UNIVERSITY OF OSLO

Recent advances in Loewner Theory

PAVEL GUMENYUK

UNIVERSITY OF STAVANGER, NORWAY

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Outline of the talk

I. Classical Loewner Theory
II. Interesting application: Stochastic Loewner Evolution
III. Abstract approach by Bracci, Contreras and Díaz-Madrigal
IV. Analogy with Lie Group Theory and representation problem

The starting point of Loewner Theory is the seminal paper by

Czech – German mathematician

Karel Löwner (1893 – 1968) known also as
Charles Loewner

Untersuchungen über schlichte konforme Abbildungen des Einheitskreises,
Math. Ann. **89** (1923), 103–121.

In this paper Loewner introduced a new method to study the famous *Bieberbach Conjecture* concerning the *so-called class* $\mathcal{S}$.

Bieberbach's Conjecture

Ludwig Bieberbach, 1916: analytic properties of conformal mappings

$$f : \mathbb{D} \xrightarrow{\text{into}} \mathbb{C}, \quad \mathbb{D} := \{z : |z| < 1\}, \qquad f(0) = 0, \; f'(0) = 1.$$

Class $\mathcal{S}$

By $\mathcal{S}$ we denote the class of all *holomorphic univalent functions*

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

the famous Bieberbach Conjecture (1916)

$$\forall f \in \mathcal{S} \;\; \forall n = 2, 3, \ldots \qquad |a_n| \leqslant n \tag{2}$$

Bieberbach (1916): $n = 2$; Loewner (1923): $n = 3$; . . .

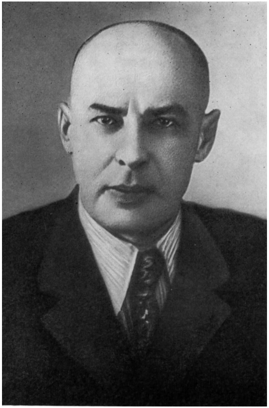
de Branges (1984): all $n \geqslant 2$ — *using Loewner's method*

Elementary properties of class $\mathcal{S}$.

- – there is no natural linear structure in the class $\mathcal{S}$;
- – the class $\mathcal{S}$ is not even a convex set in $\mathrm{Hol}(\mathbb{D}, \mathbb{C})$;

- + the class $\mathcal{S}$ is compact w.r.t. local uniform convergence in $\mathbb{D}$;
- + $\mathcal{U}_0(\mathbb{D}) :=$
  $$\left\{\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) : \varphi \text{ is univalent and } \varphi(0) = 0,\ \varphi'(0) > 0\right\}$$
  is a topological semigroup w.r.t. the composition operation $(\varphi, \psi) \mapsto \psi \circ \varphi$ and the topology of locally uniform convergence.

Development of Loewner’s method



Pavel Parfen’evich Kufarev
Tomsk (1909 – 1968)

Christian Pommerenke
(Copenhagen, 17 December 1933)

Parametric Representation of the class $\mathcal{S}$

Loewner–Kufarev ODE

$$\frac{dw}{dt} = -w(t)\,p\big(w(t),t\big), \quad w(0) = z \in \mathbb{D}, \tag{3}$$

where $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a *classical Herglotz function*, *i.e.*

- $p(z, \cdot)$ is measurable for all $z \in \mathbb{D}$;
- $p(\cdot, t)$ is holomorphic for all $t \geqslant 0$;
- $\operatorname{Re} p > 0$ and $p(0, t) = 1$ for all $t \geqslant 0$.

It is known that:

For any $z \in \mathbb{D}$, the solution $t \mapsto w(t)$ to (3) is unique and exists for all $t \geqslant 0$.

NOTATION:

$$\varphi^{p}_{0,t}(z) := w(t)$$

Parametric Representation — CONT'ED

Loewner – Kufarev ODE: $\frac{dw}{dt} = -w(t)\,p(w(t),t), \quad w(0) = z \in \mathbb{D}$ (3)

NOTATION: $\varphi^{p}_{0,t}(z) := w(t)$

Recall: $\mathcal{S} := \left\{ \mathbb{D} \ni z \mapsto f(z) = z + \sum_{n=2}^{+\infty} a_n z^n : \ f \text{ is univalent in } \mathbb{D} \right\}$

Theorem (Parametric Representation) [Loewner, 1923; Kufarev, 1943; Pommerenke, 1965-75; Gutlyanski, 1970]

(A) for any classical Herglotz function p, the limit

$$f(z) := \lim_{t \to +\infty} e^{t} \varphi^{p}_{0,t}(z) \tag{4}$$

exists for all $z \in \mathbb{D}$, and $f \in \mathcal{S}$.

(B) For any $f \in \mathcal{S}$ there exists a classical Herglotz function p such that f is represented as limit (4).

In other words, the formula $p \mapsto f[p] := \lim_{t\to+\infty} e^t\varphi^p_{0,t}$ defines a map of the *convex cone*

formed by all classical Herglotz functions p

onto the class $\mathcal{S}$.

A cornerstone of the proof of *surjectivity* is the possibility

to embed any $f \in \mathcal{S}$ as an initial element into

a *(classical) Loenwer chain*, *i.e.* a family $(f_t)_{t\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{C})$ s.t.:

LC1. for each $t \geqslant 0$, $f_t : \mathbb{D} \to \mathbb{C}$ is univalent in $\mathbb{D}$;

LC2. for each $s \geqslant 0$ and $t \geqslant s$, $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$

[the image domain is expending];

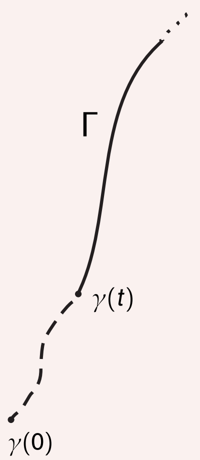
LC3. for each $t \geqslant 0$, $f_t(z) = e^t z + a_2(t)z^2 + \ldots \quad [\Leftrightarrow \ e^{-t}f_t \in \mathcal{S}]$ (5)

Loewner's construction [slit mappings]

Loewner considered the dense subclass $\mathcal{S}' \subset \mathcal{S}$ of all *slit mappings*,

$$\mathcal{S}' := \Big\{ f \in \mathcal{S}\colon f(\mathbb{D}) = \mathbb{C} \setminus \Gamma, \text{ where } \Gamma \text{ is a Jordan arc extending to } \infty \Big\}.$$

Figure 1



Loewner's construction

- Consider $f \in \mathcal{S}'$ and let $\Gamma := \mathbb{C} \setminus f(\mathbb{D})$.
- Choose a parametrization $\gamma : [0, +\infty] \to \Gamma$, $\gamma(+\infty) = \infty$.
- Consider the domains $\Omega_t := \mathbb{C} \setminus \gamma\big([t, +\infty]\big)$, $t \geqslant 0$.
- By Riem. Mapping Th'm $\forall t \geqslant 0$ $\exists!$ *conformal mapping*

$$f_t : \mathbb{D} \xrightarrow{\text{onto}} \Omega_t, \quad f_t(0) = 0, \; f_t'(0) > 0.$$

- Reparameterizing Γ: $\quad \forall\, t \geqslant 0 \;\; f_t'(0) = e^t$.
- Note that (f_t) is a *Loewner chain* and $\boxed{f_0 = f}$.

In this case [of a "slit" Loewner chain]:

☞ $(z, t) \mapsto f_t(z)$ satisfies the *Loewner PDE*

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z)\, p(z, t), \tag{6}$$

☞ and $\varphi_{s,t} := f_t^{-1} \circ f_s \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, $t \geqslant s \geqslant 0$,

solves the I.V.P. for the *Loewner ODE*

$$\frac{d\varphi_{s,t}(z)}{dt} = -\varphi_{s,t}(z)\, p\big(\varphi_{s,t}(z), t\big), \quad t \geqslant s, \quad \varphi_{s,s} = \mathrm{id}_{\mathbb{D}}. \tag{7}$$

☞ The function p has the form $p(z, t) := \dfrac{1 + z e^{-iu(t)}}{1 - z e^{-iu(t)}}$, (8)

where $u : [0, +\infty) \to \mathbb{R}$ is a continuous function *uniquely defined* by the slit Γ.

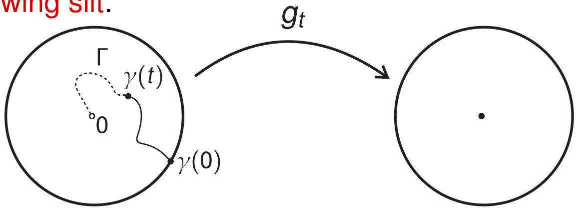
Remarks

- ✔ The family (f_t) is C^1 although *no regularity* is assumed *for the slit* Γ;
- ✔ The "driving function" $u(t)$ encodes the information about the slit,
- ✔ and hence we obtain a kind of conformally invariant *coordinates in the set of all Jordan arcs* in $\overline{\mathbb{C}}$.

This coordinates proved to be very useful for applications.

Consider a similar situation,

but with a growing slit:



Loewner ODE: $$\frac{dg_t(z)}{dt} = +g_t(z)\frac{1+e^{-iu(t)}g_t(z)}{1-e^{-iu(t)}g_t(z)}, \quad g_0(z) = z. \qquad (9)$$

Application: SLE

Again, all the information about Γ is encoded in the driving f-n $u(t)$.

In 2000, Oded Schramm introduced
the so-called *Stochastic Loewner Evolution*

by plugging $$u(t) := \sqrt{\kappa}\mathcal{B}_t,$$

where $(\mathcal{B}_t)$ is the standard Brownian motion and $\kappa = \text{const} > 0$,
to the Loewner ODE for the decreasing chains,

$$\frac{dg_t(z)}{dt} = g_t(z)\frac{1 + e^{-iu(t)}g_t(z)}{1 - e^{-iu(t)}g_t(z)}, \quad g_0(z) = z. \tag{10}$$

Main contribution of Schramm: he understood that

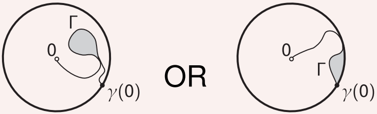
the *Wiener measure* over the driving functions u corresponds via (10) to a *measure over the slits* Γ, which arises as a
scale limit in many lattice models of Statistical Physics.

! FIELDS MEDALS: W. Werner (2006), S. Smirnov (2010)

Universitetet i Stavanger

Subtle point: ¿slit or not?

- P.P. Kufarev, 1947: Not every continuous driving function u corresponds to a slit!



- P.P. Kufarev, 1946: the first sufficient condition for generation of a slit.
- D.E. Marshall & St. Rohde, 2005; J. Lind, 2005: $||u||_{\mathrm{Lip}(1/2)} \leqslant 4 \Rightarrow$ slit solutions;
- for SLE: slit solutions a.s. **iff** $\kappa \leqslant 4$;
- interesting: this 4 was a kind of "predicted" by Kufarev in 1946 . . .
- many papers on relation between regularity of the slit Γ and that of the driving f-n u.

Small cheating

- ☞ The slits Γ we considered join a point $a \in \partial\mathbb{D}$ with $b = 0$. [RADIAL CASE]
- ☞ In many applications Γ is a cross-cut, *i.e.* both end-points $a, b \in \partial\mathbb{D}$. [CHORDAL CASE]

The chordal case is usually considered in the half-plane $\mathbb{H} := \{z\colon \operatorname{Im} z > 0\}$, with $b := \infty$.
The *chordal Loewner ODE* takes the form

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z \in \mathbb{H}, \tag{11}$$

where $\lambda : [0, +\infty) \to \mathbb{R}$ is a continuous driving function.

This equation appeared for the 1st time in a paper of 1946 by Kufarev and was rediscovered in 2000 by Schramm.

Chordal, radial, autonomous,. . .

Abstract approach

In 2008 Filippo Bracci, Manuel D. Contreras and Santiago Díaz-Madrigal suggested an abstract approach,
which *includes both radial and chordal versions* as very special cases.

What is common in the radial and chordal Loenwer equations in $\mathbb{D}$?

Radial: $dw/dt = -w\, p(w,t), \qquad w(0) = z,$

Chordal: $dw/dt = (1-w)^2 p(w,t), \quad w(0) = z,$

where in both cases, $\operatorname{Re} p \geqslant 0.$

A clear hint is given in the *theory of one-parameter semigroups*.

One-parameter semigroups

Definition

A *one-parameter semigroup* $(\phi_t) \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$

is a continuous semigroup homomorphism

$$[0,+\infty) \ni t \mapsto \phi_t \in \mathrm{Hol}(\mathbb{D},\mathbb{D}),$$

i.e. (i) $\phi_0 = \mathrm{id}_{\mathbb{D}}$, (ii) $\phi_t \circ \phi_s = \phi_s \circ \phi_t = \phi_{t+s}$ for any $s,\, t \geqslant 0$,

(iii) $\phi_t(z) \to z$ as $t \to 0^+$.

Theorem (E. Berkson, H. Porta, 1978)

(A) Any one-param. semigroup (ϕ_t) is the semiflow of the holomorphic vector filed $G(z) := \lim_{t\to 0^+}(\phi_t(z) - z)/t$, called the *infinitesimal generator* of (ϕ_t).

(B) $G \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ is the infinitesimal generator of *some* one-param. semigroup if and only if

$$G(z) = (\tau - z)(1 - \overline{\tau} z)\, p(z), \qquad (12)$$

where $\tau \in \overline{\mathbb{D}}$ and $p \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ with $\mathrm{Re}\, p \geqslant 0$.

Herglotz vector fields

The Berkson – Porta formula $G(z) = (\tau - z)(1 - \overline{\tau} z)\, p(z)$ resembles:

- ✔ radial Loewner – Kufarev equation $G(z) = -z\, p(z)$ if $\tau = 0 \in \mathbb{D}$;
- ✔ chordal Loewner equation $G(z) = (1 - z)^2 p(z)$ if $\tau = 1 \in \partial\mathbb{D}$.
- ✔ Loewner equation are *non-autonomous*.

Bracci, Contreras and Díaz-Madrigal studied *non-autonomous* analogues of infinitesimal generators G and one-param. semigroups (ϕ_t) [ArXiv 2008; Crelle's journal 2012]

Definition (*essentially* from Carathéodory's theory of ODEs)

A function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is said to be a *Herglotz vector field* if:

HVF1. for a.e. $t \geqslant 0$ fixed, the function $G(\cdot, t)$ is an infinitesimal generator;

HVF2. for each $z \in \mathbb{D}$ fixed, the function $G(z, \cdot)$ is measurable on $[0, +\infty)$;

HVF3. for each compact set $K \subset \mathbb{D}$, $t \mapsto \sup_{z \in K} |G(z, t)|$ is L^1_{loc}.

Evolution families

Definition (*"non-autonomous" semigroups* — intrinsic def-tion)

A family $(\varphi_{s,t})_{t\geqslant s\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ is called an *evolution family* if:

EF1. $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for all $s \geqslant 0$;

EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $t \geqslant u \geqslant s \geqslant 0$;

EF3. for any $z \in \mathbb{D}$, the maps $[s,+\infty) \ni t \mapsto \varphi_{s,t}(z)$ are locally absolutely continuous *uniformly w.r.t.* $s \geqslant 0$.

Theorem (Bracci, Contreras and Díaz-Madrigal, 2008)

The general Loewner ODE

$$d\varphi_{s,t}(z)/dt = G\big(\varphi_{s,t}(z),\, t\big),\quad t \geqslant s \geqslant 0,\quad \varphi_{s,s}(z) = z, \tag{13}$$

establishes an (essentially) *1-to-1 correspondence* between *Herglotz vector fields* G and *evolution families* $(\varphi_{s,t})$.

This includes uniqueness and global existence for solutions to (13).

Note: (13) is to be understood as a Carathéodory ODE.

General Loewner chains

Definition

A Loewner chain is a family of functions (f_t), $t \geqslant 0$, such that:

LC1. each $f_t : \mathbb{D} \to \mathbb{C}$ is *holomorphic* and *univalent*;

LC2. $\Omega_s := f_s(\mathbb{D}) \subset \Omega_t := f_t(\mathbb{D})$ whenever $t \geqslant s \geqslant 0$;

LC3. $t \mapsto f_t(z)$ is loc. abs. continuous *loc. uniformly w.r.t.* $z \in \mathbb{D}$.

Theorem (M. D. Contreras, S. Díaz-Madrigal and **P. Gum.**, 2010)

(A) For any Loewner chain (f_t), the *transition maps* $\varphi_{s,t} := f_t^{-1} \circ f_s$, $t \geqslant s \geqslant 0$, form an evolution family.

(B) Conversely, every evolution family $(\varphi_{s,t})$ is formed by transition maps of some Loewner chain (f_t), which is unique up to biholomorphisms of $\Omega := \cup_{t \geqslant 0} f_t(\mathbb{D})$.

(C) Every Loewner chain (f_t) satisfies the general Loewner PDE

$$(\partial/\partial t) f_t(z) = -f_t'(z) G(z,t), \tag{14}$$

where G is the Herglotz v.f. of the evolution family $(\varphi_{s,t}) \sim (f_t)$.

Some remarks

Remarks

☞ According to the abstract approach by Bracci *et al*, the essence of Loewner Theory resides in the interplay among

- Loewner chains,
- evolution families and
- Herglotz vector fields.

There is essentially 1-to-1 correspondence among them.

☞ The generality of this approach might seem to be excessive from the viewpoint of certain application.

☞ BUT: it is *intrinsic* and extends naturally to *complex manifolds*.
[Bracci, Contreras and Díaz-Madrigal, Math. An. (2009)]
[Arosio, Bracci, Hamada and Kohr, J. Anal. Math. (2013)]

Loewner chains = Abstract basins

In higher dim's, Loewner chains take values
in some *abstract manifold* = the so-called abstract basins of attraction
considered by J.E. Fornæss and B. Stensønes, 2004,
in connection with the Bedford Conjecture on stable manifolds.

For an evolution family $(\varphi_{s,t}) \subset \mathrm{Hol}(X, X)$, the *abstract basin* Ω
(aka "tail space") is formed by

- ☞ all trajectories $[s, +\infty) \ni t \mapsto \gamma_{z,s}(t) := \varphi_{s,t}(z) \in X$
- ☞ modulo: $\gamma_1 \sim \gamma_2$ iff $\gamma_1 = \gamma_2$ on their common domain.

- ☞ the maps $f_s : X \ni z \mapsto [\gamma_{z,s}] \in \Omega$ form a Loewner chain $\sim (\varphi_{s,t})$
- ☞ Ω is simply connected & non-compact $\Rightarrow$ in $\dim = 1$, $\Omega \cong \mathbb{D}$ or $\mathbb{C}$

L. Arosio, F. Bracci, E. Fornæss Wold, 2013:
$X \subset \mathbb{C}^n$ starlike [hyperbolic complete] domain $\Rightarrow$ $\Omega \hookrightarrow \mathbb{C}^n$.

Analogy with Lie groups

Denote: (ϕ_t^G) the one-param. semigroup ~ an inf. generator G.
For $U \subset \mathrm{Hol}(\mathbb{D},\mathbb{D})$ let $\mathcal{G}[U] := \{\text{inf. generators } G\colon (\phi_t^G) \subset U\}$.

Analogue of the Lie exponential map

$$\mathcal{G}[U] \ni G \mapsto \mathrm{Exp}_{\mathrm{Lie}}(G) := \phi_1^G \in U \subset \mathrm{Hol}(\mathbb{D},\mathbb{D}) \text{ (subsemigroup)}$$

- ☺ For Lie groups, the Exp-map recovers the group (*at least* locally)
- ☹ However in our case, typically $\mathrm{Exp}_{\mathrm{Lie}}\big(\mathcal{G}[U]\big) \neq U, \neq \mathcal{O}_U(\mathrm{id}_{\mathbb{D}})$.

Loewner's idea: Instead of (ϕ_t)'s satisfying the autonomous ODE

$$d\phi_t(z)/dt = G(\phi_t(z)), \qquad t \geqslant 0, \qquad \phi_0(t) = z \in \mathbb{D}, \quad (15)$$

consider two-parameter families $(\varphi_{s,t})_{t\geqslant s\geqslant 0}$, generated by its *non-autonomous analogue*:

$$d\varphi_{s,t}(z)/dt = G\big(\varphi_{s,t}(z), t\big), \quad t \geqslant s \geqslant 0, \quad \varphi_{s,s}(z) = z \in \mathbb{D}, \quad (16)$$

where $G(\cdot, t) \in \mathcal{G}[U]$ for a.e. $t \geqslant 0$.

Development of Loewner's idea

- The ODE $\frac{d\varphi_{s,t}(z)}{dt} = G\big(\varphi_{s,t}(z), t\big), \;\; t \geqslant s \geqslant 0, \;\; \varphi_{s,s}(z) = z,$ (16) is in fact the *general Loewner equation* by Bracci *et al*;
- the functions $G : [0, +\infty) \ni t \mapsto G(\cdot, t) \in \mathcal{G}[U]$ are *Herglotz vector fields*.
- the families $(\varphi_{s,t})$ are *evolution families*.
- $\mathcal{G}[U]$ will be called the *infinitesimal structure* of U.

We would wish to reconstruct the semigroup U from its infinitesimal structure $\mathcal{G}[U]$ using the general Loewner ODE (16):

Definition

We say that a semigroup $U \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ *admits a Loewner – type representation* if the union $\mathcal{R}[U]$ of all evolution families $(\varphi_{s,t})$ generated by Herglotz vector fields G with $G(\cdot, t) \in \mathcal{G}[U]$ for a.e. $t \geqslant 0$ coincides with U.

Self-maps with BRFPs

Problem: *construct a Loewner – type parametric representation for semigroups formed by univalent self-maps with given fixed points.*

Let $\mathcal{F}$ be a finite set of points on $\mathbb{T} := \partial\mathbb{D}$.

First family of semigroups

$$\mathcal{U}(\mathbb{D},\mathcal{F}) := \left\{\varphi \in \mathcal{U}(\mathbb{D}) : \text{each } \sigma \in \mathcal{F} \text{ is a BRFP of } \varphi\right\}$$

"BRFP"="*boundary regular fixed point*":

A point $\sigma \in \partial\mathbb{D}$ is said to be *BRFP* of $\varphi \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ if

$$\exists \angle \lim_{z\to\sigma} \varphi(z) = \sigma \quad \text{and} \quad \exists\, \varphi'(\sigma) := \angle \lim_{z\to\sigma} \frac{\varphi(z)-\sigma}{z-\sigma} \neq \infty.$$

Denjoy – Wolff point

Fix additionally $\tau \in \overline{\mathbb{D}} \setminus \mathcal{F}$.

Second family of semigroups

$$\mathcal{U}_\tau(\mathbb{D}, \mathcal{F}) := \{\mathrm{id}_{\mathbb{D}}\} \cup \left\{\varphi \in \mathcal{U}(\mathbb{D}, \mathcal{F}) \setminus \{\mathrm{id}_{\mathbb{D}}\} : \tau \text{ is the DW-point of } \varphi\right\}$$

"DW-point"="*Denjoy – Wolff point*" [Denjoy – Wolff Theorem]

For any $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\mathrm{id}_{\mathbb{D}}\}$,

$\exists!$ (boundary regular) fixed point $\tau \in \overline{\mathbb{D}}$ such that $|\varphi'(\tau)| \leqslant 1$.

Moreover, if φ is *not* an elliptic automorphism of $\mathbb{D}$,

then $\varphi^{\circ n} \to \tau$ l.u. in $\mathbb{D}$ as $n \to +\infty$.

This point τ is called the *Denjoy – Wolff point* of φ.

Motivation

- Loenwer's idea potentially can work in the general setting of an abstract semigroup with "compatible diffeology".
  However, no criteria for such a semigroup
  to admit a parametric representation is known.
  So it is interesting to study more examples.

- In Geometric Function Theory there has been
  considerable interest to study self-maps with given BRFP's

  H. Unkelbach, 1938, 1940; C. Cowen, Ch. Pommerenke, 1982;

  Ch. Pommerenke, A. Vasil'ev, 2000; J.M. Anderson, A. Vasil'ev, 2008;

  M. Elin, D. Shoikhet, N. Tarkhanov, 2011;

  V.V. Goryainov [talk at Steklov Math. Inst., Moscow, 26/12/2011];

  A. Frolova, M. Levenshtein, D. Shoikhet, A.Vasil'ev, ArXiv:1309.3074, 2013.

- The infinitesimal structure of $\mathcal{U}(\mathbb{D},\mathcal{F})$ and $\mathcal{U}_\tau(\mathbb{D},\mathcal{F})$ is well-studied.

Results

Theorem (P. Gum. — work in progress)

Let $\mathcal{F} \subset \mathbb{T}$ be a finite set, $n := \mathrm{Card}(\mathcal{F})$, and $\tau \in \overline{\mathbb{D}} \setminus \mathcal{F}$.
The following semigroups U admit
the Loewner-type parametric representation, *i.e.* $\mathcal{R}[U] = U$:

- ✔ $U = \mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ for $\tau \in \mathbb{D}$ and $n = 1$; [Unkelbach and Goryainov]
- ✔ $U = \mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ for $\tau \in \mathbb{T}$ and $n \leqslant 2$;
- ✔ $U = \mathcal{U}(\mathbb{D}, \mathcal{F})$ for $n \leqslant 3$.

H. Unkelbach, 1940: an attempt to give
the Loewner-type parametric representation for $\mathcal{U}_0(\mathbb{D}, \{1\})$;
V.V. Goryainov, approx. 2013 (to appear in *Mat. Sb.*):
the complete proofs.

Conjecture and open problem

Conjecture [know how to prove]

If $\tau \in \mathbb{D}$, then the semigroup $\mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ admits
the Loewner type representation for *any finite set* $\mathcal{F} \subset \mathbb{T}$.

Open problem

Given a finite $\mathcal{F} \subset \mathbb{T}$ with $\mathrm{Card}(\mathcal{F}) = n$,

¿ Does the semigroups $\mathcal{U}_\tau(\mathbb{D}, \mathcal{F})$ admits
the Loewner type representation for $\tau \in \mathbb{T}$ and $n > 2$?

¿ Does the semigroups $\mathcal{U}(\mathbb{D}, \mathcal{F})$ admits
the Loewner type representation for $n > 3$?

My conjecture is that the correct answer for both questions is NO.

Last phrase . . .

Tusen takk så mye !!!