COMPUTING THE EULER CHARACTERISTIC OF GENERALIZED KUMMER VARIETIES

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Abstract. We give an elementary proof of the formula $\chi(K^nA) = n^3\sigma(n)$ for the Euler characteristic of the generalized Kummer variety K^nA, where $\sigma(n)$ denotes the sum of divisors function.

1. Introduction

Let A be an abelian surface and let $n \geq 2$ be a natural number. Beauville [1] introduced the *generalized Kummer variety* K^nA (see Definition 2.2), as an example of a compact irreducible symplectic variety. In this note we will give an almost elementary proof of the following:

Theorem 1.1 (Göttsche [6])**.** *The topological Euler characteristic of the generalized Kummer variety is given by*

$$\chi(K^nA) = n^3\sigma(n)$$

where $\sigma(n)$ denotes the sum of divisors function $\sigma(n) = \sum_{d|n} d$.

This formula was first found by Göttsche [6, Corollary 2.4.13], as a corollary of his computation of the Betti numbers of K^nA. If we aim only at the Euler characteristic, however, a much simpler argument is possible. Indeed, Debarre [3] gave an alternative proof of the theorem, using a Lagrangian fibration of a certain (subvariety of a) relative Jacobian of curves on A, together with various geometric constructions relating the relative Jacobian to K^nA. Our proof utilizes a Lagrangian fibration of K^nA itself, in the case where A is a product of elliptic curves.

The structure of our argument can be outlined as follows: It is enough to consider the case where $A = E \times E'$ is a product of elliptic curves (Proposition 3.1). In this case, K^nA admits a Lagrangian fibration (Section 4), and only the most degenerate fibres contribute to the Euler characteristic (Lemma 4.2). Using the formula of Ellingsrud and Strømme (Theorem 2.1) for the Euler characteristic of the punctual Hilbert scheme, the computation of $\chi(K^nA)$ can be reduced to the computation of the Euler characteristic of certain varieties parametrizing effective divisors on E (Lemma 4.3). This computation is carried out in Section 5, with the help of a well known combinatorial formula, recalled in Section 2.3.

1.1. **Notation.** We work with schemes over $\mathbb{C}$ throughout. By a map we mean a morphism in the category of schemes. By a variety we mean a reduced, not necessarily irreducible, separated scheme of finite type. Whenever $a \in A$ is a point on an abelian variety, we write $T_a\colon A \to A$ for the translation map, and we denote by $0 \in A$ the identity element for the group law.

2. Preliminaries

2.1. **Topology.** We are concerned with the Euler characteristic χ defined using cohomology with compact support. It has two friendly properties.

Firstly, χ is additive: If X is a variety, and $U \subset X$ is open, we have

$$\chi(X) = \chi(X \setminus U) + \chi(U).$$

By a point-set-topological argument it follows that

$$\chi(X) = \sum_i \chi(U_i)$$

whenever $X = \bigcup_i U_i$ is a disjoint union of locally closed subsets.

Secondly, χ is multiplicative: If $f\colon X \to Y$ is a map of algebraic varieties, such that all fibres $f^{-1}(y)$ have equal Euler characteristic, we have

$$\chi(X) = \chi(Y)\chi(f^{-1}(y)), \quad (\text{any } y \in Y).$$

This follows from the well known multiplicative property for topological fibrations, together with the existence of a stratification of Y into locally closed strata, such that f is locally trivial (in the transcendent topology) above each stratum [10, Corollaire 5.1].

2.2. **Geometry.** Given a surface X, we write $X^{[n]}$ for the Hilbert scheme parametrizing finite subschemes $\xi \subset X$ of length n, and $X^{(n)}$ for the symmetric product parametrizing positive zero-cycles on X of degree n. There exists a map [8], the *Hilbert-Chow morphism*,

$$\rho\colon X^{[n]} \to X^{(n)}$$

which on the level of sets sends a subscheme $\xi \in X^{[n]}$ to its cycle.

Fix a point $p \in X$. The *punctual Hilbert scheme* is the reduced subvariety $H(n) \subset X^{[n]}$ consisting of subschemes supported at p. We suppress both the point p and the surface X from the notation, as the isomorphism class of $H(n)$ is independent of these choices. We will make essential use of the following:

Theorem 2.1 (Ellingsrud and Strømme [4])**.** *The Euler characteristic of the punctual Hilbert scheme $H(n)$ equals the number $p(n)$ of partitions of n.*

In the case where $X = A$ is an abelian surface, we may compose the Hilbert-Chow morphism with the n-fold addition map on A to obtain a map

$$A^{[n]} \to A.$$

The fibre over $0 \in A$ is the Kummer variety of Beauville:

Definition 2.2. Given A and n, the *generalized Kummer variety* K^nA is the closed subset

$$K^nA = \left\{ \xi \in A^{[n]} \,\middle|\, \sum_{x \in \xi} m_x x = 0 \right\}$$

(where m_x denotes the multiplicity of x in ξ) together with its reduced induced structure.

From now on, we will drop the word "generalized", and simply refer to K^nA as a Kummer variety.

2.3. **Combinatorics.** We will need an expression for the sum of divisors function $\sigma(n)$ in terms of the number of partitions function $p(n)$. Our starting point is the well known formula

$$p(n) = \frac{1}{n} \sum_{k=1}^{n} \sigma(k) p(n-k), \tag{1}$$

which may be proved either using Euler's generating function for $p(n)$ or by a counting argument [7, Theorem 6, Chapter 12].

We denote by $\alpha = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ the partition of $n = \sum i\alpha_i$ in which i occurs α_i times. We use the notation $\alpha \vdash n$ to signify that α is a partition of n.

Solving (1) for $\sigma(n)$ we find by induction the formula

$$\sigma(n) = \sum_{\alpha \vdash n} \prod_i p(i)^{\alpha_i} c(\alpha)$$

for integers $c(\alpha)$ satisfying the recursion

$$c(\alpha) = \begin{cases} n & \text{if } \alpha = (n^1) \\ -\sum_i c(1^{\alpha_1} \cdots i^{\alpha_i - 1} \cdots) & \text{otherwise.} \end{cases} \tag{2}$$

In this formula, the partitions on the right hand side are obtained from $\alpha = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ by lowering the i'th exponent by one. If α_i is already zero, we interpret $c(1^{\alpha_1} \cdots i^{\alpha_i - 1} \cdots)$ as being zero.

3. Deformation to a product

We will in fact calculate the Euler characteristic of K^nA in the special case where $A = E \times E'$ is a product of elliptic curves. To conclude that the resulting formula will be valid also for an arbitrary abelian surface, we use a deformation argument.

Recall [2, Theorem 8.3.1 and Proposition 8.8.2] that there exists an *irreducible* moduli space $\mathcal{A}_{g,d}$ for polarized abelian varieties $(A,\mathcal{L})$ of type d with level structure, where the type $d=(d_1,\dots,d_g)$ is a tuple of natural numbers, and a level structure is an isomorphism

$$K(\mathcal{L}) \cong \bigoplus_{i=1}^{g} (\mathbb{Z}/d_i\mathbb{Z})^2.$$

Here, the group $K(\mathcal{L})$ consists of the points $a \in A$ such that $T_a^*\mathcal{L} \cong \mathcal{L}$. Furthermore, if $d_1 \geq 3$, then $\mathcal{A}_{g,d}$ carries a universal family.

Now, a product $A = \prod_{i=1}^{g} E_i$ of elliptic curves admits polarizations of any type. In fact, denoting by $pr_i \colon A \to E_i$ the projection to the i'th factor, the sheaf

$$\mathcal{L} = \bigotimes_{i=1}^{g} pr_i^* \mathcal{O}_{E_i}(D_i)$$

defines a polarization of type $d=(d_1,\dots,d_g)$, whenever D_i is a divisor of degree d_i on E_i. From this it follows that, whenever A is a product of g elliptic curves and A' is an arbitrary g-dimensional abelian variety, there exists an abelian scheme

$$X \to S$$

over a nonsingular, irreducible curve S, with A and A' among its fibres. Namely, we may take S to be the (normalization of) any irreducible curve through the two points in $\mathcal{A}_{g,d}$ corresponding to A and A', equipped with level structures of the same type, and X to be the pullback of the universal family.

Proposition 3.1. *Let $A = E \times E'$ be a product of elliptic curves, and let A' be an arbitrary abelian surface. Then the associated Kummer varieties K^nA and K^nA' are deformation equivalent via a smooth deformation, and hence diffeomorphic. In particular, their Euler characteristics are equal.*

Proof. Let $X \to S$ be an abelian scheme of relative dimension two over a nonsingular, irreducible curve, with A and A' among its fibres. Let $X_S^{[n]}$ and $X_S^{(n)}$ denote the relative Hilbert scheme and the relative symmetric product. We have the Hilbert-Chow morphism [8]

$$\rho\colon X_S^{[n]} \to X_S^{(n)}$$

and the n-fold addition map

$$\mu\colon X_S^{(n)} \to X$$

over S, and both these maps commute with base change. Form the fibred product K

$$\begin{array}{ccc} K & \longrightarrow & X_S^{[n]} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mu\circ\rho} \\ S & \xrightarrow{\sigma} & X \end{array}$$

where σ is the zero section. Then the fibres of $K \to S$ are the Kummer varieties of the fibres of $X \to S$, and in particular we have found the required deformation between K^nA and K^nA'.

It only remains to check that the deformation is smooth. For this we basically follow Beauville's proof [1] for the nonsingularity of the Kummer variety: It is straight forward to verify that there is a cartesian diagram

$$\begin{array}{ccc} K \times_S X & \xrightarrow{\nu} & X_S^{[n]} \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle \mu\circ\rho} \\ X & \xrightarrow{n_X} & X \end{array}$$

where ν is induced by the natural action of X on the Hilbert scheme by translation, q is the second projection and n_X denotes multiplication by the natural number n. Since n_X is étale, so is ν, and by Fogarty's result [5, Theorem 2.9], the Hilbert scheme $X_S^{[n]}$ is smooth over S. We conclude that $K \times_S X$ is smooth over S.

In particular, both X and $K \times_S X$ are flat over S. It follows that $K \times_S X$ is flat over X via second projection q. By pulling back q over the zero section σ, we recover the structure map $K \to S$, which thus is flat. To prove it is smooth it is therefore enough to prove that every geometric fibre is nonsingular. In other words, we may replace S with the spectrum of an algebraically closed field k, in which case the fact that $K \times_k X$ is nonsingular implies that K is nonsingular. $\square$

4. Projection maps

From now on, let $A = E \times E'$ be a product of elliptic curves. Let $pr\colon A \to E$ denote the first projection. The restriction of the composed map

$$A^{[n]} \xrightarrow{\rho} A^{(n)} \xrightarrow{pr^{(n)}} E^{(n)}$$

to the Kummer variety $K^nA \subset A^{[n]}$ is a map

$$\pi\colon K^nA \to E^{(n)}.$$

Let $P \subset E^{(n)}$ denote the image of π. By definition of K^nA, P is precisely the set of effective divisors of degree n mapping to the zero element $0 \in E$ under the n-fold addition map

$$E^{(n)} \to E.$$

Remark 4.1. Although we will not need this, we can identify P as follows: Since the points of an effective divisor D of degree n on E sums to zero if and only it is linearly equivalent to the divisor $n\cdot 0 = 0+\cdots+0$, we see that P is just the linear system $|n\cdot 0| \cong \mathbb{P}^{n-1}$. It is easy to check that a generic fibre of π is connected, and it follows from results of Matsushita [9] that π is an example of a Lagrangian fibration. Together with Proposition 3.1, this provides a simple example of the fact that after deformation, any Kummer variety admits a Lagrangian fibration.

In any case, we have a "projection to the first factor"

$$\pi\colon K^nA \to P \subset E^{(n)}$$

and similarly a "projection to the second factor"

$$\pi'\colon K^nA \to P' \subset E'^{(n)},$$

mapping onto the sets P, resp. P', of effective degree n divisors summing to zero on E, resp. E'. We first examine the fibres of π'.

Lemma 4.2. *Let $\pi'\colon K^nA \to P'$ be the map defined above. If $D \in P'$ is not of the form $n\cdot a$ for some point $a \in E'$, then*

$$\chi(\pi'^{-1}(D)) = 0.$$

Thus, the Euler characteristic of K^nA is

$$\chi(K^nA) = n^2\chi(F)$$

where F is the fibre $\pi'^{-1}(n\cdot 0)$.

Proof. Choose a point $a \in \operatorname{Supp} D$, and write

$$D = D' + k\cdot a$$

where k is the multiplicity of a in D. By assumption, D' is nonzero. A point $\xi \in K^nA$ in the fibre $\pi'^{-1}(D)$ can be uniquely written as a disjoint union

$$\xi = \xi_1 \cup \xi_2$$

where ξ_1 has length k and is supported in $E\times\{a\}$. Taking the sum on E of the points in ξ_1, with multiplicities, we get a map

$$\nu\colon \pi'^{-1}(D) \to E.$$

We claim that all fibres of ν are isomorphic. In fact, given $p \in E$ we may choose $q, r \in E$ such that

$$kq = p, \quad (n-k)r = -p.$$

Then, decomposing $\xi = \xi_1 \cup \xi_2$ as above, the map

$$\theta\colon \nu^{-1}(p) \to \nu^{-1}(0), \quad \theta(\xi) = T^{-1}_{(q,0)}(\xi_1) \cup T^{-1}_{(r,0)}(\xi_2)$$

defines an isomorphism between the fibre over p and the fibre over 0. Since the base space E for ν has Euler characteristic zero, we conclude that the total space $\pi'^{-1}(D)$ has Euler characteristic zero also.

For the last part of the lemma, let $U \subset P'$ denote the set of divisors with at least two distinct supporting points. We have just shown that over U, all fibres of π' have Euler characteristic zero. Using the multiplicative property of the Euler characteristic (Section 2.1), we conclude that $\pi'^{-1}(U)$ has Euler characteristic zero also. Now U is the complement of the set of divisors of the form $n \cdot a$, where a is an n-division point on E', hence there are n^2 points outside U. Furthermore, any fibre $\pi'^{-1}(D)$ over a divisor of the form $D = n \cdot a$ is isomorphic to the fibre F over $n \cdot 0$, via translation by $(0, a)$. Thus the formula $\chi(K^n A) = n^2 \chi(F)$ follows. □

We next study the fibre $F = \pi'^{-1}(n \cdot 0)$ by means of the *first* projection map $\pi \colon K^n A \to P$. Note that F consists of those subschemes $\xi \in A^{[n]}$, supported in $E \times \{0\}$, such that the sum of its support points, with multiplicities, equals 0. This set can be stratified according to the multiplicities of the points in ξ. Let $\alpha = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ be a partition of n, and define the locally closed subset $V(\alpha) \subset F$ by

$$V(\alpha) = \{\xi \in F \,|\, \xi \text{ has } \alpha_i \text{ points of multiplicity } i\}.$$

We define a corresponding locally closed subset $W(\alpha) \subset P$ by

$$W(\alpha) = \{D \in P \,|\, D \text{ has } \alpha_i \text{ points of multiplicity } i\}.$$

Then we can reduce the computation of the Euler characteristic of F to the Euler characteristic of each $W(\alpha)$:

Lemma 4.3. *With $W(\alpha)$ as above we have*

$$\chi(F) = \sum_{\alpha \vdash n} \prod_i p(i)^{\alpha_i} \chi(W(\alpha)).$$

Proof. Clearly, the projection map $\pi \colon K^n A \to P$ maps $V(\alpha)$ to $W(\alpha)$. Let

$$\pi_\alpha \colon V(\alpha) \to W(\alpha)$$

denote the restricted map. The divisors $D \in W(\alpha)$ are of the form

$$D = \sum_i \big(i \sum_{j=1}^{\alpha_i} p_{ij}\big)$$

where the $p_{ij} \in E$ are distinct points. Hence the fibre of π_α above D consists of subschemes of the form $\xi = \cup_{ij} \xi_{ij}$, where each component ξ_{ij} has length i and is supported at $(p_{ij}, 0) \in A$. Thus every fibre of π_α is isomorphic to a product $\prod_i H(i)^{\alpha_i}$ of punctual Hilbert schemes. By Theorem 2.1 we conclude

$$\chi(V(\alpha)) = \prod_i p(i)^{\alpha_i} \chi(W(\alpha)).$$

Finally, since $F = \bigcup_{\alpha \vdash n} W(\alpha)$ is a disjoint union of locally closed subsets, we get the result by summing the last formula over all partitions of n. □

5. The recursion

Comparing Lemmas 4.2 and 4.3 with Section 2.3, wee see that Theorem 1.1 follows if we can show that $\frac{1}{n}\chi(W(\alpha))$ satisfies the recurrence relation (2).

Lemma 5.1. *We have* $\chi(W(n^1)) = n^2$ *for every* n.

Proof. $W(n^1)$ consists of the divisors of the form $D = n \cdot a$, where a is an n-division point on E. Hence we can identify $W(n^1)$ with the set of n-division points $E_n \cong (\mathbb{Z}/n\mathbb{Z})^2$, which is a finite group of order n^2. $\square$

Lemma 5.2. *Let* $\alpha = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ *be a partition of* n*, not equal to* (n^1)*, and let* i *be an index such that* $\alpha_i \neq 0$*. Let*

$$\alpha' = (1^{\alpha_1} \cdots i^{\alpha_i - 1} \cdots)$$

denote the partition of $n - i$ *obtained from* α *by lowering the* i*'th exponent by one. Then*

$$\chi(W(\alpha)) = -\frac{n^2(\sum_j \alpha_j - 1)}{\alpha_i (n-i)^2}\chi(W(\alpha')).$$

Proof. Basically, we would like to compare $W(\alpha)$ and $W(\alpha')$ by means of the incidence variety

$$\{(a, D) \,|\, D \text{ has multiplicity } i \text{ at } a\} \subset E \times W(\alpha).$$

However, if we remove from D the component supported at a, we do get an effective divisor of degree $n - i$, but the sum of its points under the group law on E is no longer zero. Thus there is no natural map from the incidence variety to $W(\alpha')$.

Instead, we let

$$Y = \left\{ (a, b, D) \,\middle|\, \begin{array}{c} D \text{ has multiplicity } i \text{ at } a \\ \text{and } (n-i)b = ia \text{ on } E \end{array} \right\} \subset E \times E \times W(\alpha).$$

It is clearly an algebraic subset. There are maps

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & W(\alpha) \\ {\scriptstyle\psi}\downarrow & & \\ W(\alpha') & & \end{array}$$

where ϕ is induced by projection to the third factor, and

$$\psi(a, b, D) = T_b(D - i \cdot a).$$

Here, $D - i \cdot a$ denotes the effective divisor obtained from D by removing the component supported at a. Note that the sum of the supporting points of $T_b(D - i \cdot a)$, with multiplicities, is zero, so ψ is indeed a map to $W(\alpha')$.

We want to calculate the Euler characteristic $\chi(Y)$ twice, using each of the maps ϕ and ψ, and equate the results.

First, let $D \in W(\alpha)$ and consider the fibre

$$\phi^{-1}(D) \cong \left\{ (a,b) \,\middle|\, \begin{array}{c} D \text{ has multiplicity } i \text{ at } a \\ \text{and } (n-i)b = ia \text{ on } E \end{array} \right\} \subset E \times E.$$

Now D has α_i points of multiplicity i. Let us denote them a_j with $j = 1, \ldots, \alpha_i$. Then $\phi^{-1}(D)$ is just the disjoint union of the α_i sets

$$\{b \,|\, (n-i)b = ia_j\} \subset E$$

and each of these consists of $(n-i)^2$ points. Thus every fibre of ϕ is a discrete set of $\alpha_i(n-i)^2$ points. In particular we have

$$\chi(Y) = \alpha_i(n-i)^2\chi(W(\alpha)). \tag{3}$$

Next, the fibre over a point $D' \in W(\alpha')$ can be described as

$$\psi^{-1}(D') \cong \left\{ (a,b) \,\middle|\, \begin{array}{c} a+b \notin D' \text{ and} \\ (n-i)b = ia \end{array} \right\} \subset E \times E.$$

This identification comes about since, if $\psi(a,b,D) = D'$, then the divisor D is uniquely determined by the pair (a,b) as

$$D = T_b^{-1}(D') + i \cdot a,$$

and this divisor has multiplicity i at a if and only if $a \notin T_b^{-1}(D')$, or equivalently $a + b \notin D'$.

Now $\psi^{-1}(D')$ is contained in the slightly bigger set

$$B = \{(a,b) \,|\, (n-i)b = ia\} \subset E \times E, \tag{4}$$

which has Euler characteristic zero, as can be seen by projecting to e.g. the second factor, and noting that all fibres are isomorphic (in fact, they are discrete sets of i^2 points).

It remains to count the pairs $(a,b) \in B$ with $a + b \in D'$. For each point c in the support of D', the set

$$\{(a,b) \,|\, (n-i)b = ia \text{ and } a+b = c\} \cong \{b \,|\, nb = ic\} \subset E$$

consists of n^2 points. Since there are $\sum_j \alpha_j - 1$ distinct points $c \in D'$, we see that $\psi^{-1}(D')$ is the complement in B to $n^2(\sum_j \alpha_j - 1)$ points. Hence we have

$$\chi(Y) = -n^2(\sum_j \alpha_j - 1)\chi(W(\alpha'))$$

and equating with (3) gives the result. $\square$

We can now finish the proof of the theorem by verifying that $\frac{1}{n}\chi(W(\alpha))$ satisfies the relation (2), that is,

$$\frac{1}{n}\chi(\alpha) = \begin{cases} n & \text{if } \alpha = (n^1) \\ -\sum_i \frac{1}{n-i}\chi(1^{\alpha_1} \cdots i^{\alpha_i - 1} \cdots) & \text{otherwise.} \end{cases}$$

where we use the shorthand $\chi(\alpha) = \chi(W(\alpha))$. In fact, the first equality is Lemma 5.1, and by Lemma 5.2 we have

$$\begin{aligned} -\sum_i \frac{1}{n-i}\chi(1^{\alpha_1}\cdots i^{\alpha_i - 1}\cdots) &= \sum_i \frac{\alpha_i(n-i)}{n^2(\sum_j \alpha_j - 1)}\chi(\alpha) \\ &= \frac{1}{n}\chi(\alpha)\frac{\sum_i \alpha_i(n-i)}{n(\sum_j \alpha_j - 1)} \\ &= \frac{1}{n}\chi(\alpha) \end{aligned}$$

since $n = \sum_i i\alpha_i$, α being a partition of n.

Remark 5.3. The recursion in Lemma 5.2 is easier to solve than the one in Section 2.3. In fact, we find

$$c(\alpha) = \frac{1}{n}\chi(\alpha) = (-1)^{\sum_i \alpha_i - 1} n \frac{(\sum_i \alpha_i - 1)!}{\prod_i (\alpha_i!)}$$

giving a closed solution to (2).

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