrt{\frac{2[\tilde{W}(\tilde{R}) + \tilde{I}(\tilde{R})]}{\tilde{\Lambda}}}, \quad \text{(61)}$$

where the argument of the $\tilde{W}$– and $\tilde{I}$–functions is

$$\tilde{R} = \tilde{R}_v + (\tilde{R}_l - \tilde{R}_v)(1 - S_{vr})(1 - S_v^*), \quad \text{(62)}$$

and obtain

$$\left[\frac{\partial \tilde{P}^b}{\partial \tilde{x}}\right]_{\text{eq}} = -\tilde{g}\tilde{R}_v\gamma(S_v^*). \quad \text{(63)}$$

We now postulate that

$$\frac{\partial \tilde{p}_v}{\partial \tilde{x}} = -\tilde{g}\tilde{R}_v\gamma(S_v^*), \quad \text{(64a)}$$

$$\frac{\partial \tilde{p}_l^*}{\partial \tilde{x}} = -\tilde{g}\tilde{R}_v\gamma(1 - S_v^*). \quad \text{(64b)}$$

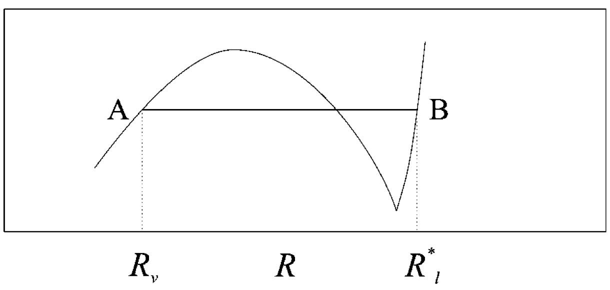

Fig. 8—P^b+P^c vs. density, corresponding to the *W*+*I* curve of Fig. 5 (broken line). Compare with Fig. 1.

Using these equations, together with Eqs. 59 and Eqs. 52, we obtain

$$k_{rv}=\frac{S_v^*}{1+\gamma(S_v^*)}, \quad \ldots\ldots (65a)$$

$$k_{rl}=\frac{1-S_v^*}{1+(R_v/R_l^*)\gamma(1-S_v^*)}. \quad \ldots\ldots (65b)$$

These equations express relative permeabilities in terms of the equation of state and the capillary pressure and, in addition, of the parameters $\tilde{g}$, $\tilde{\Lambda}$, $\tilde{R}_v$, $\tilde{R}_l$, and S_{vr}.

Note that, when the capillary pressure vanishes, then the γ-function vanishes, and the relative permeability curves reduce to two straight lines. We thus recover the conventional result.

Discussion

The Incomplete Wetting Approximation. We presented previously the incomplete-wetting approximation and assumed that it holds for the calculations presented in the paper. We have, however, used a capillary pressure vs. saturation correlation (Eq. 33), which holds for a completely vapor-wet medium. This was done in the interest of simplicity and has led to results that make sense both mathematically and physically, leading to the conclusion that the model presented is robust. The conclusions are restricted, however, to cases in which the incomplete wetting approximation holds (i.e., essentially cases in which the permeability is considerably greater than the millidarcy level).

Work is in progress on the strictly incomplete-wetting case, with a capillary pressure correlation of the form[14]

$$P^c=C_v\left(\frac{1-S_{vr}}{S_v-S_{vr}}\right)^{a_v}+C_l\left(\frac{1-S_{lr}}{S_l-S_{lr}}\right)^{a_l}.$$

Qualitative Behavior of the Relative Permeabilities. The new relative permeability vs. saturation functions presented in this paper depend, as Eqs. 65 and 61 show, on the following parameters: $\tilde{R}_v$, $\tilde{R}_l$, S_{vr}; also $\tilde{g}$, $\tilde{\Lambda}$, and, through the capillary pressure function, the inverse a of the pore size distribution index. Referring to the Nomenclature, we see that the definition of the dimensionless parameters above requires the knowledge of the critical density R_c and the critical pressure P_c. (It is easily seen that knowledge of the unit of length L is not necessary, because it drops out of the product $\tilde{g}\sqrt{\tilde{\Lambda}}$.) No additional parameters are brought in by the $\tilde{I}$–function (see Eq. 32), but some could be brought in by the $\tilde{W}$–function through the equation of state that is used to calculate it (see Eq. 20).

To show the qualitative behavior of the new relative permeability functions for different values of the parameters, we introduce a number of simplifications. As a first step, we choose the very simple $\tilde{W}$–function given by Eq. 41 and use the assumptions of Eqs. 45. In this way, we do not introduce new parameters, and we reduce the dependence on $\tilde{R}_v$ and $\tilde{R}_l$ to a dependence on r_0. Further, we let $\tilde{g}=0.05$ and $\tilde{\Lambda}=0.01$: The numerical experiment summarized in Fig. 6 has shown that these numerical values, together with the order of magnitude of r_0 (a number between 0 and 1, not too close to 0 or 1), give reasonable values for the width of the transition region and for the velocity of the traveling wave. In addition, we take $a=0{:}5$ because we want to compare the new relative permeability curves with the ones given by the standard Corey-Burdine formulas, which include essentially the one parameter a.

We have thus simplified the problem of investigating the dependence of the new relative permeability functions on the parameters to that of plotting these functions for different values of two parameters, r_0 and S_{vr}. Further, it is known that, in general, S_{vr} decreases as the temperature approaches the critical value. As pointed out previously (see Eqs. 45 and the statements following these equations), r_0 has essentially the same behavior, and it is reasonable to assume that S_{vr} is proportional to r_0 at some positive power. Because our only aim is to illustrate the qualitative behavior of the new relative permeability functions, we choose here the following simple relation:

$$S_{vr}=\frac{r_0}{2}. \quad \ldots\ldots (66)$$

A plot of the curves given by Eqs. 65 is shown in **Fig. 9,** for four values of r_0, showing the increasing concavity with increasing departure from the critical temperature. Included as a standard reference are the Corey-Burdine relative permeabilities, which, with our notation for the normalized vapor saturation, are given by

$$k_{rl}=[1-(S_v^*)^{2a+1}][1-S_v^*]^m, \quad \ldots\ldots (67a)$$

$$k_{rv}=(S_v^*)^{2a+1+m}, \quad \ldots\ldots (67b)$$

where $m=$the tortuosity exponent. The standard value $m=2.0$ suggested by Burdine (see the discussion in Ref. 15) has been adopted, together with $a=0.5$. We see that the new relative permeability curves are in the vicinity of the Corey-Burdine curves for $r_0\approx0.6$, but the deviations are important for other values of the parameter. Even though the present study is qualitative, it tends to show that wetting effects have a significant effect on relative permeabilities.

The Entry Pressure in Terms of Residual Saturation. The entry pressure C is given above by Eq. 35. It depends on the equation of state of the fluid through W (see Eq. 20) and on the capillary pressure correlation through J (see Eqs. 33 and 34b). It can easily be seen that C depends on the fluid parameters R_l and R_v. More importantly, C is a function of two variables, a and S_{vr}.

For the sake of a simple illustration, we have chosen the $\tilde{W}$–function given by Eq. 41 and have plotted the resulting dimensionless $\tilde{C}$–function in **Fig. 10** for $\tilde{R}_v=0.4$ and $\tilde{R}_l=1.6$. This plot can be used for a quick determination of the residual vapor saturation in the case of a vapor-wet porous medium, whenever the constant a and the entry pressure C are either known or can easily be determined experimentally.

Conclusions

We have presented a new model for two-phase flow in porous media for a fluid consisting of one chemical component. Two important characteristics of this model, as far as the present paper is concerned, are as follows: First, it does not need relative permeabilities, and second, it assumes that the wetting properties of the rock are described through an interaction energy, I, which is assumed to be part of the input to the model.

Information available in the traditional two-phase model has been used to calculate the I-function. The new model has then been used to throw some light on the relative permeability concept.

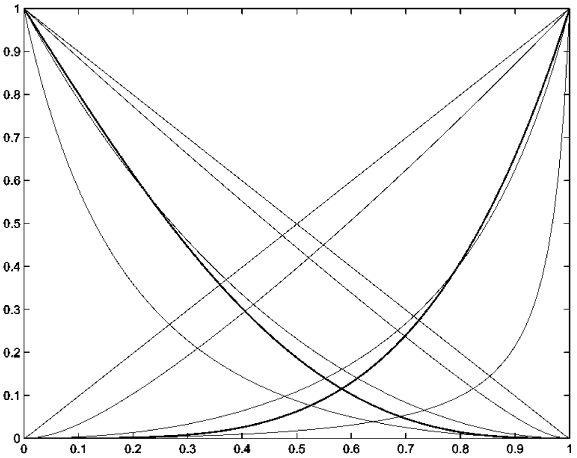

Fig. 9—Relative permeability curves given by Eqs. 65, plotted vs. normalized vapor saturation S_v^* (thin lines), for r_0 = 0.2 to 0.8, by steps of 0.2. The smallest r_0 gives almost straight lines, and the concavity increases with r_0. The thick lines are the Corey-Burdine relative permeability curves of Eqs. 67. For the numerical values of the various parameters, see the text preceding Eq. 66.

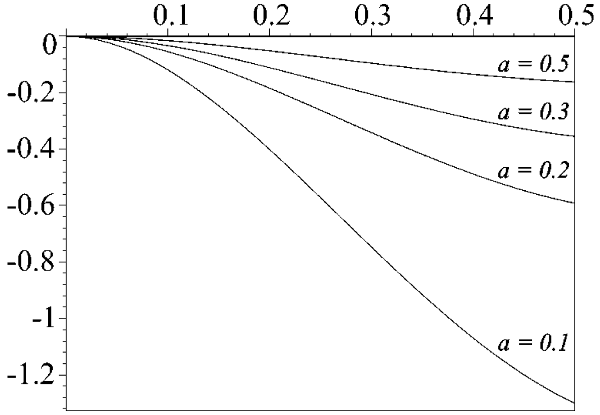

Fig. 10—The dimensionless entry pressure $\tilde{C}$ vs. S_{vr}, for different values of *a*. See text for details.

It has been shown that relative permeabilities can be defined if one assumes a stabilized flow type; that is, a flow type that can be characterized as a traveling wave. Formulas for the relative permeabilities as functions of saturation have been given, allowing the calculation of these functions when the capillary pressure correlation and the equation of state of the fluid are known; there is an additional parameter, Λ, proportional to the width of the transition region, which must also be known.

A number of additional statements can be made in the framework of the new model.

1. Capillary pressure is, strictly speaking, not a function of state in the thermodynamical sense. It becomes an equation state in the incomplete wetting approximation—that is, when the wetting angles at the pore level are between zero and 180°.
2. The relative permeability curves are straight lines for neutral wetting.
3. There is a relation between the pore size distribution parameter, the entry pressure, and the residual vapor saturation. This can be used for a quick determination of the residual vapor saturation.

Nomenclature

a = see Eq. 33
A_Σ = pore surface per unit volume
C = entry pressure; see Eq. 33
$\tilde{C}$ = C/P_c
F_Σ = free energy of the Σ-fluid, Eq. 4
g = acceleration caused by gravity
$\tilde{g}$ = $(R_cL/P_c)g$
G = gravitational potential giving rise to an acceleration ∇G
$\tilde{G}$ = $(R_c/P_c)G = \tilde{g}\tilde{x}$
I = free energy of interaction between the fluid proper and the Σ-fluid, Eqs. 4 and 15
$\tilde{I}$ = I/P_c
J = see Eq. 34
K = absolute permeability
k_{rl} = relative permeability to liquid
k_{rv} = relative permeability to vapor
ℓ = thickness of transition region between phases (pore level)
L = unit of length
m = tortuosity exponent
M = Eq. 7
$\bar{M}$ = fluid chemical potential at equilibrium
M^b = chemical potential of bulk fluid (macroscopic)
$\tilde{M}^b$ = $(R_c/P_c)M^b$
M^c = chemical potential of bulk fluid, attributable to capillary effects or wetting (macroscopic)
$\tilde{M}^c$ = $(R_c/P_c)M^c$
M_Σ = Eq. 5c
M^c_Σ = Eq. 5d
$\tilde{p}^*_l$ = dimensionless pressure in averaged liquid phase (with residual vapor)
$\tilde{p}_v$ = dimensionless pressure in averaged vapor phase
$\bar{P}$ = fluid pressure at equilibrium
P^b = pressure of bulk fluid (macroscopic)
$\tilde{P}^b$ = P^b/P_c
P^c = capillary pressure (macroscopic)
$\tilde{P}^c$ = P^c/P_c
P_c = critical pressure of averaged fluid
q = $\phi/(\ell A_\Sigma)$
R = density of averaged fluid (macroscopic level)
$\tilde{R}$ = R/R_c
R_c = critical density of averaged fluid
R_l = density of averaged liquid
$\tilde{R}_l$ = R_l/R_c
$\tilde{R}_v$ = R_v/R_c
R_v = density of averaged vapor
R^*_l = density of averaged liquid with residual vapor
$\tilde{R}^*_l$ = R^*_l/R_c
$\tilde{R}_0$ = density of averaged fluid at initial time
R^*_v = density of averaged vapor with residual liquid
R_Σ = density of averaged Σ-fluid (macroscopic level)
S_l = liquid saturation, Eq. 26b. Also $(\tilde{R}-\tilde{R}_v)/(\tilde{R}_l-\tilde{R}_v)$
S_v = vapor saturation, Eq. 26a. Also $(\tilde{R}_l-\tilde{R})/(\tilde{R}_l-\tilde{R}_v)$
S^*_v = $(\tilde{R}^*_l-\tilde{R})/(\tilde{R}^*_l-\tilde{R}_v)$, normalized vapor saturation (vapor-wet medium)
S_{vr} = residual vapor saturation
S_{lr} = residual liquid saturation
t = time
$\tilde{t}$ = $[KP_c/(\phi\eta_cL^2)]t$
$\tilde{t}_i$ = particular values of $\tilde{t}(i=1,2,3)$
T = temperature (Kelvin scale)
T_c = critical temperature (Kelvin scale)
$\mathbf{V}$ = velocity (macroscopic level)
V = velocity (macroscopic level) in 1D
$\tilde{V}$ = $[\phi\eta_cL/(KP_c)]V$, dimensionless version of V
$\tilde{V}^*_l$ = dimensionless velocity of averaged liquid (with residual vapor)
$\tilde{V}_v$ = dimensionless velocity of averaged vapor
W = Eqs. 17 and 20
$\bar{W}$ = W/P_c, Eq. 41
x = space coordinate (1D system)
$\tilde{x}$ = x/L
α = integration constant
β = integration constant
γ = Eq. 61
Γ^*_v = normalized vapor momentum, Eq. 54
η = viscosity of averaged fluid, Eq. 36
$\tilde{\eta}(\tilde{R})$ = $\eta/\eta_c = (\eta_v/\eta_l)^r,\ r = (\tilde{R}_l+\tilde{R}_v-2\tilde{R})/[2(\tilde{R}_l-\tilde{R}_v)]$
η_v = viscosity of averaged vapor
η_l = viscosity of averaged liquid
η_c = $\sqrt{\eta_v/\eta_l}$
Λ = Constant (macroscopic) related to interfacial tension (microscopic)
$\tilde{\Lambda}$ = $[R_c^2/(P_cL^2)]\Lambda$
ρ = fluid density (pore level)
ρ_Σ = density of Σ-fluid (pore level)
ϕ = porosity
Ψ = Helmoltz free energy, Eqs. 4 and 15
Ψ^b = bulk free energy, Eqs. 4, 15, and 17

Subscripts

c = critical
l = liquid
v = vapor
Σ = Σ-fluid

Superscripts

$'$ = derivative; also indicates an integration variable
b = bulk
c = capillary
$*$ = normalized variable

Overstrikes

$\sim$ = dimensionless variable

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