Løsningsforslag øving 1

Oppgaver fra boka:

4.53/4.4:3

La W=fortjenesten. Ut fra opplysningene i oppgaven får vi da at $W = 1.65 \cdot X - 5 \cdot 1.20 + 0.75 \cdot 1.20 \cdot (5 - X) = 0.75 \cdot X - 1.5$.

$$\mathrm{E}(X) = \sum_x x f(x) = 0 \cdot \frac{1}{15} + 1 \cdot \frac{2}{15} + 2 \cdot \frac{2}{15} + 3 \cdot \frac{3}{15} + 4 \cdot \frac{4}{15} + 5 \cdot \frac{3}{15} = \frac{46}{15}$$

Dermed blir: $\mathrm{E}(W) = 0.75\mathrm{E}(X) - 1.5 = 0.75 \cdot \frac{46}{15} - 1.5 = \underline{\underline{0.8}}$

4.64/4.4:14

$Z = -2X + 4Y - 3$, pga uavhengighet er $\mathrm{Cov}(X, Y) = 0$ og vi får:

$$\mathrm{Var}(Z) = (-2)^2\mathrm{Var}(X) + 4^2\mathrm{Var}(Y) = 4 \cdot 5 + 16 \cdot 3 = \underline{\underline{68}}$$

4.65/4.4:15

Når variablene ikke er uavhengige må vi ta hensyn til kovariansen:

$$\mathrm{Var}(Z) = (-2)^2\mathrm{Var}(X) + 4^2\mathrm{Var}(Y) + 2 \cdot (-2) \cdot 4\mathrm{Cov}(X, Y) = 4 \cdot 5 + 16 \cdot 3 - 16 \cdot 1 = \underline{\underline{52}}$$

4.72/4.4:22

For terningkast har vi at

$$f(x) = \begin{cases} \frac{1}{6} & , x = 1, 2, 3, 4, 5, 6 \\ 0 & , \text{ellers} \end{cases}$$

som gir $\mathrm{E}(X) = \sum_{x=1}^{6} x(1/6) = 3.5$, $\mathrm{E}(X^2) = \sum_{x=1}^{6} x^2(1/6) = 91/6$ og dermed $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 91/6 - 3.5^2 = 35/12$. Vil her da også ha $\mathrm{Var}(Y) = 35/12$ og siden to terningkast er uavhengige vil $\mathrm{Cov}(X, Y) = 0$.

a)

$$\mathrm{Var}(2X - Y) = 2^2\mathrm{Var}(X) + (-1)^2\mathrm{Var}(Y) = 4\frac{35}{12} + \frac{35}{12} = \underline{\underline{\frac{175}{12}}}$$

b)

$$\mathrm{Var}(X + 3Y - 5) = \mathrm{Var}(X) + 3^2\mathrm{Var}(Y) = \frac{35}{12} + 9\frac{35}{12} = \underline{\underline{\frac{175}{6}}}$$

4.92/4, review:12

$$\rho_{XY} = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \frac{1}{\sqrt{5}\sqrt{3}} = \underline{\underline{0.258}}$$

Oppgave 1:

$$f(x) = \begin{cases} 4x(1 - x^2) & \text{for } 0 \le x \le 1 \\ 0 & \text{ellers} \end{cases}$$

a)

$$P(X < 0.5) = \int_0^{0.5} 4(x - x^3)dx = 4[\frac{1}{2}x^2 - \frac{1}{4}x^4]_0^{0.5} = 4(\frac{1}{2}0.5^2 - \frac{1}{4}0.5^4) = \underline{\underline{\frac{7}{16}}} = 0.4375$$

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x4(x - x^3)dx = 4\int_0^1 (x^2 - x^4)dx \\ &= 4[\frac{1}{3}x^3 - \frac{1}{5}x^5]_0^1 = 4[\frac{1}{3} - \frac{1}{5}] = 4 \cdot \frac{2}{15} = \underline{\underline{\frac{8}{15} = 0.533}} \\ \mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ \mathrm{E}(X^2) = &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 4(x - x^3)dx = 4\int_0^1 (x^3 - x^5)dx \\ &= 4[\frac{1}{4}x^4 - \frac{1}{6}x^6]_0^1 = 4[\frac{1}{4} - \frac{1}{6}] = 4 \cdot \frac{1}{12} = \frac{1}{3} \\ \mathrm{Var}(X) &= \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \underline{\underline{0.049}} \end{aligned}$$

b)

$$\begin{aligned} F(x) &= P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_0^x 4t(1 - t^2)dt \\ &= \int_0^x (4t1 - 4t^3)dt = [2t^2 - t^4]_0^x = \underline{\underline{2x^2 - x^4}} \\ P(X < 0.3) &= F(0.3) = 2 \cdot 0.3^2 - 0.3^4 = \underline{\underline{0.1719}} \end{aligned}$$

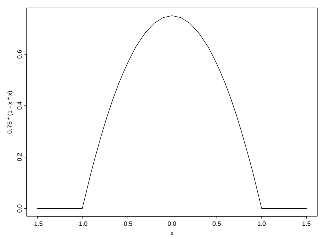
Oppgave 2:

$$f(x) = \begin{cases} k(1-x^2) & \text{for } -1 \le x \le 1 \\ 0 & \text{ellers} \end{cases}$$

For at $f(x)$ skal være en sannsynlighetstetthet, må $\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} f(x)dx = 1$.

$$\int_{-1}^{1} f(x)dx = k[x - \frac{1}{3}x^3]_{-1}^{1} = k(1 - \frac{1}{3} - (-1 + \frac{1}{3})) = k\frac{4}{3} = 1$$

Det gir $\underline{\underline{k = \frac{3}{4}}}$. Skisse av $f(x)$ i figuren under:



$$P(X \le 0.5) = \int_{-1}^{0.5} \frac{3}{4}(1-x^2)dx = \frac{3}{4}[x - \frac{1}{3}x^3]_{-1}^{0.5} = \frac{3}{4}(0.5 - 0.04167 - (-1 + 0.3333)) = \underline{\underline{0.8438}}$$

$$P(X \le 0.8 | X > 0.5) = \frac{P(X \le 0.8 \cap X > 0.5)}{P(X > 0.5)} = \frac{P(0.5 < X \le 0.8)}{P(X > 0.5)}$$

$$P(0.5 < X \le 0.8) = \int_{0.5}^{0.8} \frac{3}{4}(1-x^2)dx = \frac{3}{4}[x - \frac{1}{3}x^3]_{0.5}^{0.8} = \frac{3}{4}(0.629 - 0.458) = 0.128$$

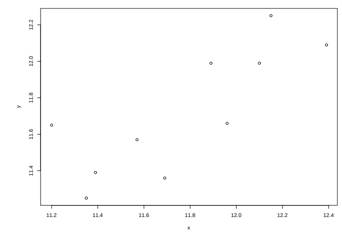
Det gir $P(X \le 0.8 | X > 0.5) = \frac{0.128}{1-0.8438} = \underline{\underline{0.821}}$

$$\begin{aligned} \mathrm{E}(Y) &= E(1+X^2) = \int_{-\infty}^{\infty}(1+x^2)f(x)dx = \int_{-1}^{1}(1+x^2)\frac{3}{4}(1-x^2)dx \\ &= \frac{3}{4}\int_{-1}^{1}(1-x^4)dx = \frac{3}{4}[x - \frac{1}{5}x^5]_{-1}^{1} = \frac{3}{4}[2 - \frac{2}{5}] = \underline{\underline{1.2}} \end{aligned}$$

Oppgave 3:

$$\begin{aligned} s_x^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{10-1}1.38029 = 0.153 \\ s_y^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{10-1}1.054 = 0.117 \\ s_{xy} &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{10-1}0.9624 = 0.107 \\ r_{xy} &= \frac{s_{xy}}{\sqrt{s_x^2 \cdot s_y^2}} = \frac{0.107}{\sqrt{0.153 \cdot 0.117}} = \underline{\underline{0.80}} \end{aligned}$$

Sterk positiv korrelasjon, bensinprisene følger hverandre i stor grad (går opp og ned i takt).



Oppgave 4:

$$\begin{aligned} \mathrm{E}(\bar{X}) &= \frac{1}{n}\mathrm{E}(\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n\mu = \underline{\underline{\mu}} \\ \mathrm{Var}(\bar{X}) &= \left(\frac{1}{n}\right)^2 \mathrm{Var}(\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}n\sigma^2 = \underline{\underline{\frac{\sigma^2}{n}}} \\ \mathrm{E}(Y) &= \mathrm{E}(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i \mathrm{E}(X_i) = \sum_{i=1}^{n} a_i\mu = \underline{\underline{\mu\sum_{i=1}^{n} a_i}} \\ \mathrm{Var}(Y) &= \mathrm{Var}(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) = \sum_{i=1}^{n} a_i^2\sigma^2 = \underline{\underline{\sigma^2\sum_{i=1}^{n} a_i^2}} \\ \mathrm{E}(Z) &= \mathrm{E}\left(\frac{\sum_{i=1}^{n} a_i X_i}{\sum_{i=1}^{n} a_i}\right) = \frac{\sum_{i=1}^{n} a_i \mathrm{E}(X_i)}{\sum_{i=1}^{n} a_i} = \frac{\sum_{i=1}^{n} a_i\mu}{\sum_{i=1}^{n} a_i} = \mu\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i} = \underline{\underline{\mu}} \\ \mathrm{Var}(Z) &= \mathrm{Var}\left(\frac{\sum_{i=1}^{n} a_i X_i}{\sum_{i=1}^{n} a_i}\right) = \frac{1}{(\sum_{i=1}^{n} a_i)^2}\mathrm{Var}(\sum_{i=1}^{n} a_i X_i) = \frac{1}{(\sum_{i=1}^{n} a_i)^2}\sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) \\ &= \frac{\sum_{i=1}^{n} a_i^2\sigma^2}{(\sum_{i=1}^{n} a_i)^2} = \underline{\underline{\sigma^2\frac{\sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i)^2}}} \end{aligned}$$