Quasiconformal extentions via the chordal Loewner equation

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Synopsis

My talk is devoted to some problems in One $\mathbb{C}$omplex Variable.

(I) PRELIMINARIES

1° Quasiconformal mappings

2° Classical Loewner Theory

3° Application of the classical Loewner Theory to quasiconformal extensions of holomorphic functions

4° Chordal variant of the Loewner Theory

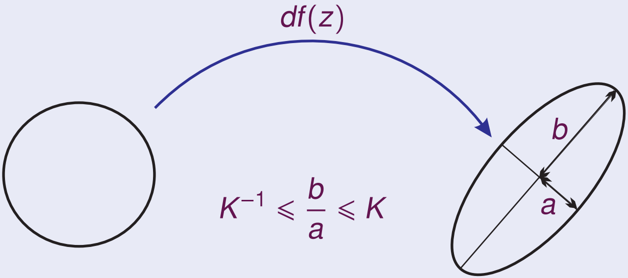
(II) NEW RESULTS (joint work with Ikkei HOTTA)

1° Quasiconformal extensions via the chordal Loewner equation

2° Sufficient conditions for quasiconformal extendibility of holomorphic functions in the half-plane

Definition (a simple one)

Let $K \geqslant 1$ be a constant and $D \subset \mathbb{C}$ a domain. A sense-preserving C^1-homeomorphism $f\colon D \xrightarrow{\text{into}} \mathbb{C}$ is said to be a *K-quasiconformal mapping* if for any $z \in D$ the differential $df(z)$ maps circles onto ellipses with the *ration of the major semiaxis to the minor one not exceeding K.*



$$K^{-1} \leqslant \frac{b}{a} \leqslant K$$

For $K = 1$ we recover the conformal mappings.

Quasiconformal mappings 2

If $f\colon D \xrightarrow{\text{into}} \mathbb{C}$ is a K-quasiconformal mapping of class C^1, then it satisfies the *Beltrami PDE*

$$\bar\partial f = \mu_f(z)\partial f, \tag{1}$$

where $\partial f := \frac{1}{2}\left(\frac{df}{dx} - i\frac{df}{dy}\right),\ \bar\partial f := \frac{1}{2}\left(\frac{df}{dx} + i\frac{df}{dy}\right),\ z = x + iy,$

the *Beltrami coefficient* μ_f satisfies $|\mu_f(z)| \leqslant k < 1$ for all $z \in D$ and $k := (K-1)/(K+1)$.

Definition (the general one)

A mapping $f\colon D \xrightarrow{\text{into}} \mathbb{C}$ is said to be K*-quasiconformal* if:

(i) f is a sense-preserving homeomorphism of D onto $f(D)$;

(ii) f is ACL in D;

(iii)
$$\bar\partial f = \mu_f(z)\partial f \qquad \text{for a. e. } z \in D$$

with some measurable μ_f s.t. $\operatorname{ess\,sup}|\mu_f(z)| \leqslant k := (K-1)/(K+1)$.

Quasiconformal mappings 3

SOME REMARKS:

- Again, a 1-quasiconformal mapping is the same as a *conformal mapping.*
- "Quasiconformal" is usually abbreviated as "q.c."
- By a *q.c.-mapping* one means a K-q.c. mapping with some (unspecified) $K \geqslant 1$.
- Quite often, abusing the language, one specifies $k < 1$, *i.e.* the upper bound for the Beltrami coefficient, instead of $K \geqslant 1$. So by a k-q.c. mapping one means K-q.c. mapping with $K := (1 + k)/(1 - k)$.
  In what follows, we will use the "k-small" notation.
- The definition of quasiconformality extends naturally to mappings between Riemann surfaces. In particular, we will be interested in q.c.-mappings of $\overline{\mathbb{C}}$ onto itself, *i.e.* q.c.-automorphisms of $\overline{\mathbb{C}}$.

Quasiconformal mappings 4

Why q.c.-mappings are interesting?

- ✔ Q.c.-mappings generalize conformal maps.
- ✔ They are more flexible. In particular, the notion of a q.c.-mapping extends naturally to $\mathbb{R}^n$, $n > 2$. In higher dimensions conformal mappings are trivial, while q.c.-mappings form a large class.
- ✔ Q.c.-mappings inherit many fundamental properties of conformal mappings, such as removability of isolated singularities, compactness principles, boundary behaviour,
  (Measurable) Riemann Mapping Theorem, etc.
- ✔ Q.c.-mappings appear naturally in many parts of Complex Analysis such as Holomorphic Dynamics, Univalent Functions, Riemann Surfaces, Kleinian Groups, etc.
- ✔ Q.c.-mappings can be seen as deformations of the complex structure. This role is played by q.c.-mappings in Teichmüller's theory of Riemann surfaces.

Quasiconformal extensions

Notation: $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$

Definition

A function $f \colon \mathbb{D} \to \mathbb{C}$ is said to be *q.c.-extendible* if there exists a q.c.-automorphism $F \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ s.t. $F(\infty) = \infty$ and $F|_{\mathbb{D}} = f$.

Clearly, q.c.-extendible functions are
univalent (= injective + holomorphic) in $\mathbb{D}$.

Definition (Normalized univalent functions)

By class $\mathcal{S}$ we mean the set of all univalent function $f : \mathbb{D} \to \mathbb{C}$ normalized by $f(0) = 0$, $f'(0) = 1$.

$$\mathcal{S}(k) := \{f \in \mathcal{S} \colon \exists \text{ a } k\text{-q.c. map } F : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \text{ s.t. } F(\infty) = \infty \text{ and } F|_{\mathbb{D}} = f\}.$$

The union $\bigcup_{k\in[0,1)} \mathcal{S}(k) = \Big\{f \in \mathcal{S} \colon f \text{ is q.c.-extendible}\Big\}$ is one of the models of the *Teichmüller universal space.*

Extremal Problems for Univ. Functions

The class $\mathcal{S}$ of all normalized univalent (= injective + holomorphic) functions $f\colon \mathbb{D} \to \mathbb{C}$, $f(0) = 0$, $f'(0) = 1$, *on its own* is a classical object of study in Geometric Function Theory.

- \+ The class $\mathcal{S}$ is compact (w.r.t. the locally uniform convergence), so it make sense to pose *Extremal Problems* for continuous functionals on $\mathcal{S}$.
- − However, $\mathcal{S}$ has no natural linear structure, and it is NOT convex in $\mathrm{Hol}(\mathbb{D}, \mathbb{C})$.
- − As a result, *the standard variational technique does not apply to the extremal problems in the class $\mathcal{S}$*.

Bieberbach's Problem, 1916

$$|a_n| \to \max \quad \text{over all} \quad f(z) = z + \sum_{n=2}^{+\infty} a_n z^n \quad \text{from} \quad \mathcal{S}$$

Extremal Problems 2

Bieberbach's Problem, 1916

$$|a_n| \to \max \quad \text{over all} \quad f(z) = z + \sum_{n=2}^{+\infty} a_n z^n \quad \text{from} \quad \mathcal{S}$$

- Bieberbach, 1916, proved that $\max_{\mathcal{S}} |a_2| = 2$ and conjectured that $\max_{\mathcal{S}} |a_n| = n$ for all $n \geqslant 2$ — the *Bieberbach Conjecture.*
- This conjecture was a major problem in Complex Analysis for a long time. Certain progress was achieved by:

  $n = 3$: Löwner (=Loewner), 1923; $|a_n| \leqslant en$: Littlewood, 1925;
  $n = 4$: Garabedian and Schiffer, 1955;
  $\limsup |a_n|/n \leqslant 1$: Hayman, 1955; $|a_n| \leqslant (1.243)n$: Milin, 1965;
  $n = 6$: Pederson, 1968; Ozawa, 1969;
  $n = 5$: Pederson and Schiffer, 1972;
  $|a_n| \leqslant (1.081)n$: FitzGerald, 1972; $|a_n| \leqslant (1.07)n$: Horowitz, 1978

- de Branges, 1984, completely proved the Bieberbach Conjecture.
- The cornerstone of his proof is essentially the same method as the one introduced by Charles Loewner (=Karel/Karl Löwner) in 1923, known as *(Loewner's) Parametric Representation*.

Definition

A *classical Herglotz function* is a function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ s.t.:

(M) $p(z, \cdot)$ is measurable for all $z \in \mathbb{D}$;

(H) $p(\cdot, t)$ is holomorphic for all $t \geqslant 0$;

(Re) $\operatorname{Re} p > 0$ and $p(0, t) = 1$ for all $t \geqslant 0$.

Given a classical Herglotz function p,

the (classical radial) *Loewner–Kufarev ODE*

$$\frac{d}{dt} w(z,t) = -w(z,t)\, p\big(w(z,t), t\big), \quad \big(\forall\, z \in \mathbb{D}\big)\;\; w(z,0) = z, \qquad (3)$$

has a unique solution $w = w_p : \mathbb{D} \times [0, +\infty) \to \mathbb{D}$.

Parametric Method 2

Again, for a classical Herglotz function p,
we denote by w_p the unique solution to the (I.V.P. for the)

(classical radial) Loewner – Kufarev ODE

$$\frac{d}{dt}w(z,t) = -w(z,t)\,p\big(w(z,t),t\big), \quad \big(\forall\, z \in \mathbb{D}\big)\;\; w(z,0) = z, \tag{4}$$

Theorem (Pommerenke, 1965-75; Gutlyanskii, 1970)

(I) *A function $f\colon \mathbb{D} \to \mathbb{C}$ belongs to $\mathcal{S}$ if and only if $\exists$ a classical Herglotz function p s.t.*

$$f(z) = \lim_{t\to+\infty} e^t w_p(z,t) \quad \text{for all } z \in \mathbb{D}. \tag{5}$$

(II) *$\forall$ classical Herglotz function p the limit (5) exists and it is attained locally uniformly in $\mathbb{D}$.*

In other words, formula (5) defines a surjective mapping $p \mapsto f$ of the convex cone of all classical Herglotz functions onto the class $\mathcal{S}$.

Conditions for q.c.-extendibility

- ☞ Parametric Representation had been introduced and used as an effective instrument to solve Extremal Problems in the class $\mathcal{S}$.
- ☺ in 1972 Becker found a construction that allows one to apply the Loewner – Kufarev equations to obtain q.c.-extensions of holomorphic functions in $\mathbb{D}$.
- ☞ In this way he was able to deduce several *sufficient conditions* for q.c.-extendibility:

Let $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ and $k \in [0, 1)$. Each of the following conditions is sufficient for f to be k-q.c. extendible:

(a) $|1 - f'(z)| \leqslant k$ for all $z \in \mathbb{D}$;

(b) $\left|zf''(z)/f'(z)\right| \leqslant \frac{k}{1-|z|^2}$ for all $z \in \mathbb{D}$;

(c) $|Sf(z)| \leqslant \frac{2k}{(1-|z|^2)^2}$ for all $z \in \mathbb{D}$, where $Sf(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$ is the Schwarzian derivative.

Consider the *characteristic PDE* of the Loewner – Kufarev ODE:

Loewner – Kufarev PDE

$$\frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z}p(z,t), \qquad z \in \mathbb{D}, \quad t \geqslant 0. \tag{6}$$

The unique solution $(z,t) \mapsto f_t(z)$ to (6) that is:

- ✔ well-defined and univalent in $\mathbb{D}$ for all $t \geqslant 0$;
- ✔ normalized by $f_0(0) = 0$, $f_0'(0) = 1$,

is given by the formula

$$f_s(z) = \lim_{t\to+\infty} e^t w_p(z;s,t), \tag{7}$$

where $t \mapsto w_p(z;s,t)$ is the unique solution to the Loewner – Kufarev ODE $dw/dt = -w\,p(w,t)$ with the I.C. $w_p(z;s,s) = z$ for all $z \in \mathbb{D}$.

NOTE: The initial condition is now given at $t = s$.

Theorem (Pommerenke, 1965)

The formula $$f_s(z) = \lim_{t\to+\infty} e^t w_p(z;s,t), \tag{8}$$

where $t \mapsto w_p(z;s,t)$ *solves the (I.V.P. for the) Loewner – Kufarev ODE*

$$dw/dt = -w\,p(w,t), \quad w_p(z;s,s) = z \quad \text{for all } z \in \mathbb{D}, \tag{9}$$

est'shes a 1-to-1 relation between the classical Herglotz functions p *and the (so-called) classical radial Loewner chains* (f_t).

Definition (Pommerenke's book "Univalent functions")

A family $(f_t)_{t\geqslant 0} \subset \mathrm{Hol}(\mathbb{D},\mathbb{C})$ is said to be a *classical radial Loewner chain* if the following conditions hold:

- all f_t's are *univalent* in $\mathbb{D}$;
- $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $t \geqslant s \geqslant 0$;
- $f_t(0) = 0$ and $f_t'(0) = e^t$ $(\Longleftrightarrow e^{-t} f_t \in \mathcal{S})$ for all $t \geqslant 0$.

Becker's q.c.-extension

Becker's construction of q.c.-extensions is given in the following thrm.

Theorem (Becker, 1972 (*J. Reine Angew. Math.*))

Fix $k \in [0,1)$ and let (f_t) be a classical radial Loewner chain. If the associated classical Herglotz function p satisfies

$$p(z,t) \in U(k) := \left\{\zeta \in \mathbb{C} \colon \left|\frac{\zeta - 1}{\zeta + 1}\right| \le k\right\} \quad (\forall\, z \in \mathbb{D} \text{ and a.e. } t \geqslant 0), \tag{10}$$

then:

(A) *All f_t's extend continuously to $\partial\mathbb{D}$.* (B) *The function*

$$\tilde{f}(z) := \begin{cases} f_0(z), & z \in \mathbb{D}, \\ f_{\log|z|}\,(z/|z|)\,, & z \in \mathbb{C}\setminus\mathbb{D}, \\ \infty, & z = \infty, \end{cases} \tag{11}$$

is a k-q.c. automorphism of $\overline{\mathbb{C}}$. In particular, $f_0 \in \mathcal{S}(k)$, i.e. f_0 is k-q.c. extendible.

Chordal Loewner Equation

- ☞ The vector field $G(w,t) := -w\,p(w,t)$ in the r.h.s of the Loewner – Kufarev ODE has a zero at $w = 0$.
- ☞ Correspondingly the solutions $w(\cdot; s,t) \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ have an attracting fixed point at $z = 0$.
- ☞ The *chordal Loewner equation* is an analogue of the (classical radial) Loewner – Kufarev equation for the case of a boundary attracting fixed point (= boundary Denjoy – Wolff point).
- ☞ A particular case of the chordal Loewner ODE seems to be known since 1946 (Kufarev), but we will use the general form due to Bracci, Contreras and Díaz-Madrigal, 2012 (*J. Reine Angew. Math.*).
- ☺ Let us pass to $\mathbb{H} := \{z \colon \mathrm{Re}\, z > 0\}$.

Chordal Loewner Equation 2

Chordal Loewner ODE

$$\frac{dw(z,t)}{dt} = p\big(w(z,t),t\big), \quad t \geqslant 0, \tag{12}$$

where $p : \mathbb{H} \times [0,+\infty) \to \mathbb{C}$ is a Herglotz function in $\mathbb{H}$.

Definition

A Herglotz function in $\mathbb{H}$ is a function $p : \mathbb{H} \times [0,+\infty) \to \mathbb{C}$ s.t.:

(M) $p(z,\cdot)$ is measurable for all $z \in \mathbb{H}$;

(H) $p(\cdot,t)$ is holomorphic for all $t \geqslant 0$;

(Re) $\operatorname{Re} p \geqslant 0$; and

($\int$) $t \mapsto p(z_0,t)$ is locally integrable on $[0,+\infty)$ for some (and hence all) $z_0 \in \mathbb{H}$.

Chordal Loewner Equation 3

Definition

A *chordal Loewner chain* $(f_t)_{t\geqslant 0}$ associated with a Herglotz function $p : \mathbb{H}\times[0,+\infty)\to\mathbb{C}$ is a solution to the chordal Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -\frac{\partial f_t(z)}{\partial t}p(z,t), \tag{13}$$

s.t. f_t is well-defined and univalent in $\mathbb{H}$ for all $t\geqslant 0$.

Remark

It is known that, given a Herglotz function p in $\mathbb{H}$,

☺ the associated chordal Loewner chain exists,

☹ but it does NOT need to be unique.

Proposition (P. Gum., Ikkei HOTTA)

Let $p : \mathbb{H} \times [0, +\infty) \to \mathbb{C}$ be a Herglotz function in $\mathbb{H}$. Suppose that there exists a locally $\int$-ble function $M : [0, +\infty) \to [0, +\infty)$ s.t.

(i) $\displaystyle\int_{[0,+\infty)} M(t)\,\mathrm{d}t = +\infty$, and

(ii) $C_1 M(t) \leqslant \operatorname{Re} p(z,t) \leqslant C_2 M(t)$ for a.e. $t \geqslant 0$ and all $z \in \mathbb{H}$,

where $C_1, C_2 > 0$ are some constants.

Then

$$\bigcup_{t \geqslant 0} f_t(\mathbb{H}) = \mathbb{C}$$

for any chordal Loewner chain (f_t) associated with p, and hence the associated chordal Loewner chain (f_t) is unique up to affine maps, i.e. if (g_t) is another chordal Loewner chain associated with p, then $g_t = af_t + b$, $t \geqslant 0$, for some $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$.

Theorem (**P. Gum.**, Ikkei HOTTA)

Fix $k \in [0,1)$ and let (f_t) be a chordal Loewner chain associated to a Herglotz function $p : \mathbb{H} \times [0,+\infty) \to \mathbb{C}$.

If

$$p(z,t) \in U(k) := \left\{\zeta \in \mathbb{C} \colon \left|\frac{\zeta-1}{\zeta+1}\right| \le k\right\} \ (\forall\ z \in \mathbb{H} \text{ and a.e. } t \geqslant 0), \quad (15)$$

then:

(A) *All f_t's extend continuously to $\partial\mathbb{H}$.* (B) *The function*

$$\tilde{f}(z) := \begin{cases} f_0(z), & z \in \mathbb{H}, \\ f_{-\operatorname{Re} z}(i \operatorname{Im} z), & z \in \mathbb{C}\setminus\mathbb{H}, \\ \infty, & z = \infty, \end{cases} \quad (16)$$

is a k-quasiconformal extension of f_0 to $\overline{\mathbb{C}}$.

Suff. conditions for q.c.-extendibility

Hyperbolic (Poincaré) distance in $\mathbb{H}$ is given by

$$\operatorname{dist}_{hyp,\mathbb{H}}(z_1,z_2) := \frac{1}{2}\log\frac{1+\rho_{\mathbb{H}}(z_1,z_2)}{1-\rho_{\mathbb{H}}(z_1,z_2)},$$

$$\text{where } \rho_{\mathbb{H}}(z_1,z_2) := \frac{|z_1-z_2|}{|z_1+\overline{z}_2|} \quad \text{for all } z_1,z_2\in\mathbb{H}. \tag{17}$$

Theorem (**P. Gum.**, Ikkei HOTTA)

Fix $k\in[0,1)$ and let $K:=(1+k)/(1-k)$. Let $D\subset\mathbb{H}$ be a closed hyperbolic disk of radius $\frac{1}{2}\log K$. Finally, let $f\in\operatorname{Hol}(\mathbb{H},\mathbb{C})$. Each of the following conditions is sufficient for f to be k-q.c. extendible:

(a) $\left|\dfrac{f''(z)}{f'(z)}\right| \leqslant \dfrac{k}{2\operatorname{Re} z}$ *for all $z\in\mathbb{H}$ [Becker & Pommerenke, 1984]*

(b) $f'(\mathbb{H})\subset D$;

(c) $[f'(z)]^{-1}(f(z)+a)-z\in D$ *for all $z\in\mathbb{H}$ and some $a\in\mathbb{C}$.*

From part (b) of the previous theorem we obtained
a corollary for functions in $\mathbb{D}$.

Corollary (P. Gum., Ikkei HOTTA)

Fix $k \in [0, 1)$ and let $K := (1 + k)/(1 - k)$. Let $D \subset \mathbb{H}$ be a closed hyperbolic disk of radius $\frac{1}{2} \log K$. If $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ satisfies

$$\frac{zf'(z)}{f(z)} \in D \quad \text{for all } z \in \mathbb{D} \setminus \{0\}, \tag{18}$$

then f is k-q.c. extendible.

This corollary extends a classical result, in which $D := U(k)$.

The End