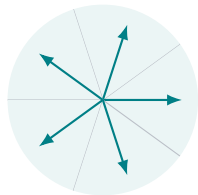


Clusters, Twistors and Stability

From decay conditions to twistor geometry

$$y''(x) = (x^3 + ax + b)y(x)$$
$$x \in \mathbb{C}$$



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Joint work with Tom Bridgeland

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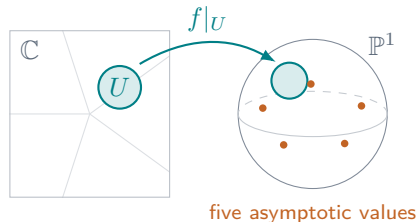
The equation is a projective structure

Solutions y_1, y_2 of $y'' = Q y$, $Q = x^3 + ax + b$, are entire; $W = y_1' y_2 - y_1 y_2' \neq 0$ is constant.

$$f = \frac{y_1}{y_2} : \mathbb{C} \longrightarrow \mathbb{P}^1, \quad f' = \frac{W}{y_2^2} \neq 0.$$

Charts $f|_U$; a new basis gives $g \circ f$,
 $g \in \mathrm{PGL}_2(\mathbb{C})$: a **projective structure** on $\mathbb{C}$.
 Conversely,

$$\{f, x\} = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 = -2 Q(x).$$

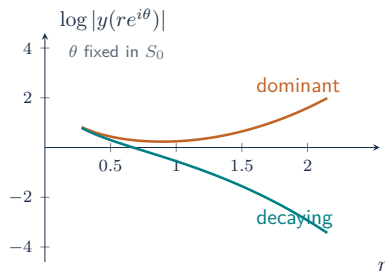
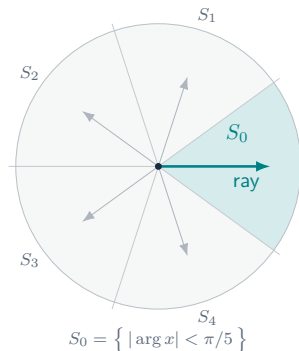


$$\mathbb{C}_{a,b}^2 \cong \{ \text{framed projective structures} \\ \text{on } \mathbb{C} \text{ with cubic Schwarzian} \}$$

The ODE *is* the projective structure; (a, b) are its coordinates.

Fixed polar type: pole of order 7 at ∞ (Gupta–Mj's $\mathcal{P}(3)$); degree $n + 1$ gives A_n . Any entire Q : infinite-dimensional.

Decay depends on direction



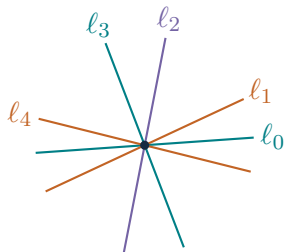
$$y \sim r^{-3/4} \exp(\pm \frac{2}{5} r^{5/2}) \text{ along a ray in } S_0$$

Each sector S_j contains a unique subdominant solution, up to scale.

Sibuya (1975). Leading asymptotics for $a = b = 0$; in general the exponent is $\frac{2}{5}x^{5/2} + O(x^{1/2})$.

Five lines in a two-dimensional solution space

$$V_{a,b} = \{y : y'' = (x^3 + ax + b)y\} \cong \mathbb{C}^2.$$



solution lines (schematic)

$$S_j \longmapsto \ell_j \in \mathbb{P}(V_{a,b}) \cong \mathbb{P}^1.$$

Adjacent lines are distinct:

$$\ell_j \neq \ell_{j+1} \quad (j \bmod 5).$$

Equivalently: the asymptotic values of $f = y_1/y_2$, called *crown tips*.

Sibuya (1975); Allegretti, Adv. Math. 337 (2018), introduction.

Sibuya's map

$$M : \mathbb{C}^2 \longrightarrow \frac{\{(p_0, \dots, p_4) \in (\mathbb{P}^1)^5 : p_j \neq p_{j+1}\}}{\mathrm{PGL}_2(\mathbb{C})}$$
$$(a, b) \longmapsto [\ell_0, \dots, \ell_4].$$

Source

Two coefficients of a monic, centred cubic.

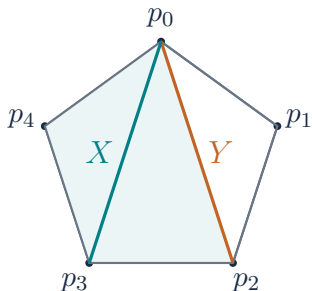
Target

Five cyclically labelled crown tips, up to projective equivalence.

A holomorphic local biholomorphism between complex surfaces.

Sibuya; Bakken, Amer. J. Math. 99 (1977); Allegretti (2018). Nonadjacent points may coincide.

Cross-ratio coordinates



$$\mathrm{CR}(a, b, c, d) = -\frac{(a-b)(c-d)}{(b-c)(d-a)}.$$

One cross-ratio per diagonal of a triangulation:

$$X = \mathrm{CR}(p_0, p_2, p_3, p_4), \quad Y = \mathrm{CR}(p_0, p_1, p_2, p_3).$$

Chart: $(p_0, \dots, p_4) = (\infty, -1, 0, Y, Y(1+X))$,
 $X, Y \in \mathbb{C}^*$. Flip p_0p_3 to p_2p_4 and relabel:

$$\mu(X, Y) = (Y(1+X), X^{-1}).$$

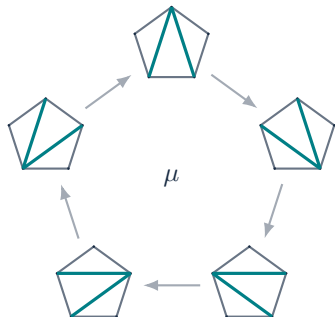
Flipping a diagonal changes the coordinates by the mutation μ .

Fock–Goncharov, Publ. Math. IHES 103 (2006); Allegretti (2018), Section 3.

The target is the A_2 cluster variety

Five triangulations, five tori, glued by μ :

$$\mathcal{X}_{\mathbb{C}}(A_2) = \bigcup_T (\mathbb{C}^*)^2_T, \quad \boxed{\mu^5 = \text{id}}$$



This is the target of Sibuya's map:

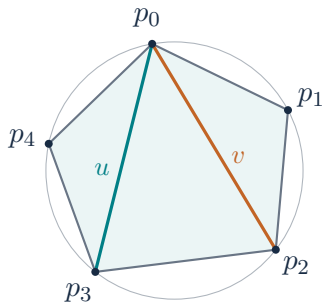
$$\mathcal{X}_{\mathbb{C}}(A_2) = \frac{\{p_j \neq p_{j+1}\}}{\text{PGL}_2(\mathbb{C})} \cong \overline{M}_{0,5} \setminus \bigcup_j D_{j,j+1}.$$

Here $\overline{M}_{0,5} = dP_5$, and the five curves $D_{j,j+1}$, where adjacent points collide, form a pentagon.

Sibuya's map lands in dP_5 minus a pentagon of (-1) -curves.

Fock–Goncharov; Fomin–Zelevinsky; Allegretti (2018). Quiver $1 \rightarrow 2$. The fixed point (φ, φ^{-1}) of μ , φ the golden ratio, is the cubic $y'' = x^3 y$.

Positive points and the log mutation



ideal pentagon (Klein model)

If $X, Y > 0$ in one chart, then in every chart: μ is subtraction-free.

Positive points: $p_0, \dots, p_4$ in cyclic order on $\mathbb{RP}^1$, that is, ideal pentagons in $\mathbb{H}^2$.

In logarithms $u = \log X$, $v = \log Y$ (shear coordinates):

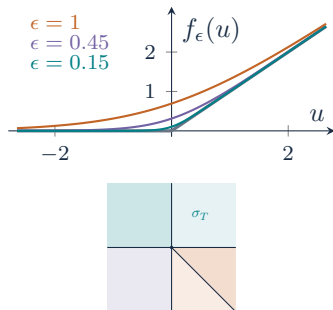
$$\mu_{\log}(u, v) = (v + \log(1 + e^u), -u).$$

$$\text{reminder: } \mu(X, Y) = (Y(1 + X), X^{-1})$$

Each chart is global on positive points: $\mathcal{X}_{\mathbb{R}_{>0}} \cong \mathbb{R}^2$.

Fock (shear coordinates); Fock–Goncharov; Bridgeland–Ruddat, Part I.

Continuous tropicalization



Rescale: $X = e^{u/\epsilon}$, $Y = e^{v/\epsilon}$, $\epsilon > 0$.

$$\mu_\epsilon(u, v) = (v + f_\epsilon(u), -u),$$

$$f_\epsilon(u) = \epsilon \log(1 + e^{u/\epsilon}) \xrightarrow{\epsilon \rightarrow 0} \max(0, u).$$

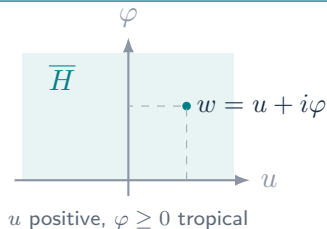
$$\mu_{\text{trop}}(u, v) = (v + \max(0, u), -u).$$

Glued by μ_{trop} : the **tropical cluster variety**
 $\mathcal{X}_{\mathbb{R}^{\text{trop}}} \cong \mathbb{R}^2$, one cone per triangulation.

Real-analytic gluing at $\epsilon = 1$, piecewise-linear gluing at $\epsilon = 0$.

Integer tropical points are integral laminations (Fock–Goncharov).

The log cluster space



One closed half-plane per diagonal of a triangulation T ; the cells $\overline{H}_T^2$ are glued along their boundaries by

$$\mu_{\mathcal{L}}(w_1, w_2) = (w_2 + \log(1 + e^{w_1}), -w_1).$$

$$\Phi_T : \overline{H}_T^2 \rightarrow \mathcal{X}_{\mathbb{R}_{>0}} \times \mathcal{X}_{\mathbb{R}^{\text{trop}}}, \quad w \mapsto (\operatorname{Re} w, \operatorname{Im} w),$$

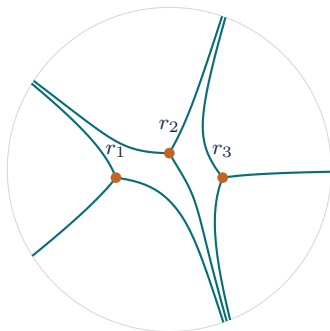
and $\exp_{\mathcal{L}} : \mathcal{L} \rightarrow \mathcal{X}_{\mathbb{C}}, x_j = e^{w_j}.$

Theorem (Bridgeland–Ruddat, Part I). $\mathcal{L}$ is a complex manifold, the Φ_T glue to a homeomorphism $\mathcal{L} \rightarrow \mathcal{X}_{\mathbb{R}_{>0}} \times \mathcal{X}_{\mathbb{R}^{\text{trop}}}$, and $\exp_{\mathcal{L}}$ is a holomorphic local homeomorphism, not a covering. For A_2 : $\mathcal{L} \cong \mathbb{C}_{a,b}^2$, and $\exp_{\mathcal{L}}$ is Sibuya's map.

Proof of the last claim: identify $\mathcal{L}$ with projective structures on $\mathbb{C}$ by grafting (Gupta–Mj); then $\exp_{\mathcal{L}}$ is the crown-tip map.

Cluster data alone reconstruct Sibuya's map.

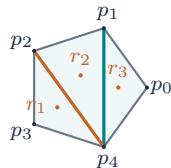
Quadratic differentials also produce triangulations



$$P(x) = x^3 - x + 0.35i$$

$$\phi = P(x) dx^2, \quad \text{Im} \int^x \sqrt{P(t)} dt = \text{const},$$

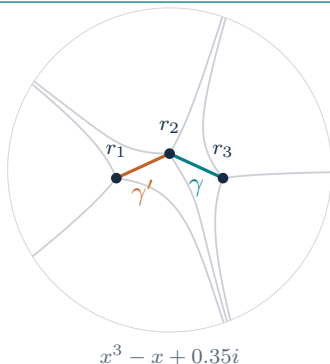
for $P(x) = x^3 + ax + b$. Separating trajectories cut out strips; a dual arc across each strip is a diagonal:



Here: the fan at p_4 , one zero per triangle.

Bridgeland–Smith; Allegretti. Vertex p_j : the direction $\arg x = 2\pi j/5$.

The small-parameter limit sees periods



$$\hbar^2 y'' = (x^3 + ax + b) y.$$

$$Z_\gamma = \oint_\gamma \sqrt{x^3 + ax + b} dx,$$

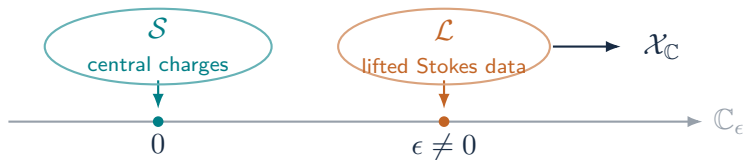
where γ runs from one zero of the cubic to another and back on the other sheet of the square root: a closed path on its Riemann surface. X_γ is the cross-ratio of the diagonal that γ crosses (slide 11).

$$\hbar \log X_\gamma(\hbar) \longrightarrow Z_\gamma$$

The cluster coordinate, rescaled, becomes a period of $\sqrt{x^3 + ax + b}$.

As $\hbar \rightarrow 0$ in suitable sectors. At leading order the condition $X = -1$ becomes the quantisation rule $Z_\gamma \approx 2\pi i \hbar (n + \frac{1}{2})$ (backup). Iwaki–Nakanishi; Allegretti.

The stability space is the limit $\epsilon \rightarrow 0$



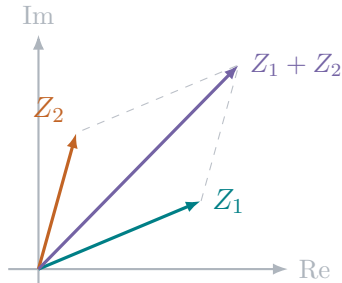
$$\mathcal{S} \subset \mathcal{X}_{\mathbb{R}^{\text{trop}}} \times \mathcal{X}_{\mathbb{R}^{\text{trop}}} \xleftarrow{\epsilon \rightarrow 0} \mathcal{L} \cong_{\text{top}} \mathcal{X}_{\mathbb{R}_{>0}} \times \mathcal{X}_{\mathbb{R}^{\text{trop}}}$$

On cells $w = \epsilon \log x \in \overline{H}^2$ tends to $z \in \tilde{H}^2$, $\tilde{H} = \{z \neq 0 : 0 \leq \arg z < \pi\}$: the real part tropicalizes (slide 9), the gluing becomes linear, and z is the period Z_γ of slide 12.

$$\mathcal{S} \cong \text{Stab}^\circ(\mathcal{D}_{A_2})/\text{Br} \cong \{x^3 + ax + b : 4a^3 + 27b^2 \neq 0\}$$

Positive coordinates tropicalize; the central charges are the coordinates on $\mathcal{S}$.

Periods become central charges



A stability condition assigns central charges to classes of objects:

$$Z : K_0(\mathcal{D}_{A_2}) \longrightarrow \mathbb{C}.$$

Their arguments are phases; stability determines the allowed stable constituents.

$$\mathcal{S} \cong \text{Stab}^\circ(\mathcal{D}_{A_2}) / \text{Br}(\mathcal{D}_{A_2}) \cong \mathbb{C}^2 \setminus \Delta.$$

WKB chambers become stability chambers.

Bridgeland, Ann. of Math. (2007); Bridgeland–Smith, Publ. Math. IHES 121 (2015). Here $\Delta = \{4a^3 + 27b^2 = 0\}$ and $\mathcal{D}_{A_2}$ is the associated CY_3 category.

The main theorem

Theorem (Bridgeland–Ruddat). For every ADE quiver Q there is a complex manifold $\mathcal{Z}$ with a holomorphic submersion $\pi : \mathcal{Z} \rightarrow \mathbb{C}$ such that

- $\pi^{-1}(0) \cong \mathcal{S} \cong \text{Stab}^\circ(\mathcal{D}_Q)/\text{Br}(\mathcal{D}_Q)$, the space of central charges;
- $\pi^{-1}(\epsilon) \cong \mathcal{L}$ for $\epsilon \neq 0$, and $\exp_{\mathcal{L}} : \mathcal{L} \rightarrow \mathcal{X}_{\mathbb{C}}$ is a local biholomorphism;
- $\mathbb{C}$ acts with $\pi(s \cdot z) = e^s \pi(z)$, and $s = \pi i$ acts as the DT element [1].

Built from half-planes $\text{Im}(w/\epsilon) \geq 0$, glued by $w_j \mapsto w_j + \epsilon v_{ij} \log(1 + e^{\pm w_i/\epsilon})$.

Type A_2 : $\mathcal{S} = \{x^3 + ax + b : 4a^3 + 27b^2 \neq 0\}$; $\mathcal{L} = \{y'' = (x^3 + ax + b)y\}$, grafted pentagons; $\exp_{\mathcal{L}} = \text{Sibuya's map}$.

Stability conditions at $\epsilon = 0$, Stokes data at $\epsilon \neq 0$: one complex manifold.

Bridgeland–Ruddat, Parts I–II. *Twistor*, after Penrose: sections of π are to be the twistor lines of a Joyce structure; $\epsilon = \infty$ is Part III.

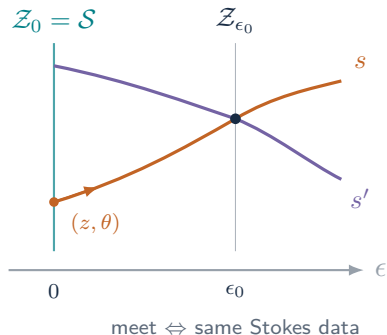
Stability of one side is complex moduli of the other

1. **Categories.** Kontsevich: $D^b\mathrm{Fuk}(X) \simeq D^b\mathrm{Coh}(Y)$, hence the stability spaces agree.
2. **Moduli.** Stability conditions on $D^b\mathrm{Coh}(Y)$ contain the Kähler moduli of Y ; mirror symmetry makes these the complex moduli of the partner X , with central charges $\int_L \Omega_X$. Here that space is $\mathcal{S}$, with the periods as coordinates.
3. **The family.** HMS does not relate $\mathcal{S}$ to the Betti side, the cluster variety $\mathcal{X}_{\mathbb{C}}$ of Stokes data and local systems. $\mathcal{Z}$ does: central charges at $\epsilon = 0$, cluster coordinates for $\epsilon \neq 0$. In type A the $\epsilon \neq 0$ fibre is the space of cubic projective structures.

HMS identifies the categories; a Joyce structure is the family from $\mathcal{S}$ to $\mathcal{X}_{\mathbb{C}}$.

Kontsevich (1994); Thomas–Yau (2002); Bridgeland (2007); Smith, Quiver algebras as Fukaya categories (2015).

Penrose's picture: twistor lines give TS



- A **twistor line**: a section s of π over $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1)^{\oplus n}$, $n = \dim \mathcal{S}$ (A_2 : $\mathcal{O}(1) \oplus \mathcal{O}(1)$).
- Its jet at $\epsilon = 0$ is a point $(z, \theta) \in TS$: twistor lines $\leftrightarrow$ points of TS , locally.
- Two lines meet over ϵ iff their Stokes data agree there: a foliation $\mathcal{F}_\epsilon$ of TS ; in the Joyce picture $\mathcal{Z}_\epsilon = TS/\mathcal{F}_\epsilon$.
- This pencil of flat non-linear connections is the **Joyce structure**: a non-linear Frobenius manifold, with GL_n replaced by Poisson automorphisms of a torus (DT wall-crossing).

Points of TS are curves in $\mathcal{Z}$; how they meet is the geometry.

Penrose (1976); Bridgeland (2025). A_2 : deformed cubic oscillators, Painlevé I (backup); $\epsilon = \infty$: Part III.