

Spontaneous imbibition of water into cylindrical cores with high aspect ratio: Numerical and experimental results

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Abstract

Spontaneous imbibition of water into oil-saturated porous chalk having a cylindrical shape with the top and bottom faces closed for fluid exchange was modeled by solving a diffusion-like equation with the assumption of constant capillary diffusivity coefficient (CDC). The calculated oil recoveries vs. time curves have the same shape as the experimental curves and fit them very well over the whole saturation range. Furthermore, the analytical solution of the diffusion-like equation describes water saturation profiles, which are in qualitative agreement with results reported in the literature. The model presented is able to account for different wettabilities of the rock material due to variations in the capillary diffusivity coefficient. The model is a useful tool when analyzing oil recovery rates measured from cylindrical cores in the laboratory.

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1. Introduction

Modeling spontaneous imbibition (SI) of water into oil-saturated porous media in accordance with experimental results is important when evaluating oil recovery rates from fractured water-wet reservoirs, where SI is an important process. The first attempt to quantify oil recovery vs. time on laboratory core samples was an article written by Aronofsky et al. (1958). These authors considered the following problem: Submerge a strongly water-wet rock sample containing oil at some saturation totally in water at time zero. Due to strong capillary forces water will force its way into the rock sample and expel oil. By placing constraints on the functional

relationship (1. oil production from the rock sample is a continuous monotonic function of time and it will converge to a finite limit 2. none of the properties which determine the rate of convergence change sufficiently during the process to affect the oil production rate or the limit) describing oil volume produced vs. time Aronofsky et al. (1958) arrived at the following expression:

$$\frac{V_o(t)}{V_i(t=0)} = R \cdot (1 - e^{-\lambda t}) \quad (1)$$

where, $V_o(t)$ =Volume oil produced at time t (m^3); $V_i(t=0)$ =Volume oil initially in place (m^3); R =Maximum oil recovery; λ =Constant describing the rate of convergence (s^{-1}); t =Time the rock sample has been exposed to water (s).

Later experimental and theoretical work has shown that the expression derived by Aronofsky et al. (1958) describes the oil expulsion rate reasonably well. Exper-

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imental data, however, shows that the exponential relationship tends to underestimate the oil recovery rate at early times which can lead to serious error when evaluating production rates. Because of the complexity of the spontaneous imbibition process, analytical or semi-analytical solutions to the flow equations have only been derived for a limited number of cases, usually for simplified geometries. Li et al. (2003) showed that the flow equations describing 1-D linear counter-current flow of oil and water could be solved by a similarity transformation and an analytical expression could be obtained. Kashchiev and Firoozabadi (2002) solved the non-linear diffusivity equation obtaining approximate analytical solutions for the transient stage when water imbibed counter-currently into linear, cylindrical and spherical geometries. Chen et al. (1990) solved the non-linear diffusivity equation semi-analytically using an integral equation formulation.

The objective of this paper is to present an analytical solution to the problem arising when water is allowed to imbibe freely due to capillary forces into an oil-saturated cylindrical core having the top and the bottom faces closed for fluid exchange being at its initial water saturation. This boundary condition implies that the analytical solution also should be approximately valid for cylindrical cores having high aspect ratio, i.e. cylindrical cores where the core length L is much longer than the core diameter d ($L > 2.5 \cdot d$). The approach chosen to solve the problem is to model the SI as a diffusion process assuming that the so-called capillary diffusivity coefficient (CDC) is constant and independent of water saturation. Results obtained from the analytical solution will be compared to experimental data performed on rock samples having both the top and the bottom faces closed for fluid exchange. The method presented here will be useful when interpreting spontaneous imbibition rates into cylindrical cores.

2. Experimental section and theoretical model

2.1. Experimental section

Details related to the experimental procedures, core handling, etc. are reported elsewhere (Standnes, 2004) and only a brief description is given here.

2.1.1. Fluids

n-Decane from Riedel-deHäen with grade >95% was used as oil phase. This oil does not modify the wettability of the porous medium due to the absence of polar components. The density of the oil phase is 0.731 g/cm^3 at $20 \text{ }^\circ\text{C}$. Distilled water was used as the imbib-

ing water phase. Fluid properties for the SI tests are summarized in Table 1.

2.1.2. Porous medium

Outcrop chalk samples from Denmark were used as the porous medium. The samples were taken from the same block and porosities were all close to 42% and absolute permeability was determined to be approximately 2 mD. No-flow surface boundary conditions were obtained by coating the top and bottom faces of the cylindrical cores with Glasfiberspackel (polyester) from Hagmans Kemi AB Sweden. Rock properties for all tests are shown in Table 2. It should be noticed that the test numbers are taken directly from the original work (Standnes, 2004).

2.1.3. n-Decane-water IFT

n-Decane-water IFT was determined at (46 mN/m) ambient temperature using a ring-tensiometer.

2.1.4. Rock sample geometry

The geometry and boundary conditions are shown Fig. 1. The cylindrical rock samples are referred to as CY, whereas the boundary conditions are referred to as 2 FC (2 Faces coated).

2.1.5. Spontaneous imbibition tests

All SI tests were performed at ambient temperature ($\approx 20 \text{ }^\circ\text{C}$). The tests were performed using Amott cells except the tests performed on the cylindrical cores with diameter 6.00 and 10.00 cm. Tests in the Amott cells were performed by introducing 100% oil-saturated cores into the cell, adding distilled water and monitoring oil production as a function of time. The oil production could be measured with an accuracy of $\pm 0.1 \text{ ml}$. In case of the cylindrical cores having $d=6.00$ and 10.00 cm , the oil-saturated rock samples were suspended from a balance and immersed in distilled water. The change in the weight of the core as a function of time was monitored. The accuracy of the balance was 0.01 g corresponding to an accuracy of approximately $\pm 0.02 \text{ ml}$ water imbibed. The mass of the rock sample was determined at the end of each

Table 1
Fluid properties

Liquid phase	Density at $20 \text{ }^\circ\text{C}$ (g/cm^3)	Viscosity at $20 \text{ }^\circ\text{C}$ ($\text{mPa} \cdot \text{s}$)	IFT using decane as oil phase at $20 \text{ }^\circ\text{C}$ (mN/m)
Distilled water	0.9982	1.0	46
n-Decane	0.731	0.95	–

Table 2
Core data from SI tests

Test #	Sample shape-boundary condition	Diameter (d) and radius (a) (m)	Height (H) (m)	$V_b \times 10^{-6}$ (m ³)	Porosity (%)	PV $\times 10^{-6}$ (m ³)
3	CY-2FC	0.0199 0.00995	0.05	15.55	43.0	6.68
4	CY-2FC	0.0339 0.01695	0.0496	44.77	42.6	19.06
5	CY-2FC	0.06 0.03	0.032	90.48	42.4	38.36
6	CY-2FC	0.1 0.05	0.0397	311.8	43.2	134.7

run to check material balance and good agreement between measured volume of water imbibed and the weight of the rock sample was obtained in all tests. The total mass of water imbibed was used to calibrate the volume of water imbibed during the imbibition test to correct for the effect of the downward movement (approximately 0.6 cm) of the oil-water contact during each run (the movement will induce decreasing buoyancy forces).

2.2. Theoretical model

2.2.1. Water imbibition rate into the cylindrical core sample

Consider a cylindrical core with an initial water or oil saturation (S_{wi} or S_{oi}) with the top and the bottom faces closed for fluid exchange. If the porous medium is water-wet and the core is totally immersed in water at time $t=0$, the normalized oil saturation S at any point (x, y, z) in the rock sample at time $t \geq 0$ will be described by the following hyperbolic partial differential equation (Aziz and Settari, 1985):

$$\phi \frac{\partial S}{\partial t} = \frac{\partial}{\partial x} D(S_w) \frac{\partial S}{\partial x} + \frac{\partial}{\partial y} D(S_w) \frac{\partial S}{\partial y} + \frac{\partial}{\partial z} D(S_w) \frac{\partial S}{\partial z} \quad (2)$$

where

$$D(S_w) = - \frac{k k_{ro}}{\mu_o} \cdot \frac{1}{1 + \frac{k_{ro}}{k_{rw}} \cdot \frac{\mu_w}{\mu_o}} \cdot \frac{dP_c}{dS_w} \quad (3)$$

has units m²/s and is called capillary diffusivity coefficient (CDC). Furthermore, k =Absolute permeability (m²); k_{ro} =Relative permeability to oil (fraction); k_{rw} =Relative permeability to water (fraction); μ_o =Oil viscosity (Pa s); μ_w =Water viscosity (Pa s); P_c =Capillary pressure (Pa); S_w =Normalized water saturation (m³/m³); t =Imbibition time (s); ϕ =Fractional porosity (m³/m³).

The radial geometry calls for the application of cylinder-coordinates. Hence Eq. (2) transforms into:

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial S}{\partial r} = \frac{\phi}{D} \cdot \frac{\partial S}{\partial t} = \frac{1}{\alpha^2} \cdot \frac{\partial S}{\partial t} \quad (4)$$

where, $\alpha^2 = D/\phi$ (m²/s) is assumed to be constant and independent of S_w .

The capillary diffusivity coefficient, $D(S_w)$, is in general a function of water saturation (Kashchiev and Firoozabadi, 2002). Assuming D to be constant (a kind of average CDC) Eq. (2) becomes linear and the normalized oil saturation in the matrix block $S=S(r, t)$ can be found analytically with the boundary and the initial condition specified in Appendix 1. The solution to Eq. (2) expressing oil saturation as a function of time (since oil recovered is equal to the amount of water imbibed (incompressible two-phase flow)) can be written as an infinite series:

$$S_{AV}(t) = 4 \sum_{n=1}^{\infty} \frac{1}{z_n^2} \cdot e^{-\frac{D}{\alpha^2} \cdot \left(\frac{z_n}{a}\right)^2 t} \quad (5)$$

where, z_n =Zeros of the Bessel function $J_0(-)$, $n=1, 2, 3, \dots$; a =Radius of the cylindrical core (m); S_{AV} =Average oil saturation in the core at time t (m³/m³).

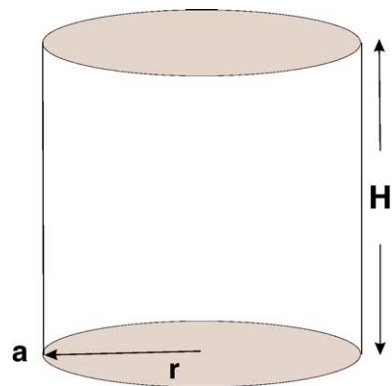


Fig. 1. Core geometry for the spontaneous imbibition tests. The top and bottom faces are closed for fluid exchange as indicated on the drawing.

Since water volume imbibed is equal to oil volume produced the following expression can be obtained for average normalized water saturation in the core as a function of time:

$$S_{wAV}(t) = 1 - S_{AV}(t) = 1 - 4 \sum_{n=1}^{\infty} \frac{1}{z_n^2} \cdot e^{-\frac{D}{\phi} \cdot \left(\frac{z_n}{a}\right)^2 t} \quad (6)$$

S_{wAV} Average normalized water saturation in the core at time t (m^3/m^3).

The solution to Eq. (1) given in Eq. (5) is subject to the following assumptions:

- The capillary diffusivity coefficient is assumed to be constant and independent of water saturation;
- The water and the oil phase flow as incompressible fluids;
- Gravity forces are neglected.

The solution given in Eq. (6) should also be valid for rock samples having a wettability different from strongly water-wet, because the geometry with top and bottom phases closed will reduce the impact of gravity forces on the oil production rate. Capillary forces (magnitude of D) will therefore govern the oil expulsion rate even if these are reduced compared to strongly water-wet conditions. The model should therefore be of great interest when analyzing oil recovery rates due to differences in rock sample wettability as none strongly water-wet conditions currently are regarded as the most usual case for most reservoir core material.

2.2.2. Water saturation profile

The normalized oil saturation (fraction of recoverable oil) as a function of position r and time t can be found from Eq. (A17):

$$S(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_0\left(z_n \cdot \frac{r}{a}\right)}{z_n J_1(z_n)} \cdot e^{-\frac{D}{\phi} \cdot \left(\frac{z_n}{a}\right)^2 t} \quad (7)$$

where,

$J_1(z_n)$ Bessel function of the first kind and order one evaluated at z_n , $n=1, 2, 3, \dots$

The normalized water saturation profile inside the core at time $t=t^*$ can be found by plotting $S_w(r)$ as a

Table 3

The first forty zeros of the Bessel-function $J_0(x)$ (Watson, 1980)

Zeros (z_n) of the Bessel-function $J_0(x)$			
n	$J_0(z_n)=0$	n	$J_0(z_n)=0$
1	2.4048256	11	33.7758202
2	5.5200781	12	36.9170984
3	8.6537279	13	40.0584258
4	11.7915344	14	43.1997917
5	14.9309177	15	46.3411884
6	18.0710640	16	49.4826099
7	21.2116366	17	52.6240518
8	24.3524715	18	55.7655108
9	27.4934791	19	58.9069839
10	30.6346065	20	62.0484692
21	65.1899648	31	96.6052680
22	68.3314693	32	99.7468199
23	71.4729816	33	102.8883743
24	74.6145006	34	106.0299309
25	77.7560256	35	109.1714896
26	80.8975559	36	112.3130503
27	84.0390908	37	115.4546127
28	87.1806298	38	118.5961766
29	90.3221726	39	121.7377421
30	93.4637188	40	124.8793089

function of r where the normalized water saturation is given by:

$$S_w(r) = 1 - S(r, t^*) = 1 - 2 \sum_{n=1}^{\infty} \frac{J_0\left(z_n \cdot \frac{r}{a}\right)}{z_n J_1(z_n)} \cdot e^{-\frac{D}{\phi} \cdot \left(\frac{z_n}{a}\right)^2 t^*} \quad (8)$$

Forty terms have been used in the infinite series in Eqs. (6) and (8) due to limited access to the zeros of the Bessel-function J_0 . The values of these zeros are given in Table 3 (Watson, 1980). The error introduced by applying only forty terms is approximately 1%. Hence, forty terms should be sufficient to give reliable results within the accuracy of the experimental uncertainty.

3. Results and discussion

The results comparing calculated oil recovery vs. time from Eq. (6) to experimental data will first be presented before looking at the development of the water saturation profile inside the porous medium as a function of r and t .

3.1. Comparison of calculated oil recovery vs. time with experimental results

Oil recovery vs. time calculated from Eq. (6) is plotted in Fig. 2 for cores with diameters equal to 1.99, 3.39, 6.0 and 10.0 cm together with experimental curves for cores having the same diameters (the cylindrical cores had the top and bottom faces closed for fluid exchange). Ideally, the same CDC should be used to calculate all the oil recovery curves shown in the figure, because the porous medium and fluid systems were the same in all experiments. Variation and uncertainty in experimental parameters may allow for variations of the CDCs used in the calculation of oil recovery vs. time for each

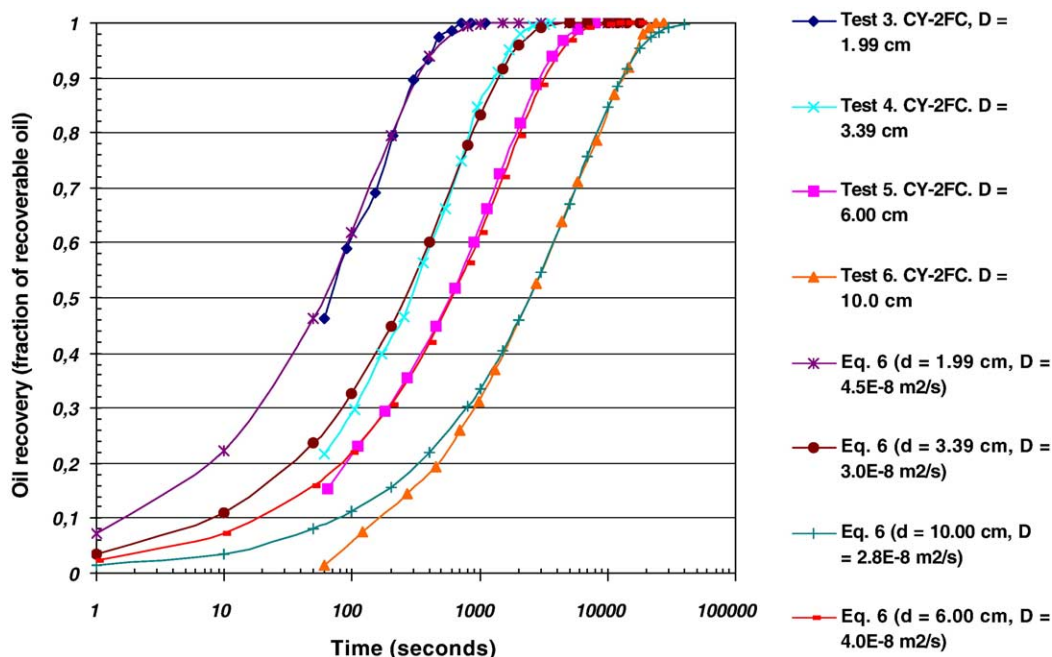


Fig. 2. Modeled and experimental oil recovery curves for cylindrical cores having the top and bottom face closed for fluid exchange (logarithmic time scale). Optimal capillary diffusivity coefficient value used is indicated for each calculated curves.

experiment. After performing a best-fit approximation the variation in the CDCs applied varied from 2.8 to $4.5 \times 10^{-8} \text{ m}^2/\text{s}$ for the 10.0 and 1.99 cm diameter core, respectively (the criterion as when to optimize the curves is that the best-fitted model and the experimental curves should superimpose for normalized water saturations greater than approximately 0.25 , see below). The results using optimal CDCs show that the calculated oil recovery curves vs. time fit the experimental curves extremely well. All four curves superimposed with the corresponding experimental curve over the whole saturation range, except for a small discrepancy for very small times where the experimental uncertainty is maximum. The same data set is re-plotted in Fig. 3 on a linear time scale, because the logarithmic time scale has a tendency to suppress the differences when comparing two different curves, especially at large times. The calculated curves in Fig. 3 still fit the experimental curves very well, confirming the ability for Eq. (6) to model oil recovery vs. time satisfactorily.

These results are quite surprising given the simplification introduced by assuming that the CDC is independent of water saturation. In the literature CDC has been shown to be a bell-shaped curve of water saturation with a maximum approximately at normalized water saturation equal to 0.5 (Kashchiev and Firoozbadi, 2002). The value of CDC approaches zero at high and small values of the normalized water saturation.

Assuming that the approximation of using a constant CDC is best for intermediate values of the normalized water saturation and using these values to optimize the fit between modeled and experimental curves, largest discrepancy between experimental and calculated oil recovery curves vs. time should therefore be expected for normalized water saturation close to zero and unity. It is difficult to interpret discrepancies between experimental and modeled curves at early time of imbibition as the experimental uncertainty is at maximum in this range due to the practical problem of releasing the produced oil from the rock sample and measure the oil volume accurately. Looking at the experimental and modeled curves for large times in Figs. 2 and 3, i.e. for normalized water saturations close to unity there are still not much differences. The modeled results therefore seem to fit the experimental data well for normalized water saturations where the experimental data are expected to have the highest degree of reliability. These results justify the assumption that 1-D radial counter-current SI can be modeled fairly well using constant CDC.

The predicted oil recovery vs. time for the Aro-nofsky model given by Eq. (1) is plotted in Fig. 4 as well as the functional relationship for the square root of time for recoveries of oil up to 0.5 . Comparison is made for the experimental result for the core sample having $d=3.39 \text{ cm}$ and the corresponding theoretical curve for

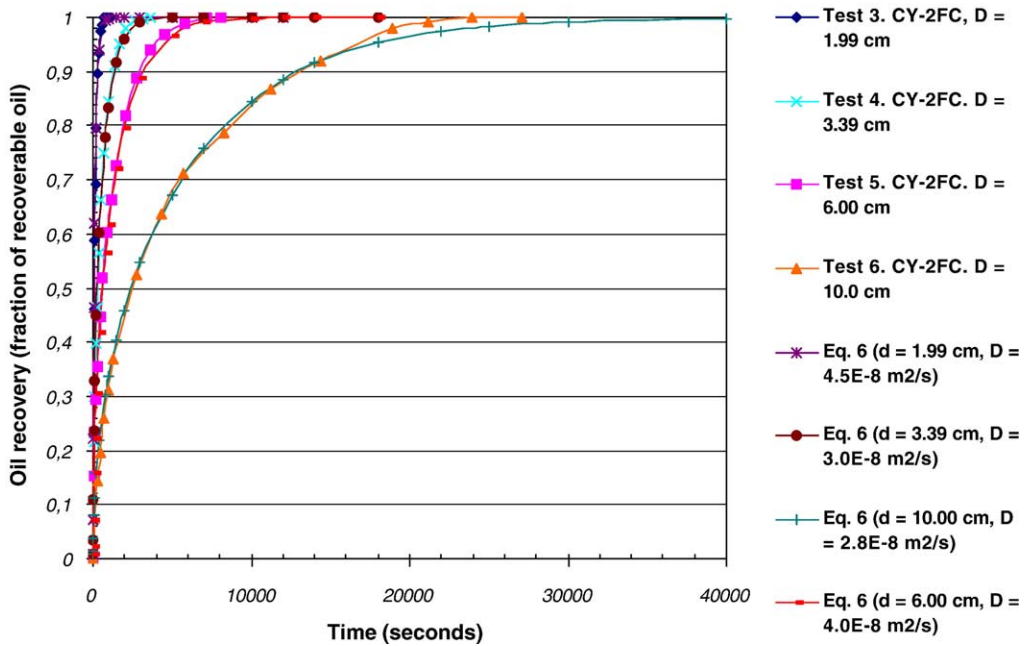


Fig. 3. Modeled and experimental oil recovery curves for cylindrical cores having the top and bottom face closed for fluid exchange (linear time scale). Optimal capillary diffusivity coefficient value used is indicated for each calculated curves.

this rock sample from Eq. (6). It can be seen that the Aronofsky model deviates considerably from the analytical solution given in Eq. (6). It underestimates oil recovery at early time and over predict at late times.

Hence, the shape of the oil recovery vs. time curve is significantly better described by using the analytical solution in Eq. (6). Several authors have shown that oil recovery vs. time is more accurately described by a

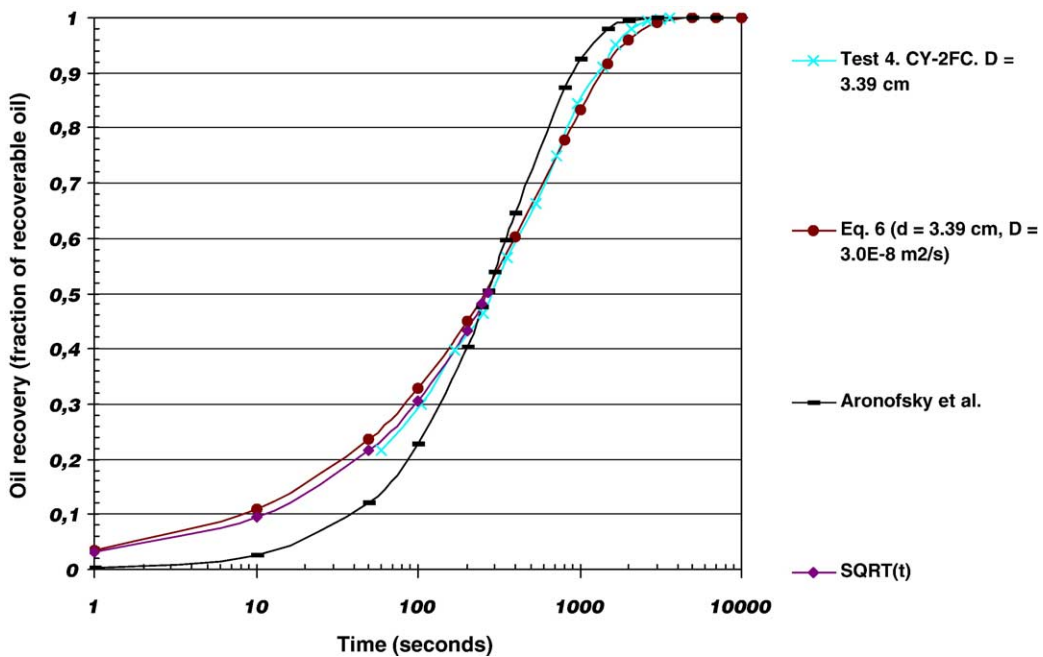


Fig. 4. Modeled and experimental oil recovery curves for the case $d=3.39$ cm compared to Eq. (1) (Aronofsky et al., 1958) and the square root of time function.

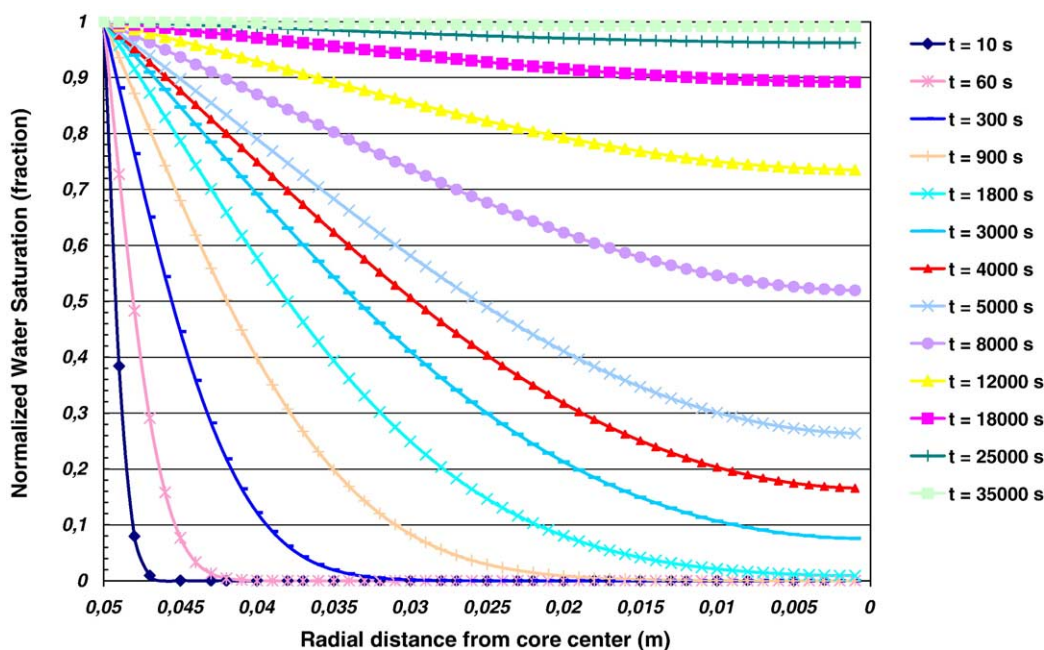


Fig. 5. Calculated water saturation profiles from Eq. (8) for the core with $d=10.0$ cm for different times.

square root of time relationship for early times (Li et al., 2003; Reis and Cil, 1993). Fig. 5 shows that the square root function fits the experimental result for the core sample with $d=3.39$ cm much better than the Arofnosky model and it is close to the analytical solution given by Eq. (6).

3.2. Water saturation profiles

The normalized water saturation profiles calculated from Eq. (8) for the core having $d=10.0$ cm is depicted in Fig. 5 for different times. The curves show that the water saturation inside the core plug decreases almost linearly with the distance from the lateral side and there are no sharp waterfronts. The latter observation is not in accordance with most experimental and simulated results, which indicate there should be a sharp waterfront when water is imbibing into 100% oil-saturated core (Baldwin and Spinler, 1999; Kashchiev and Firsozabadi, 2002). It is, however, interesting to notice that Chen et al. (1995) based on data by Kazemi et al. have published water saturation profiles very much similar to the profiles presented in Fig. 5. The reason for the absence of a sharp waterfront for the curve shown in Fig. 5 is probably because the properties inherent in the diffusivity equation. It can be observed that according to Eq. (6), the water saturation will start to increase instantaneously for all r as soon as t

becomes bigger than zero, however small. That means that the water is diffusing into the core with infinite speed, which is obviously not correct (Greenberg, 1978). The water saturation profiles shown in Fig. 5 describe the increase in water saturation in the rock sample qualitatively correct without including the correct sharp water front. This conclusion is based on the fact that the shapes of the oil recovery vs. time curves are almost identical for the experimental and modeled cases.

Baldwin and Spinler (1999) observed quite different water saturation profiles into chalk samples depending on initial water saturations and wettability using MRI. They concluded that the sharp waterfront observed for SI into 100% oil-saturated cores decreased with either increase in the initial water saturation or with decreasing water-wetness of the rock material. The case investigated here ($S_{wi}=0\%$) is a rather artificial situation and may not apply directly to SI of water under real reservoir conditions. Real reservoir material is expected to have S_{wi} above 10–15% and the wettability of the material is probably not strongly water-wet. The modeling approach should fit experimental data even better when modeling oil recovery vs. time and water saturation profiles using real reservoir core material and fluids. The model presented opposing most SI models also has the potential to accounts for variations in rock wettability through the value of the CDC.

4. Conclusions

The following conclusions can be drawn from this work:

- The analytical solution, Eq. (6), gives oil recovery curves vs. time fits well to experimental spontaneous imbibition data over the whole saturation range for cores used in the experiments (chalk cores having diameter in the range from 1.99 to 10.0 cm).
- The water saturation profiles predicted by the analytical solution (Eq. (8)) give results in qualitatively agreement with other reported results in the literature. However, Eq. (8) cannot account for the expected sharp invading waterfront for cores having $S_{wi}=0$.
- The present model also has a potential for describing spontaneous imbibition of water into rock material having wettability different from strongly water-wet, which is the case for most reservoir rocks.

Nomenclature

A	Arbitrary constant
a	Radius of cylindrical core (m)
B	Arbitrary constant
C	Arbitrary constant
CDC	Capillary diffusivity coefficient
D	Capillary diffusion coefficient (m^2/s)
d	Diameter (m)
E_n	Expansion coefficient
$f(r)$	Function orthogonal to the function $g(r)$ with respect to the inner product (f, g)
$g(r)$	Function orthogonal to the function $f(r)$ with respect to the inner product (f, g)
H	Height of the cylindrical rock sample (m)
$J_\nu(x)$	Bessel-functions of first kind and order ν
$-k^2$	Separation constant
k	Absolute permeability (m^2)
k_n	$z_n/a \cdot (m^{-1})$
k_{ro}	Relative permeability to oil (–)
k_{rw}	Relative permeability to water (–)
MRI	Magnetic resonance imaging
PV	Pore volume (cm^3)
P_c	Capillary pressure (Pa)
Q_o	Volume oil produced at time t (m^3)
R	Maximum oil recovery level as a fraction of recoverable oil or function only depending on the r -coordinate ($R=R(r)$)
r	Coordinate in r -direction
S	Normalized oil saturation (fraction, m^3/m^3)
S_o	Oil saturation in the position (r, θ, z) at time t (fraction, m^3/m^3)
S_{AV}	Average oil saturation at time t (fraction, m^3/m^3)

S_w	Water saturation at time t in position (r, θ, z) (fraction, m^3/m^3)
S_{wAV}	Average water saturation at time t (fraction, m^3/m^3)
S_{wi}	Initial water saturation (fraction, m^3/m^3)
S_{oi}	Initial oil saturation (fraction, m^3/m^3)
S_{of}	Minimum oil saturation at maximum water saturation (fraction, m^3/m^3)
SI	Spontaneous imbibition
T	Function only depending on time ($T=T(t)$)
t	Imbibition time (s)
V_b	Bulk volume (m^3)
$V_i(t=0)$	Initial oil volume in place (m^3)
$V_o(t)$	Oil volume after time t (m^3)
x	Coordinate
$Y_0(x)$	Bessel-function of second kind and order zero
y	Coordinate
z	Coordinate
z_n	Zero of the Bessel-function J_0 for $n=1, 2, 3, \dots$
α^2	$D/\phi(m^2/s)$
σ	Oil–water interfacial tension (mN/m)
ϕ	Fractional porosity (m^3/m^3)
λ	Factor determine the rate of convergence (s^{-1})
μ_o	Oil viscosity (Pa. s)
μ_w	Water viscosity (Pa. s)
θ	Coordinate in θ -direction

Appendix A

Consider a cylindrical core saturated with oil and water at initial oil saturation (S_{oi}) or irreducible water saturation (S_{wi}). The core material is strongly water-wet and the rock sample is submerged totally in water at $t=0$. Due to strong capillary forces water will imbibe spontaneously into the core and the normalized oil saturation at time t can be described by the following partial differential equation (Aziz and Settari, 1985):

$$\phi \frac{\partial S}{\partial t} = \frac{\partial}{\partial x} D(S_w) \frac{\partial S}{\partial x} + \frac{\partial}{\partial y} D(S_w) \frac{\partial S}{\partial y} + \frac{\partial}{\partial z} D(S_w) \frac{\partial S}{\partial z} \quad (A1)$$

where

$$S = \frac{S_o - S_{of}}{S_{oi} - S_{of}} \quad S_{of} \leq S_o \leq S_{oi} \quad \text{and} \quad 0 \leq S \leq 1$$

and, S =Normalized oil saturation at time t (m^3/m^3); S_{of} =Residual oil saturation when water imbibition has ceased (m^3/m^3); S_{oi} =Initial oil saturation at irreducible

water saturation (m^3/m^3); S_o =Oil saturation in the point (x, y, z) at time t (m^3/m^3); t =Imbibition time (s).

Due to the geometry of the problem it is convenient to re-express Eq. (A1) in cylindrical coordinates. Hence,

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial S}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 S}{\partial \theta^2} + \frac{\partial^2 S}{\partial z^2} = \frac{\phi}{D} \cdot \frac{\partial S}{\partial t} \quad (A2)$$

The core is assumed to have the top and bottom faces closed for fluid exchange so the problem turns out to be essentially a 1-D radial flow problem where the fluid exchange only takes place on the lateral side of the rock sample. Furthermore, it will be assumed that the CDC is constant and independent of water saturation. There is no variation in θ direction due to symmetry and no z -dependency due to the boundary condition (no fluid exchange through the top and bottom surfaces). (A2) therefore reappear in the following form:

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial S}{\partial r} = \frac{\phi}{D} \cdot \frac{\partial S}{\partial t} = \frac{1}{\alpha^2} \cdot \frac{\partial S}{\partial t} \quad (A3)$$

where

$$\alpha^2 = D/\phi \text{ (m}^2/\text{s)}$$

and the initial and boundary conditions read:

$$S(r, 0) = 1 \quad \text{for } r < a \text{ (the core is 100\% oil - saturated at } t = 0)$$

$$S(a, t) = 0 \quad \text{for } t \geq 0 \text{ (only water at the boundary for } t \geq 0)$$

Eq. (A3) can be solved by separation of variables seeking the solution as a product of the two functions R and T such that:

$$S(r, t) = R(r) \cdot T(t) \quad (A4)$$

Insert (A4) into (A3) gives the following two ordinary differential equations:

$$rR'' + R' + k^2 rR = 0 \quad \text{and} \quad T' + \alpha^2 k^2 T = 0 \quad (A5)$$

where, the separation constant has been set equal to $-k^2$. The general solutions to the equations in (A5) are:

$$R(r) = AJ_0(kr) + BY_0(kr) \quad \text{and} \quad T = Ce^{-\alpha^2 k^2 t} \quad (A6)$$

where A, B and C are constants and $J_0(kr)$ and $Y_0(kr)$ are Bessel-function of first and second kind (both of

order zero), respectively (Greenberg, 1978). Since the solution should be finite everywhere, B is put equal to zero because $Y_0(kr)$ approaches infinity as $r \rightarrow 0$. Thus, the solution to (A1) is:

$$S(r, t) = E \cdot J_0(kr) \cdot e^{-\alpha^2 k^2 t} \quad (A7)$$

where, $E \equiv A \cdot C$ =New constant.

The boundary condition requires that:

$$S(a, t) = 0 = E \cdot J_0(ka) \cdot e^{-\alpha^2 k^2 t} \quad (A8)$$

so (ka) should coincide with the zeros of the Bessel function J_0 . Hence,

$$k_n = \frac{z_n}{a}, n = 1, 2, 3, 4, \dots \quad (A9)$$

where, z_n is the zeros of the Bessel function J_0 . The solution to (A1) obeying the boundary condition can therefore be written:

$$S(r, t) = \sum_{n=1}^{\infty} E_n \cdot J_0(k_n r) \cdot e^{-\alpha^2 k_n^2 t} \quad (A10)$$

The initial condition requires:

$$S(r, 0) = 1 = \sum_{n=1}^{\infty} E_n \cdot J_0(k_n r) \quad (A11)$$

Using the fact that the eigenfunctions $J_0(k_n r)$, $n = 1, 2, 3, \dots$, are orthogonal with respect to the inner product:

$$(f, g) = \int_0^a f(r)g(r)rdr \quad (A12)$$

the expansion coefficients can be written (Greenberg, 1978):

$$E_n = \frac{\int_0^a J_0(k_n r)rdr}{\int_0^a [J_0(k_n r)]^2 rdr} \quad (A13)$$

Using the integral formulas

$$\int_0^a [J_0(k_n r)]^2 rdr = \frac{a^2 [J_1(k_n a)]^2}{2} \quad (A14)$$

and

$$\int x^v J_{v-1} dx = x^v J_v + C \quad (A15)$$

gives the following solution for the oil saturation $S(r, t)$ as a function of r and t :

$$S(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_0(k_n r)}{k_n a \cdot J_1(k_n a)} \cdot e^{-\alpha^2 k_n^2 t}. \quad (\text{A16})$$

Substituting $k_n = \frac{z_n}{a}$, $\alpha^2 = \frac{D}{\phi}$ gives:

$$S(r, t) = 2 \sum_{n=1}^{\infty} \frac{J_0(z_n \cdot \frac{r}{a})}{z_n J_1(z_n)} \cdot e^{-\frac{D}{\phi} \cdot (\frac{z_n}{a})^2 \cdot t}. \quad (\text{A17})$$

Volume oil in the rock sample at time t can be found by volume integration. Thus,

$$Q_o = \int_0^H z \int_0^a S(r, t) \cdot 2\pi r dr. \quad (\text{A18})$$

Performing the integration gives:

$$Q_o = 4\pi a^2 H \sum_{n=1}^{\infty} \frac{1}{z_n^2} \cdot e^{-\frac{D}{\phi} \cdot (\frac{z_n}{a})^2 \cdot t}. \quad (\text{A19})$$

Dividing (A19) with the bulk volume ($\pi a^2 H$) gives the average oil saturation in the core sample at time t :

$$S_{AV}(t) = 4 \sum_{n=1}^{\infty} \frac{1}{z_n^2} \cdot e^{-\frac{D}{\phi} \cdot (\frac{z_n}{a})^2 \cdot t}. \quad (\text{A20})$$

The increase in average normalized water saturation as a function of time (equal to decrease in oil saturation) is therefore given by:

$$S_{wAV}(t) = 1 - S_{AV}(t) = 1 - 4 \sum_{n=1}^{\infty} \frac{1}{z_n^2} \cdot e^{-\frac{D}{\phi} \cdot (\frac{z_n}{a})^2 \cdot t}, \quad (\text{A21})$$

which is similar to Eq. (6).

Appendix B

From a mathematical point of view it is interesting to notice that (A20) implies that the sum of the reciprocal zeros of the Bessel-function J_0 squared is a convergent

series with sum equal to $1/4$. Since the normalized oil saturation is assumed to be unity at $t=0$, (A20) implies:

$$S_{AV}(0) = 4 \sum_{n=1}^{\infty} \frac{1}{z_n^2} = 1 \quad (\text{A22})$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{z_n^2} = \frac{1}{4} \quad (\text{A23})$$

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