Loewner chains: the case of boundary Denjoy – Wolff point

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joint research with

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Notation and definitions

$\mathbb{D} := \{z : |z| < 1\}$, $\mathbb{T} := \partial\mathbb{D}$, $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$,

$\operatorname{Hol}(D_1, D_2)$ is the set of all holomorphic functions $f : D_1 \to D_2$.

Definition 1. A family $(\varphi_{s,t})_{0\le s\le t<+\infty} \subset \operatorname{Hol}(\mathbb{D},\mathbb{D})$ is called a *(generalized) evolution family of order* $d \in [1,+\infty]$ if:

EF1. $\varphi_{s,s} = id_{\mathbb{D}}$,

EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \le s \le u \le t < +\infty$,

EF3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0,T],\mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k_{z,T}(\xi)d\xi \quad \text{whenever } 0 \le s \le u \le t \le T.$$

Definition 2. A family $(f_t)_{0\le t<+\infty} \subset \mathrm{Hol}(\mathbb{D},\mathbb{C})$ is called a *(generalized) Loewner chain of order* d with $d \in [1,+\infty]$ if:

LC1. each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,

LC2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \le s < t < +\infty$,

LC3. for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \le \int_s^t k_{K,T}(\xi)d\xi$$

for all $z \in K$ and all $s,t \in [0,T]$, $s \leqslant t$.

Convention: for shortness we will omit attribute "*generalized* " for these notions.

Theorem A. *Every Loewner chain* (f_t) *of order* d *generates an evolution family* $(\varphi_{s,t})$ *of the same order, defined by the formula*

$$\varphi_{s,t} = f_t^{-1} \circ f_s, \quad t \geqslant s \geqslant 0. \tag{1}$$

Furthermore, for each evolution family $(\varphi_{s,t})$ *of order* d*, there exists a Loewner chain* (f_t) *of the same order s. t.* (1) *holds.*

In situation of the above Theorem, we will say that

- $(\varphi_{s,t})$ is *the evolution family of the Loewner chain* (f_t) and that
- the Loewner chain (f_t) is *associated* with the evolution family $(\varphi_{s,t})$.

Remark 1. It has been proved by F. Bracci, M. D. Contreras and S. Díaz-Madrigal that evolution families can be thought as solutions to the *generalized Loewner ODE* of the form

$$\frac{dw}{dt} = G(w,t), \quad w|_{t=s} = z, \tag{2}$$

where $G(w,t) = \big(\tau(t) - w\big)\big(1 - \overline{\tau(t)}w\big)p(w,t)$ is a so-called (generalized) *Herglotz vector field* in $\mathbb{D}$. Associated Loewner chains (f_t) satisfy, in their turn, the *generalized Loewner PDE*

$$\frac{\partial f_t(z)}{\partial t} = -G(z,t)\frac{\partial f_t(z)}{\partial z}. \tag{3}$$

A Loewner chain (f_t) contains three pieces of information:

- Geometry: collection of domains $\mathcal{D} := \{f_t(\mathbb{D}) : t \in [0, +\infty)\}$;
- Parameterization: an increasing mapping $[0, +\infty) \to \mathcal{D}$; $t \mapsto D_t$ such that $f_t(\mathbb{D}) = D_t$;
- Normalization by which the conformal mappings $f_t : \mathbb{D} \to D_t$ are uniquely chosen.

General problem we discuss is *to relate geometry of a Loewner chain with the properties of the corresponding evolution family.*

We introduce following

Definition 3. Let $\mathcal{D}$ be a collection of simply connected domains in the complex plane. Let us call $\mathcal{D}$ an *inclusion chain* if:

IC1. As an ordered set, $\mathcal{D}$ is isomorphic to $[0, +\infty)$, i.e. there is a bijective mapping $D : [0, +\infty) \to \mathcal{D}$ such that $D(t_1) \subsetneq D(t_2)$ whenever $t_1 < t_2$;

IC2. Each $\Omega \in \mathcal{D}$ is a connected component of $\mathrm{int}(\Omega^+)$, where $\mathrm{int}(\cdot)$ stands for the interior of a set and Ω^+ stands for the intersection of all $\Omega' \in \mathcal{D} \setminus \{\Omega\}$ which contains Ω;

IC3. Each $\Omega \in \mathcal{D}$ coincides with the union Ω^- of all $\Omega' \in \mathcal{D} \setminus \{\Omega\}$ which are contained in Ω, provided Ω is not the minimal element in $\mathcal{D}$.

Remark 2. For any Loewner chain (f_t) the collection of domains $\mathcal{D}[(f_t)] := \{f_t(\mathbb{D}) : t \in [0, +\infty)\}$ is an inclusion chain.

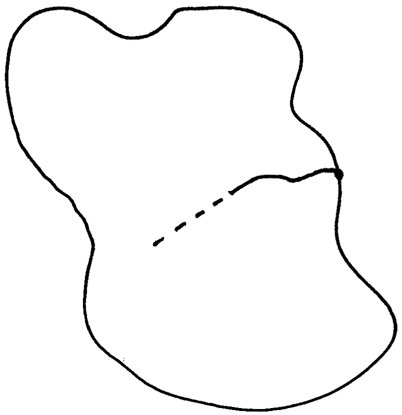
We will call $\mathcal{D}[(f_t)]$ it the *inclusion chain of the Loewner chain* (f_t).

Definition 4. Given a holomorphic self-mapping $\varphi \neq \mathrm{id}_{\mathbb{D}}$ of the unit disk, there exist a unique point $\tau \in \overline{\mathbb{D}}$ s.t. $\varphi(\tau) = \tau$ and $|\varphi'(\tau)| \leqslant 1$. This point is called the *Denjoy – Wolff point (DW-point)* of the function φ. We will say that an evolution family $(\varphi_{s,t})$ *has a common DW-point* τ, if it is a DW-point for each of the functions $\varphi_{s,t}$ (excluding identity).

Problem 1. *Describe the inclusion chains of all those Loewner chains whose evolution families has a common DW-point on the unit circle, say* $\tau = 1$.

Answer for the case of the internal DW-point is: *every inclusion chain is the inclusion chain of some Loewner chain whose evolution family* $(\varphi_{s,t})$ *satisfy* $\varphi_{s,t}(0) = 0$, $\varphi'_{s,t}(0) > 0$.

P. P. Kufarev, 1943;
Ch. Pommerenke, 1965.



"Slit-erasing" inclusion chains: *every inclusion chain obtained by erasing a slit in a simply connected domain, is the inclusion chain of some Loewner chain whose evolution family has a common DW-point on the boundary* $\mathbb{T} := \partial\mathbb{D}$.

Extension of Loewner's parametric method to univalent functions in the half-plane with hydrodynamical normalization

I. A. Aleksandrov, V. V. Sobolev, 1970;
V. V. Sobolev, 1970;
O. Schramm, 2000.

Generalization of the chordal Loewner equation

V. V. Goryainov and I. Ba, 1992;
R. O. Bauer, 2005.

Notation: $\mathcal{P}(D)$ is the *Carathéodory boundary* of a domain D, i. e. the set of all *prime ends* of D;

Definition 5. Let G be a simply connected subdomain of $\mathbb{D}$, $\psi : \mathbb{D} \to G$ a conformal mapping, $P \in \mathcal{P}(G)$, and $\zeta_0 := \psi^{-1}(P)$. We will say that *G is embedded in $\mathbb{D}$ conformally at the prime end P* if ζ_0 is a regular contact point of the function ψ ,i. e., the following conditions hold:

(i) $\exists \angle \lim\limits_{\zeta \to \zeta_0} \psi(\zeta) := z_0 \in \mathbb{T}$;

(ii) $\angle \lim\limits_{\zeta \to \zeta_0} \psi'(\zeta) \neq \infty$.

Definition 5 can be extended to the case of two arbitrary hyperbolic simply connected domains $\Omega_1 \subset \Omega_2$ by means of a conformal mapping $\phi_2 : \Omega_2 \to \mathbb{D}$:

Definition 6. A domain Ω_1 is said to be *embedded in a domain Ω_2 conformally at the prime end $P_1 \in \mathcal{P}(\Omega_1)$* if the domain $G := \phi_2(\Omega_1)$ is embedded in $\mathbb{D}$ conformally at the prime end $P := \phi_2|_{\Omega_1}(P_1) \in \mathcal{P}(G)$.

Theorem 1. *Let $\mathcal{D}$ be an inclusion chain with the minimal element Ω_0, and $P_0 \in \mathcal{P}(\Omega_0)$. Suppose that Ω_0 is embedded in each $\Omega \in \mathcal{D}$ conformally at P_0. Then there exists a Loewner chain (f_t) such that $\mathcal{D}[(f_t)] = \mathcal{D}$ and the corresponding evolution family $\varphi_{s,t} = f_t^{-1} \circ f_s$, $t \geqslant 0$, $s \in [0, t]$, has a common parabolic DW-point at $\tau = 1$.*

The converse of this theorem is immediate: *Let (f_t) be a Loewner chain. If all the functions in its evolution family $(\varphi_{s,t})$ have regular boundary fixed point $\tau = 1$, then the domain $f_0(\mathbb{D})$ is embedded in each $f_t(\mathbb{D})$ conformally at the prime end $f_0(1)$.*

It can be interesting to require some extra regularity of an evolution family at the DW-point. Since we have a distinguished point at the boundary, it is convenient to change the reference domain and work with evolution families and Loewner chains in *the upper half-plane $\mathbb{H}$ instead of the those in the unit disk, with the DW-point placed at ∞.*

Consider the following class of functions

$\varphi \in \mathfrak{R} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \varphi : \mathbb{H} \to \mathbb{H}$ is holomorphic and univalent

in $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$ and there exists $C > 0$ such that

$$|\varphi(z) - z| \leqslant C/\operatorname{Im} z, \qquad z \in \mathbb{H}. \tag{4}$$

The class $\mathfrak{R}$ is a semigroup w.r.t. the operation of composition and the functional

$$\ell(\varphi) := \min\Big\{C > 0 : |\varphi(z) - z| \leqslant C/\operatorname{Im} z \text{ for all } z \in \mathbb{H}\Big\} \tag{5}$$

is additive.

Our interest to the class $\mathfrak{R}$ is connected to the generalization of the chordal Loewner equation for slit mappings with hydrodynamical normalization given by V. V. Goryainov & I. Ba, 1992, and R. O. Bauer, 2005.

The type of evolution families they discuss can be defined as follows.
Definition 7. An evolution family $(\varphi_{s,t})$ in the half-plane $\mathbb{H}$ is said to be a *Goryainov–Ba chordal evolution family* if $(\varphi_{s,t}) \subset \mathfrak{R}$ and the function $\lambda(t) := \ell(\varphi_{0,t})$ is locally absolute continuous on $[0, +\infty)$.

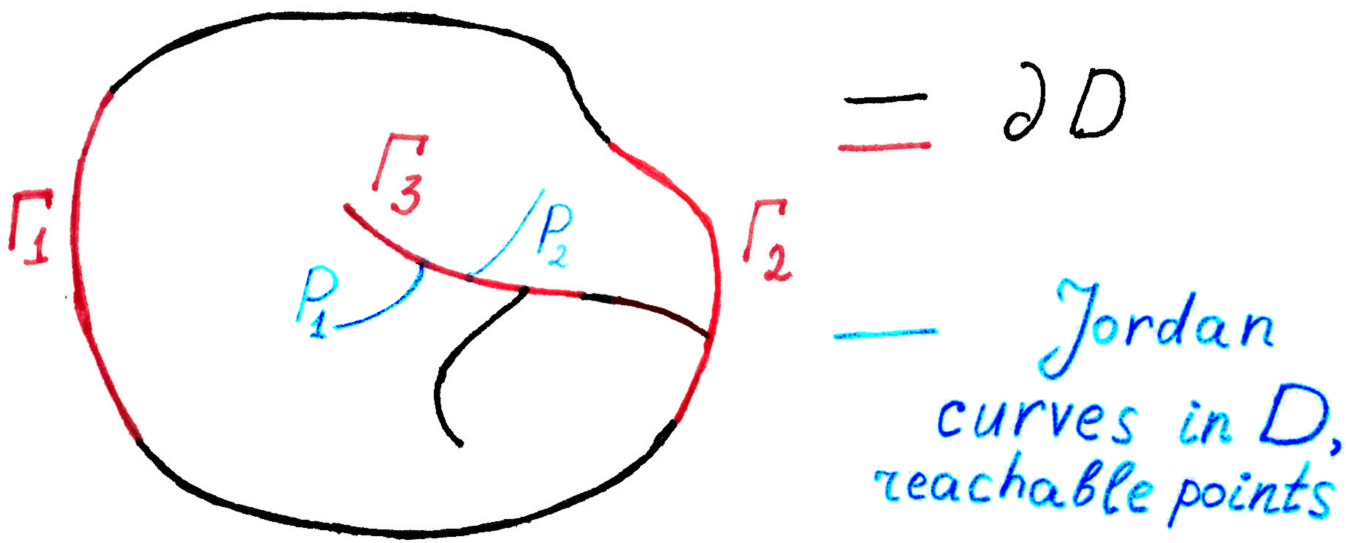
Problem 2. Describe the inclusion chains of those Loewner chains in $\mathbb{H}$ which generate *Goryainov–Ba chordal evolution families*.

Definition 8. Let D be a domain. A Jordan curve $\Gamma \subset \partial D$ is called *free Jordan arc* if the endpoints of Γ can be joined by a Jordan curve Γ' in D such that $\Gamma \cup \Gamma'$ bounds a subdomain D' of D.

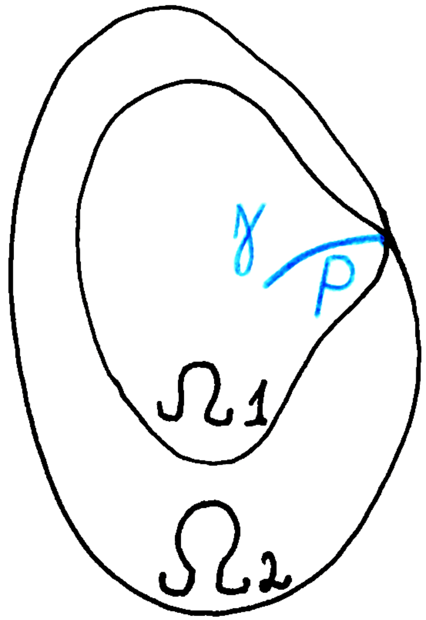
Further, let P be a reachable point of the domain D. We will say that the free Jordan arc Γ *contains P as an interior point* if

(i) the impression of P is an interior point of Γ;

(ii) as an equivalence class of curves, P contains a Jordan curve lying in D'.



Remark 3. Consider two domains $\Omega_1 \subset \Omega_2$. Let P be a reachable point of the domain Ω_1 and $\gamma \subset \Omega_1$ a Jordan curve defining reachable point P. Suppose that the impression $I(P)$ of P is a common boundary point of both domains Ω_1 and Ω_2. Then γ defines also a reachable point for Ω_2. *This reachable point does not depend on the choice of γ and will be denoted, suppressing the language, also by P.*



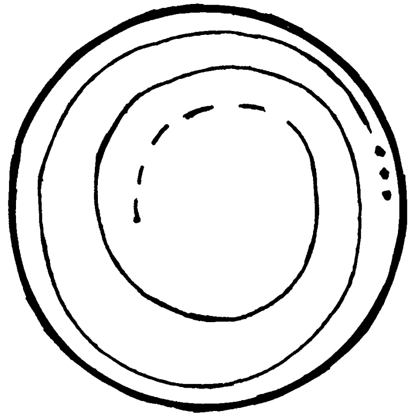
Theorem 2. *Let $\mathcal{D}$ be an inclusion family with the minimal element Ω_0 and P_0 a reachable point of $\partial\Omega_0$. Suppose that each $\Omega \in \mathcal{D}$ has a free Jordan arc Γ_Ω on the boundary that contains P_0 as an interior point and that Γ_Ω is $C^{3,\alpha}$–smooth for some $\alpha > 0$ (which can depend on Ω). If the curves Γ_Ω, $\Omega \in \mathcal{D}$, have second-order contact at the impression of P_0, then there is a Loewner chain (f_t) in $\mathbb{H}$ such that $\mathcal{D}[(f_t)] = \mathcal{D}$ and $(\varphi_{s,t})$ defined by $\varphi_{s,t} := f_t^{-1} \circ f_s$ is a* **Goryainov – Ba chordal evolution family.**

Proposition 1. *Let* (f_t) *be a Loewner chain in* $\mathbb{D}$*. Suppose that* $\partial f_t(\mathbb{D})$ *is locally connected for any* $t \geqslant 0$*. Then the functions* $\varphi_{s,t} := f_t^{-1} \circ f_s$ *can be extended by continuity to the unit circle* $\mathbb{T}$*, i.e.,* $(\varphi_{s,t}) \subset \mathcal{A}$*, where*

$$\mathcal{A} := \Big\{\varphi \in \mathsf{Hol}(\mathbb{D}, \mathbb{C}) : \Big(\exists \tilde{\varphi} \in C(\overline{\mathbb{D}})\Big)\ \varphi = \tilde{\varphi}|_{\mathbb{D}}\Big\}$$

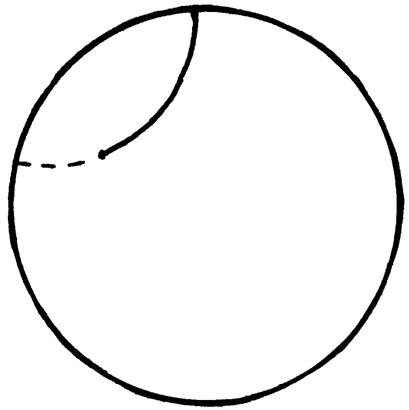
is the disk algebra.

Remark 4. The converse of the above Proposition is not true: there is an evolution family $(\varphi_{s,t}) \subset \mathcal{A}$ such that for any associated Loewner chain (f_t) the image domains $f_t(\mathbb{D})$ fail to have locally connected boundaries.



The disk algebra $\mathcal{A}$ has a natural topology, induced by the norm

$$\|\varphi\| := \sup_{z\in\mathbb{D}} |f(z)|.$$



Obviously,

$(\varphi_{s,t}) \subset \mathcal{A} \quad \not\Longrightarrow \quad (s,t) \mapsto \varphi_{s,t}$ is continuous in $\mathcal{A}$.

Kufarev's example: discontinuity at $s = 0$.

Proposition 2. *Let $(\varphi_{s,t})$ be an evolution family in $\mathbb{D}$. Suppose $(\varphi_{s,t}) \subset \mathcal{A}$. Then for any fixed $s_0 \geqslant 0$ the mapping $[s_0, +\infty) \ni t \mapsto \varphi_{s_0,t} \in \mathcal{A}$ is continuous.*