

Lagrangian torus fibrations of CY hypersurfaces

Jerusalem
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M anticanonical hypersurface of complex dim N
in smooth symplectic toric variety Z
that is a res. of sing of a toric Fano variety

e.g. $M \subseteq \mathbb{CP}^{N+1}$ smooth degree $N+2$ alg. hypers.
e.g. quintic in $\mathbb{P}^4$

Theorem (Mak, Matessi, R, Zharkov)

$\exists$ Lagrangian torus fibration $\pi: M \rightarrow B$ with N -dim^l fibers
s.t.

- (1) $B \simeq_{\text{diff}} S^N$
- (2) critical set of π is real codim 4 in M
- (3) $\exists$ Lag. section of π missing crit. set.
- (4) discriminant $D \subset B$ has codim 2
- (5) π is smooth over $\pi^{-1}(B \setminus D)$ and has simple singularities along D .

Corollary: Arnold-Liouville $\Rightarrow B \setminus D$ receives integral affine structure

§ Local models part I

Consider hypersurface $\{z_0^5 + \dots + z_4^5 = z_0 \dots z_4\} \subseteq \mathbb{CP}^4$
CY 3-fold

Cover $\mathbb{CP}^4$ by open sets where some $|z_i|$ are smaller compared to others

e.g. $\underbrace{\{z_0 = z_1 = 0\}}_{\text{near}} \subseteq \mathbb{CP}^4$
 $\cong \mathbb{CP}^2$ $|z_0|, |z_1|$ small

write M as $\frac{1}{z_2 z_3 z_4} (z_0^5 + \dots + z_4^5) = z_0 z_1$

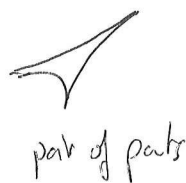
$f|_{\mathbb{CP}^2}$ is a quintic curve

affine: $z_2 = 1$

$$1 + z_3^5 + z_4^5 = 0$$



$z_2 = 1$



pair of pants

We call

$$X_0 X_1 = 1 + w_1 + w_2$$

a local model of type
 $(k, n, d) = (1, 2, 0)$

More generally type (k, n, d) is

$$X: \{z_0 \dots z_k = 1 + w_1 + \dots + w_n\} \subseteq \mathbb{C}^{k+1} \times (\mathbb{C}^*)^n \times \mathbb{C}^d$$

$$\text{Symplectic form } \omega = \frac{i}{2} \sum_{j=0}^k dz_j \wedge d\bar{z}_j + \sum_{\ell=1}^n \frac{dw_\ell \wedge d\bar{w}_\ell}{|w_\ell|^2}$$

$$f: X \rightarrow \mathbb{R}^{k+n+d}$$

$$(|z_0|^2 - |z_1|^2, \dots, |z_0|^2 - |z_k|^2, \log |w_1|, \dots, \log |w_n|)$$

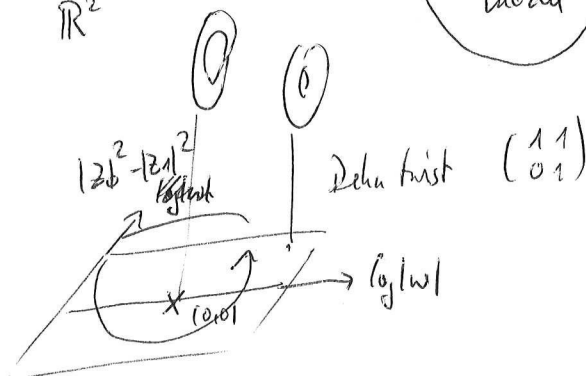
generically a torus fibration, Lagr. if $n=0,1$

Most famous $(k, n, d) = (1, 1, 0)$ Δ not for $n \geq 2$

$$\{z_0 z_1 = 1 + w\} \subseteq \mathbb{C}^2 \times (\mathbb{C}^*)^1$$

$$\downarrow f = (|z_0|^2 - |z_1|^2, \log |w|)$$

$\mathbb{R}^2$



$\exists$ global S^1 action $z_0 \mapsto g \cdot z_0$ $g \in U(1)$
 $z_1 \mapsto g^{-1} \cdot z_1$

More generally

local model of type (k, n, d) has

$$T^k \times T^d$$

$z_0 \mapsto g_i \cdot z_0$ action $\uparrow$ unused variables
 $z_i \mapsto g_i^{-1} \cdot z_0$

$$z_j \mapsto z_j$$

$$j \neq i$$

very

Basic

Approach:

- ① "Correct" f so that it becomes
 - Lagrangian also for $n \geq 2$
 - codim 2 discriminant
- ② cut M into local models of various types
- ③ glue local model fibrations

This approach exists since the 90s, no success so far!

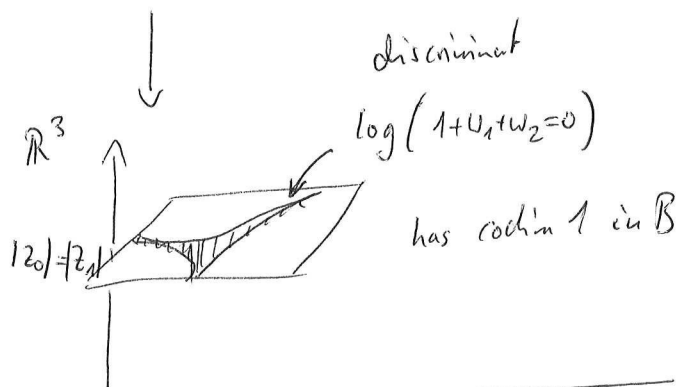
Thm (Evans-Mauri) 2019

$\exists$ Lag form fibr. of the negative vertex
 with codim 2 discriminant

Approach: • Compactify local model
 by normal crossing divisor D

"negative vertex" $(k, n, d) = (1, 2, 0)$

$$\{z_0 z_1 = 1 + w_1 + w_2\}$$



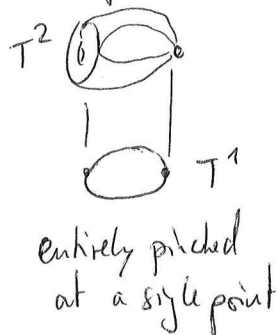
Lagrangian torus fibrations II

Jan 2026

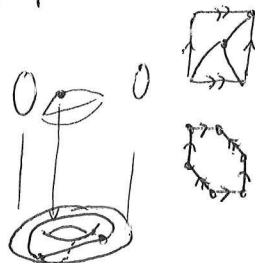
yesterday: local models, quantic, positive/negative vertex

Singular fibers

positive vertex
 T^2 -fibrations over T^1



negative vertex
 T^1 -fibrations over T^2



contains 3 circles

so we can attach

3 ways

focus-focus $\times S^1$
sing fibre

$\mathbb{Q} \times S^1$

§ Evans-Mauri model for negative vertex

$$\{z_0 z_1 = 1 + w_1 + w_2\} = U$$

(a) write down a very ad-hoc and asymmetric compactification

X so that $D = X \setminus U$ is a normal crossing divisor surface and every component of D is almost toric, so has a Lagrangian $(S^1)^2$ -fibration (Leung-Symington)

(b) smooth the fibrations of D to a fibration of the link at ∞ of U

baby example is local model $(2,0,0)$:

$$z_0 z_1 z_2 = 1 \rightsquigarrow z_0 z_1 z_2 = t$$

$t \rightarrow 0$ normal crossing

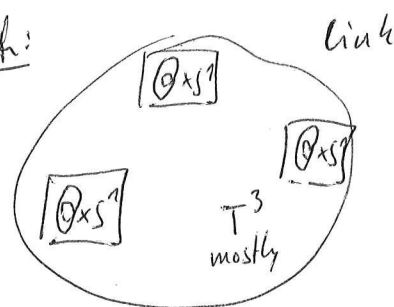
(c) The link is a real 5-fold, so we produced a T^3 -fibration of the link.

use backwards Liouville flow to

flow this fibration over entire interior

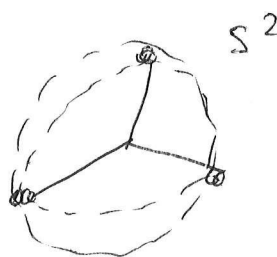
so get fibration of U

Sketch:



three singular fibres of type $(1,1,1)$

Arnold Liouville base of filling



Y-shaped discriminant codimension 2!

fibre over Liouville curve is Liouville skeleton (Morse Bott)

critical $T^2 \xrightarrow{\text{flow}}$ skeleton of pair of pants $\mathbb{Q}$ get involved with vertex fibre

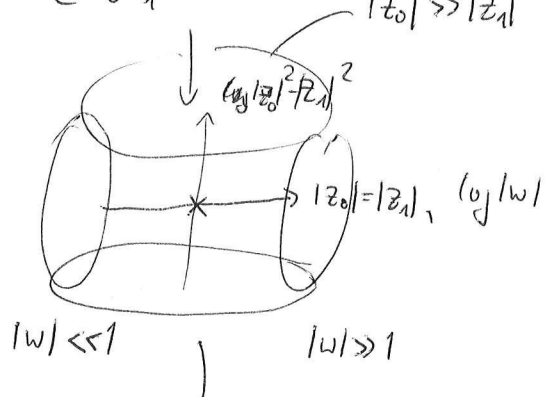
(topological work with Zhusov "tailoring pair of pants")

Problems: • does not generalize to higher dimensions by using Leung-Symington for surfaces
• how to glue EM model into other models?

§ Inductive hierarchy of local models

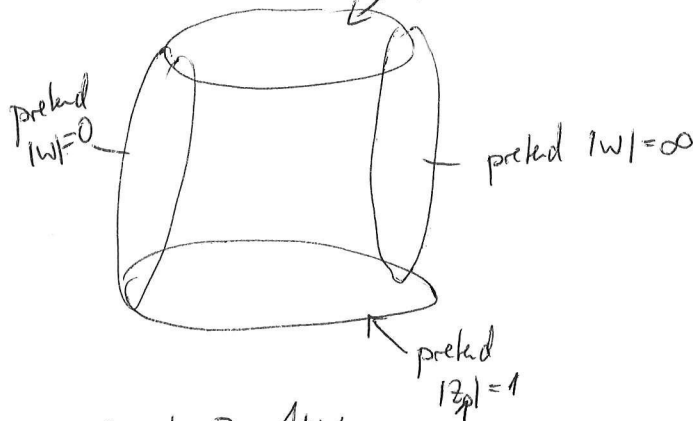
e.g. focus-focus $(1,1,0)$

$$\{z_0 z_1 = 1 + w\}$$

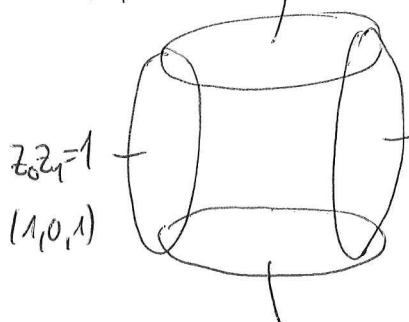


$$|z_1| \gg |z_0|$$

$$\text{pred } |z_0| = 1$$



$$(0,1,1) \quad z_1 = 1 + w$$



$$z_0 z_1 = w$$

$$\sim z_0 z_1 = 1 \quad (1,0,1)$$

$$z_0 = 1 + w$$

$$(0,1,1)$$

Upshot: the $(1,1,0)$ -model is "surrounded" by models $(a,b,1)$ with $a+b=1$

General fact: A (k,n,d) -model is enclosed by (k',n',d') -models with $d' > d$

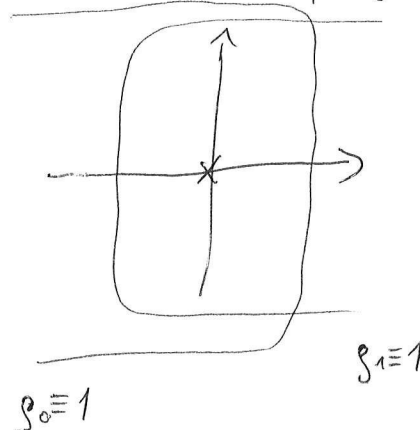
How to make "pred" rigorous?

Introduces "ironing factors" in local model equations

$$z_0 \dots z_n = g_0 + g_1 w_1 + \dots + g_n w_n$$

$$g_i = 0 \text{ if } |w_i| \ll |w_j| \text{ for } j \neq i$$

$$g_i = 1 \text{ otherwise, e.g. focus-focus}$$



Fix dimension $N = k + n + d$

Basic idea: if we had a torus fibration on all models of type (k,n,d) with $d' > d$, we could plaster the "boundary" of a model with give $d' > d$ with these, get a fibration of a link and flow that to fill the model with give $d' > d$

in short: use backwards Liouville flow inductively on local models

key difficulty:

• are these fibrations compatible/equal on overlaps?

• change of coordinates?
e.g. $z_0 z_1 = w \rightsquigarrow z_0 z_1 = 1$

Ambient toric variety $\mathbb{A}^{N+1}$, moment map T_{achl}^{N+1}

Big tons

$$(\mathbb{C}^*)^{N+1} \xrightarrow{\text{Sympl.}} Q \times \left(\frac{\mathbb{R}^{N+1}}{\mathbb{Z}^{N+1}} \right)^*$$

$$\mu: \mathbb{C} \rightarrow Q$$

$$(p, \theta) \quad \omega = \sum_{i=1}^{N+1} dp_i \wedge d\theta_i$$

How arrive at this symplectomorphism?

$$Q = \left\{ \underbrace{\lambda_i - \langle e_i, x \rangle}_{=: v_i(x)} \geq 0, \quad e_i \text{ normal to facets of } Q \right\}$$

Define $\phi: Q \rightarrow \mathbb{R}$

$$\phi = \frac{1}{2} \sum v_i \log v_i + \phi_0$$

$$\text{for } \phi_0: \mathbb{R}^{N+1} \rightarrow \mathbb{R} \quad C_\infty \quad \text{Kähler potential}$$

Complex coordinates on $Q \times \left(\frac{\mathbb{R}^{N+1}}{\mathbb{Z}^{N+1}} \right)^*$ then

$$z_j = e^{2j\phi + i\theta_j}$$

Equation for hypersurface M :

$$\sum_{a \in P} c_a e^{\langle d\phi, a \rangle + i\langle \theta, a \rangle}$$

$\nwarrow$ reflexive lattice polytope of toric Fano

$$Q \times \left(\frac{\mathbb{R}^{N+1}}{\mathbb{Z}^{N+1}} \right)^* \xrightarrow{\log |z|} (\mathbb{C}^*)^{N+1}$$

$$\searrow \quad \swarrow$$

$$d\phi \quad \mathbb{R}^{N+1}$$

We are free to choose c_a conveniently
(all choices are symplectomorphic)

Choose a triangulation T of $2P$

induced by strictly convex fuchs

(extended as star triangulation to P)

$$\eta: P \rightarrow \mathbb{R}$$

s, S real parameters, choose equation

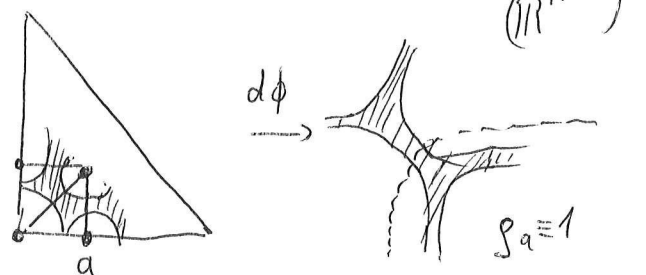
$$\sum_{a \in \text{Vol } T} s_a e^{-\eta(a)s} z^a = e^S$$

$$\text{where } z^a = e^{\langle d\phi, a \rangle + i\langle \theta, a \rangle} \text{ as before.}$$

s_a ironing factors as earlier
(make hypersurface non-algebraic)

Theorem M is still symplectic
(Abouzaid, Donaldson)

Example $P = Q = 3\Delta$



so here xy

equation looks like $xy = 1$

We explained s_a give boundary models for variables
How about 3-variables

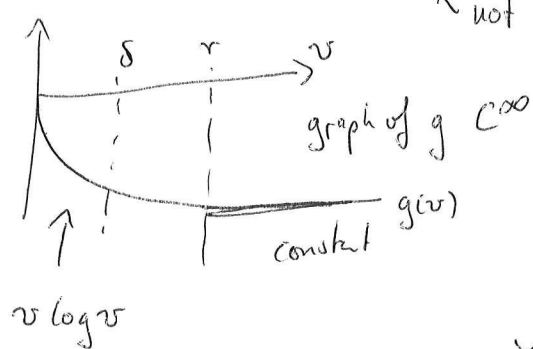
e.g.

$$z_0 z_1 z_2 = 1 + w$$

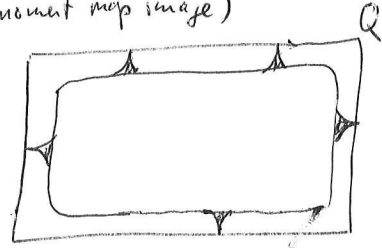
say $|z_0|^2 - |z_1|^2$ large + or small -
either way $z_0 \neq 0 \neq z_1$ so S^1 action
is free on $z_0 z_1$, treat as single monomial

We want to use a non-strictly convex potential

$$\Phi = \sum_i g(v_i) + s\psi \quad \leftarrow \text{not nec. strictly convex}$$



Amoeba of M is not (moment map image)



close to ∂Q

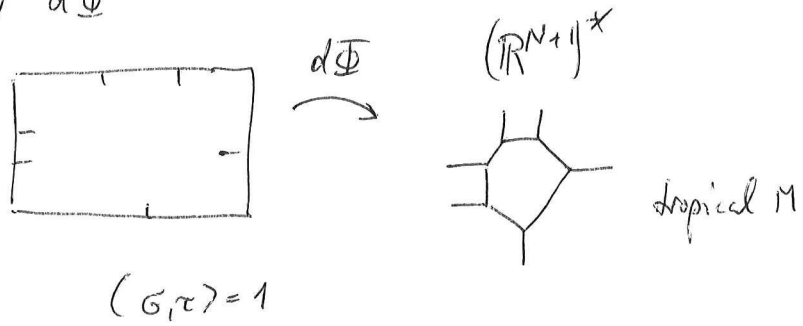
$$X = \sum_a g_a e^{-\eta(a)s}$$

$$\cdot e^{\frac{\partial_a \Phi + i \partial_a}{2a}} = e^s$$

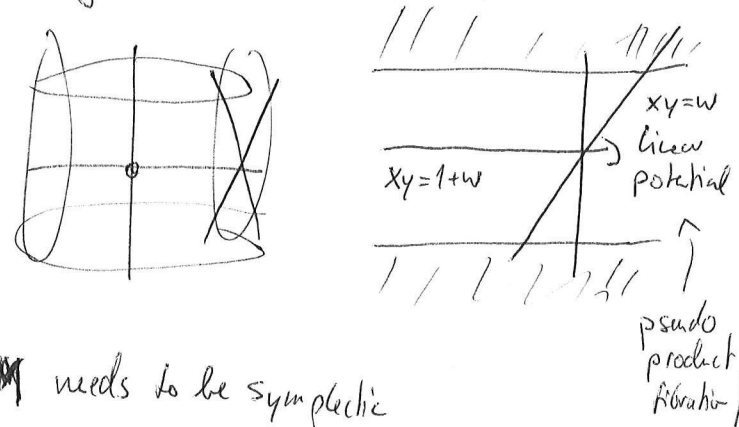
What ψ to choose?

Needs to solve 3 problems:

(1) $d\Phi$



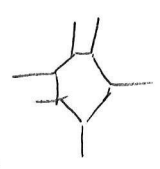
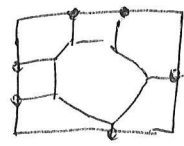
(2) Tilting in regions with linear potential



(3) M needs to be symplectic

(Lemma enough to be symplectic along moment map slice of T^k action)

Solution to all 3 problems



put the tropical variety into its Newton polygon

(can't be done linearly)

Step 1 $Q' = \text{smaller version of } Q$

$X_{Q'} * \eta \leftarrow \text{function of triangle } T$

$\nearrow$ dw. fct of Q'

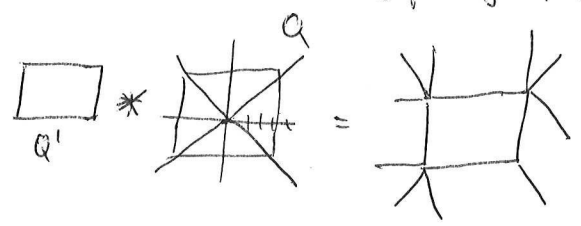
$\nwarrow$ convolution

$\equiv 1 \text{ on } Q'$

$\equiv \infty \text{ outside}$

$\lambda_1 * \lambda_2(u) = \min(\lambda_1(u) + \lambda_2(v) \mid xy=v)$

graph of $\lambda_1 * \lambda_2 =$ Minkowski sum of graphs of λ_1, λ_2



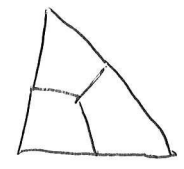
Step 2 further subdivide to get dual inside

$\nwarrow$ barycentric



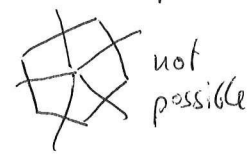
barycentric

Also



By Ceva's theorem

for simplices of classical geometry



not possible