INdAM Conference

«New Trends in Holomorphic Dynamics»

Loewner equations, evolution families and their boundary fixed points

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Bieberbach conjecture

In 1915 – 1916 Ludwig Bieberbach (1886 – 1982) studied the so-called class $\mathcal{S}$ which is formed by *univalent holomorphic functions*

$$f : \mathbb{D} := \{z : |z| < 1\} \to \mathbb{C}$$

normalized by

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Bieberbach obtained the quantitative form of several basic result on the class S, such as

- sharp upper and lower bounds for $|f(z)|$ (the Growth Theorem)
- and those for $|f'(z)|$ (the Distortion Theorem).

Bieberbach conjecture

The key point was the estimate of the second Taylor coefficient. Namely, he proved that

$$|a_2| \le 2 \quad \text{for any } f \in \mathcal{S},$$

with the equality only for the rotations of the Koebe function

$$k_\theta(z) = \frac{z}{(1-e^{-i\theta}z)^2} = z + \sum_{n=2}^{+\infty} n e^{-i(n-1)\theta} z^n, \quad \theta \in \mathbb{R}. \tag{2}$$

Bieberbach conjectured that

Bieberbach Conjecture

For any $f \in \mathcal{S}$ and any integer $n \ge 2$,

$$|a_n| \le n,$$

with the equality only for functions (2).

Bieberbach Conjecture — continued

This was the beginning of a new epoch in Geometric Function Theory, which finished in 1985 with the proof of the Bieberbach Conjecture given by Louis de Branges.

The first step on the way to this proof was done by the Czech – German mathematician Karel Löwner (1893 – 1968) known also as Charles Loewner in his paper

Untersuchungen über schlichte konforme Abbildungen des Einheitskreises,
Math. Ann. **89** (1923), 103–121.

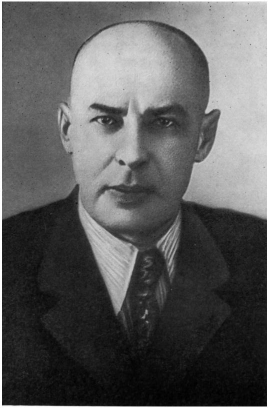
Loewner's Method

Loewner proved the Bieberbach conjecture *for* $n = 3$. What is more important, he introduced the first powerful method for systematic study of univalent functions. In particular, Loewner's method is also the cornerstone in de Branges' proof.

The main merit of Loewner is that with his method *he introduced a dynamic viewpoint in Geometric Function Theory*.

I would like to present a more modern form of Loewner's method, which is mainly due to contributions of another two prominent mathematicians:

Loewner's Method — continued



Pavel Parfen'evich Kufarev
Tomsk (1909 – 1968)

Christian Pommerenke
(Copenhagen, 17 December 1933)

Parametric representation of univalent functions

Theorem A.1 (Pommerenke, and independently V.Ya. Gutlyanskiĭ)

Let $f \in \mathcal{S}$. Then there exists a family $(f_t)_{t\geq 0}$ of holomorphic functions in $\mathbb{D}$ such that $f_0 = f$ and the following conditions hold:

LC1. for each $t \geq 0$, $f_t : \mathbb{D} \to \mathbb{C}$ is univalent in $\mathbb{D}$;

LC2. for each $s \geq 0$ and $t \geq s$, $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$;

LC3. for each $t \geq 0$,

$$f_t(z) = e^t z + a_2(t) z^2 + \dots \qquad (3)$$

Definition

A family $(f_t)_{t\geq 0}$ of holomorphic functions in $\mathbb{D}$ satisfying the above conditions LC1, LC2, and LC3 is said to be a *classical Loewner chain*.

Parametric representation — continued

Definition

A function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is said to be a *classical Herglotz function* if:

HF1. for each $t \geq 0$, $p(\cdot, t)$ is a Carathéodory function, *i.e.* it is holomorphic in $\mathbb{D}$ with $\operatorname{Re} p(\cdot, t) > 0$ and $p(0, t) = 1$;

HF2. for each $z \in \mathbb{D}$, the function $p(z, \cdot)$ is measurable on $[0, +\infty)$.

Theorem A.2

Let (f_t) be a classical Loewner chain. Then there exists essentially unique classical Herglotz function p such that (f_t) satisfies the Loewner – Kufarev PDE

$$\frac{\partial f_t(z)}{\partial t} = z \frac{\partial f_t(z)}{\partial z} p(z, t), \quad z \in \mathbb{D}, \quad t \geq 0. \tag{4}$$

Parametric representation — continued

Theorem A.2 — continued

Moreover, for any $s \geq 0$,

$$f_s = \lim_{t \to +\infty} e^t \varphi_{s,t}, \tag{5}$$

where $t \mapsto \varphi_{s,t}(z)$ for each fixed $z \in \mathbb{D}$ and $s \geq 0$ is the unique solution to

$$dw(t)/dt = -w(t)p\Big(w(t), t\Big), \quad t \geq s; \quad w(s) = z. \tag{6}$$

Equation (6) is called the Loewner – Kufarev ODE. Note that it is the characteristic ODE of the Loewner – Kufarev PDE. Hence each $\varphi_{s,t}$, $t \geq s \geq 0$, is a holomorphic self-map of $\mathbb{D}$ and

$$f_s = f_t \circ \varphi_{s,t} \quad \text{for any } s \geq 0 \text{ and any } t \geq s. \tag{7}$$

Parametric representation — continued

The converse theorem also holds:

Theorem A.3

Let p be a classical Herglotz function. Then for any $s \geq 0$ and $z \in \mathbb{D}$ following IVP

$$\frac{dw(t)}{dt} = -w(t)p\Big(w(t), t\Big), \quad t \geq s; \quad w(s) = z. \tag{6}$$

has a unique solution $w_{z,s}$ defined for all $t \geq s$ and the functions

$$\varphi_{s,t}(z) = w_{z,s}(t), \quad z \in \mathbb{D}, \quad t \geq s \geq 0, \tag{8}$$

are holomorphic univalent self-maps of $\mathbb{D}$.

Parametric representation — continued

Theorem A.3 — continued

Moreover, the formula

$$f_s = \lim_{t \to +\infty} e^t \varphi_{s,t}, \quad s \ge 0, \tag{5}$$

defines a classical Loewner chain (f_t), which satisfies the relation

$$f_s = f_t \circ \varphi_{s,t} \quad \text{for any } s \ge 0 \text{ and any } t \ge s \tag{7}$$

and the Loewner – Kufarev PDE

$$\frac{\partial f_t(z)}{\partial t} = z \frac{\partial f_t(z)}{\partial z} p(z,t), \quad z \in \mathbb{D}, \quad t \ge 0. \tag{4}$$

Parametric representation — continued

Some conclusions

- The Loewner – Kufarev equations establish 1-to-1 correspondence between classical Loewner chains and classical Herglotz functions.
- The set of the initial elements of all classical Loewner chains coincides with the class $\mathcal{S}$.
- Therefore, any extremal problem for the class $\mathcal{S}$ can be reformulated as an Optimal Control problem, where the "control" is a classical Herglotz function;
- Note that the class $\mathcal{S}$ has no natural linear structure, while the set of all classical Herglotz functions is a (real) convex cone.

This representation of the class $\mathcal{S}$ by means of classical Herglotz functions is called the *Parametric Representation* of normalized univalent functions.

Chordal Loewner Equation

P. P. Kufarev and his students constructed similar parametric representation for *univalent holomorphic self-maps f* of the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$ satisfying the so-called *hydrodynamic normalization*:

$$\lim_{z\to\infty} f(z) - z = 0, \quad \lim_{z\to\infty} z\Big(f(z) - z\Big) \neq \infty. \tag{9}$$

This normalization make sense, for example,

if $\mathbb{H} \setminus f(\mathbb{H})$ is a bounded set in $\mathbb{C}$.

To extend the normalization to a larger class of functions one has to consider angular limits instead of unrestricted ones.

Chordal Loewner Equation — continued

The role of the Loewner – Kufarev equation is played in this case by the so-called *chordal Loewner equation*, which can be written in its general form as

Chordal Loewner Equation

$$\frac{d\zeta(t)}{dt} = ip\big(\zeta(t), t\big), \tag{10}$$

where $p(\cdot, t)$ is a holomorphic function in $\mathbb{H}$ with $\operatorname{Re} p > 0$.

Rewritten in the unit disk $\mathbb{D}$ this equation take the form

Chordal Loewner Equation in $\mathbb{D}$

$$\frac{dw}{dt} = \big(1 - w(t)\big)^2 p\big(w(t), t\big), \tag{11}$$

where $p(\cdot, t)$ is again holomorphic function in $\mathbb{D}$ with $\operatorname{Re} p > 0$.

Chordal Loewner Equation and SLE

- P. P. Kufarev, 1946: a special case of chordal Loewner equation mentioned for the first time;
- N. V. Popova, 1954;
- P. P. Kufarev, V. V. Sobolev, and L. V. Sporysheva, 1968: parametric representation of slit mappings with hydrodynamic normalization;
- I. A. Aleksandrov, S. T. Aleksandrov and V. V. Sobolev: 1979, 1983: the general form of the chordal Loewner equation;
- V. V. Goryainov and I. Ba, 1992: similar results.

Unfortunately, these works did not draw a wide response.

Chordal Loewner Equation and SLE — continued

Schramm's Stochastic Loewner evolution

O. Schramm, 2000: *Stochastic chordal Loewner equation*

$$\frac{d\zeta(t)}{dt} = ip\big(\zeta(t), t\big), \quad p(\zeta, t) := \frac{2i}{\zeta - \sqrt{\kappa}\mathcal{B}_t}, \tag{12}$$

where $\kappa > 0$ is a parameter and $(\mathcal{B}_t)$ is the standard Browning motion.

NB: In Schramm's version there is "$-$" in front of the r.h.s. of (12).

This invention of Schramm proved to be extremely useful in Statistical Physics, because it describes the continuous scale limit of several classical 2D lattice models. (Two Fields Medals: Wendelin Werner in 2006 and Stanislav Smirnov in 2010.)

One-parameter semigroups

Definition

A *one-parameter semigroup* of holomorphic functions in $\mathbb{D}$ is a continuous homomorphism from $\big(\mathbb{R}_{\geq 0}, +\big)$ to $\big(\mathrm{Hol}(\mathbb{D}, \mathbb{D}), \circ\big)$. In other words, a *one-parameter semigroup* is a family $(\phi_t)_{t\geq 0} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ such that

(i) $\phi_0 = \mathrm{id}_{\mathbb{D}}$;

(ii) $\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t$ for any $t, s \geq 0$;

(iii) $\phi_t(z) \to z$ as $t \to +0$ for any $z \in \mathbb{D}$.

One-parameter semigroups appear, *e.g.* in:

- iteration theory in $\mathbb{D}$ as *fractional iterates*;
- operator theory in connection with *composition operators*;
- *embedding problem* for time-homogeneous stochastic branching processes.

Infinitesimal generators

Theorem B.1

For any one-parameter semigroup (ϕ_t) the limit

$$G(z) := \lim_{t \to +0} \frac{\phi_t(z) - z}{t}, \quad z \in \mathbb{D}, \tag{13}$$

exists. Moreover, G is a holomorphic function in $\mathbb{D}$ of the form

$$G(z) = (\tau - z)(1 - \overline{\tau} z)p(z), \tag{14}$$

where $\tau \in \overline{\mathbb{D}}$ and $p \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ with $\operatorname{Re} p(z) \geq 0$ for all $z \in \mathbb{D}$.

Definition

The function G above is called the *infinitesimal generator* of (ϕ_t).

Infinitesimal generators — continued

Theorem B.2

Any holomorphic function G of the form (14) from Theorem B.1,

$$G(z) = (\tau - z)(1 - \overline{\tau}z)p(z), \tag{14}$$

where $\tau \in \overline{\mathbb{D}}$ and $p \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ with $\mathrm{Re}\, p(z) \geq 0$ for all $z \in \mathbb{D}$, *is the infinitesimal generator* of some one-parameter semigroup (ϕ_t).
Moreover, this one-parameter semigroup (ϕ_t) is the unique solution to

$$\frac{d\phi_t}{dt} = G \circ \phi_t, \quad \phi_o = \mathrm{id}_{\mathbb{D}}. \tag{15}$$

Formula (14) is known as the *Berkson – Porta Representation*.
The point τ is called the *Denjoy – Wolff point* of the semigroup (ϕ_t).

The three ODEs

Using the Berkson – Porta formula, equation (15) from Theorem B.2 can be written for $w := \phi_t(z)$ as

ODE for 1-parameter semigroups

$$\frac{dw}{dt} = (\tau - w)(1 - \overline{\tau} w)p(w). \tag{16}$$

Classical Loewner – Kufarev ODE $(\tau = 0)$

$$\frac{dw}{dt} = -wp(w,t),$$

Chordal Loewner ODE $(\tau = 1)$

$$\frac{dw}{dt} = (1 - w)^2 p(w,t).$$

The new approach

In 2008 (to appear in *J. Reine Angew. Math.*; ArXiv 0807.1594)

Filippo Bracci, Manuel D. Contreras, and Santiago Díaz-Madrigal

suggested a new approach in Loewner Theory, according to which the three ODEs on the previous slide are special cases of a

generalized Loewner ODE

$$\frac{dw}{dt} = G(w,t), \quad t \geq s, \quad w(s) = z \in \mathbb{D}, \tag{17}$$

where $G : \mathbb{D} \times [0, +\infty)$ is the so-called *Herglotz vector field*, a kind of locally integrable family of infinitesimal generators.

The family of functions formed by the integrals to (17) is a non-autonomous analogue of one-parameter semigroups,

the so-called *evolution family*.

Definitions

In what follows

$d \in [1, +\infty]$ is a constant parameter in the time-regularity conditions.

Definition — evolution family

A family $(\varphi_{s,t})$, $0 \le s \le t < +\infty$, in $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ is an *evolution family* of order d if

EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for all $s \ge 0$;

EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \le s \le u \le t < +\infty$;

EF3 for any $z \in \mathbb{D}$ and $T > 0$ there exists $k_{z,T} \in L^d([0, T], \mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k_{z,T}(\xi) d\xi, \quad 0 \le s \le u \le t \le T. \tag{18}$$

Example

An example

Let $(\phi_t) \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ be a one-parameter semigroup.
Then $(\varphi_{s,t})$ defined by the formula

$$\varphi_{s,t} = \phi_{t-s}, \quad 0 \le s \le t < +\infty, \tag{19}$$

is an evolution family of order $d = +\infty$.

Thus the notion of an evolution family generalizes that of one-parameter semigroup.

Definitions — continued

The notion of a Herglotz vector field can be introduced in the following way.

Definition — Herglotz vector field

A Herglotz vector field of order d is a function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ of the form

$$G(z, t) = \big(\tau(t) - z\big)\big(1 - \overline{\tau(t)}z\big)p(z, t), \tag{20}$$

where

(i) $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ is measurable;

(ii) $p(\cdot, t)$ is holomorphic in $\mathbb{D}$ with $\operatorname{Re} p(\cdot, t) \geq 0$ for any $t \geq 0$;

(iii) $p(z, \cdot)$ is measurable on $[0, +\infty)$ for any $z \in \mathbb{D}$;

(iv) $p(0, \cdot)$ is of class L^d_{loc} on $[0, +\infty)$.

Herglotz vect. fields → Evolution Families

Theorem C.1 (Bracci, Contreras, Díaz-Madrigal 2008)

Let G be a Herglotz vector field of order d.
Then for any $z \in \mathbb{D}$ and $s \geq 0$ the IVP

$$\frac{dw(t)}{dt} = G\big(w(t), t\big), \quad t \geq s, \quad w(s) = z, \tag{21}$$

has a unique solution $w_{z,s} : [s, +\infty) \to \mathbb{D}$. Moreover, the formula

$$\varphi_{s,t}(z) := w_{z,s}(t), \quad z \in \mathbb{D}, \ \ 0 \leq s \leq t < +\infty, \tag{22}$$

defines an evolution family $(\varphi_{s,t})$ of the same order d.

Evolution Families → Herglotz vect. fields

The converse theorem also holds.

Theorem C.2 (Bracci, Contreras, Díaz-Madrigal 2008)

For any evolution family $(\varphi_{s,t})$ of order d there exists an essentially unique Herglotz vector field G of the same order d such that for any $z \in \mathbb{D}$ and $s \geq 0$ the function $w_{z,s} : [s, +\infty) \to \mathbb{D}$ defined by

$$w_{z,s}(t) := \varphi_{s,t}(z), \quad z \in \mathbb{D}, \;\; 0 \leq s \leq t < +\infty, \tag{23}$$

is the unique solution to IVP

$$\frac{dw(t)}{dt} = G\big(w(t), t\big), \quad t \geq s, \quad w(s) = z. \tag{21}$$

Theorems C.1 and C.2 establish a 1-to-1 correspondence between evolution families and Herglotz vector fields.

Loewner Chains

In 2010 M. D. Contreras, S. Díaz-Madrigal, and P.G. (Revista Matemática Iberoamericana **26** (2010), 975–1012) introduced

Definition — Loewner chains

A family $(f_t)_{0\leq t<+\infty}$ of holomorphic function in $\mathbb{D}$ is called a *(generalized) Loewner chain* of order d if

LC1 each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,

LC2 $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s < t < +\infty$,

LC3 for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that

$$\sup_{z\in K} |f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi)d\xi, \quad 0 \leq s \leq t \leq T. \qquad (24)$$

Remark: classical Loewner chains satisfy this definition with $d = +\infty$.

Loewner chains → evolution families

Recall that every classical Loewner chain (f_t) satisfies the Loewner – Kufarev PDE and that the corresponding Loewner – Kufarev ODE generates a family $(\varphi_{s,t}) \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ related to the Loewner chain (f_t) by the formula

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \quad 0 \leq s \leq t < +\infty. \tag{25}$$

Similar statement holds for generalized Loewner chains.

Theorem C.3 (Contreras, Díaz-Madrigal, P.G. 2010)

- For any generalized Loewner chain (f_t) of order d formula (25) defines an evolution family $(\varphi_{s,t})$ of the same order d.
- Moreover, if G is the Herglotz vector field of $(\varphi_{s,t})$, then (f_t) satisfies the generalized Loewner PDE

$$\frac{\partial f_t(z)}{\partial z} = -\frac{\partial f_t(z)}{\partial z} G(z,t), \quad t \geq 0. \tag{26}$$

Evolution Families → Loewner chains

The converse Theorem is also true. A generalized Loewner chain (f_t) is said to be *associated with* an evolution family $(\varphi_{s,t})$ if

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \quad 0 \le s \le t < +\infty. \tag{25}$$

Theorem C.4 (Contreras, Díaz-Madrigal, P.G. 2010)

- For any evolution family $(\varphi_{s,t})$ of order d there exists a generalized Loewner chain (f_t) of order d associated with $(\varphi_{s,t})$.
- If $F : \Omega \to \mathbb{C}$, $\Omega := \cup_{t\ge 0} f_t(\mathbb{D})$, is a holomorphic univalent function, then $g_t := F \circ f_t$, $t \ge 0$, form another generalized Loewner chain (g_t) of order d associated with $(\varphi_{s,t})$.
- Conversely, if (g_t) is any generalized Loewner chain associated with $(\varphi_{s,t})$, then there exists a univalent holomorphic function $F : \Omega \to \mathbb{C}$ such that $g_t = F \circ f_t$ for all $t \ge 0$.

The essence of the new approach

According to the new approach by Bracci – Contreras – Díaz-Madrigal the essence of the Loewner Theory resides in the interplay between the three basic notions:

- (generalized) Loewner Chains
- Evolution Families
- Herglotz vector fields

The results presented above were generalized to complex manifolds:

- F. Bracci, M.D. Contreras, and S. Díaz-Madrigal, *Evolution Families and the Loewner Equation II: complex hyperbolic manifolds*, Math. Ann. **344** (2009), 947–962.
- L. Arosio, F. Bracci, H. Hamada, G. Kohr, *Loewner's theory on complex manifolds*, to appear in J. Anal. Math.; ArXiv:1002.4262

Interplay between the three notions

$$\varphi_{s,t} = f_t^{-1} \circ f_s$$

Loewner Chains (f_t) ⟷ **Evolution Families** $(\varphi_{s,t})$

Generalized Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -G(z,t)\frac{\partial f_t(z)}{\partial z},$$

$$t \geq 0.$$

Generalized Loewner ODE

$$\frac{dw}{dt} = G(w,t),$$

$$w = w(t; s, z),\ t \geq s,$$

$$w|_{t=s} = z.$$

Herglotz Vector Fields $G(w,t)$

Boundary behaviour — some definitions

Let us recall some basic notions.
Let $F : \mathbb{D} \to \mathbb{C}$ be a holomorphic function and $\sigma \in \mathbb{T} := \partial\mathbb{D}$.

- The point $a \in \overline{\mathbb{C}}$ is said to be the *angular limit of F at σ* if

$$F(z) \to a \quad \text{as} \quad S \ni z \to \sigma$$

  for any Stolz angle S with vertex at σ.
- Assume that the angular limit a of F exists and finite. Then the angular limit of

$$F_1(z) := \frac{F(z) - a}{z - \sigma} \tag{27}$$

  at σ is called, if it exists, the *angular derivative* of F at σ.

In what follows we will denote the angular limit and angular derivative by $\angle F(\sigma)$ and $\angle F'(\sigma)$, respectively.

Some definitions

Definition

Let $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$. A point $\sigma \in \mathbb{T}$ is said to be a *boundary fixed point* (BFP) if $\angle\varphi(\sigma)$ exists and coincides with σ.

It is known that if σ is a BFP of $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$, then $\angle\varphi'(\sigma)$ exists and belong to $(0, +\infty) \cup \{\infty\}$.

Definition

A boundary fixed point σ of $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ is said to be *regular* (RBFP) if $\angle\varphi'(\sigma) \neq \infty$.

DW-point

Let $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\text{id}_{\mathbb{D}}\}$. It is known that there exists a unique point $\tau \in \overline{\mathbb{D}}$ that is a (boundary) fixed point of φ with $|(\angle)\varphi'(\tau)| \leq 1$. This point is the *Denjoy – Wolff* point of φ.

Common DW-point

Theorem D.1 (Bracci, Contreras, Díaz-Madrigal 2008)

All elements of an evolution family $(\varphi_{s,t})$ different from $\mathrm{id}_{\mathbb{D}}$ share the same DW-point $\tau_0 \in \overline{\mathbb{D}}$ if and only if in the Berkson – Porta type representation

$$G(z,t) = \big(\tau(t) - z\big)\big(1 - \overline{\tau(t)}z\big)p(z,t)$$

for the Herglotz vector field G of $(\varphi_{s,t})$,

$$\tau(t) = \tau_0 \quad \text{for a.e. } t \geq 0.$$

For $\tau_0 \in \mathbb{D}$, a simple calculation yields

$$\varphi'_{s,t}(\tau_0) = \exp \int_s^t G'(\tau_0, t')\, dt'. \tag{28}$$

Common DW-point — continued

Similar relation holds for the boundary DW-point.

Theorem D.2 (Bracci, Contreras, Díaz-Madrigal 2008)

Let $(\varphi_{s,t})$ be an evolution family of order d and G its Herglotz vector field. Suppose that all elements of $(\varphi_{s,t})$ different from $\mathrm{id}_{\mathbb{D}}$ share the same DW-point $\tau_0 \in \mathbb{T}$. Then

(i) for a.e. $t \geq 0$,

$$\exists \angle G(\tau_0, t) = 0, \quad \exists \angle G'(\tau_0, t) =: \lambda(t) \in (-\infty, 0];$$

(ii) the function λ is of class L^d_{loc} on $[0, +\infty)$;

(iii) $\angle \varphi'_{s,t}(\tau_0) = \exp \int_s^t \lambda(t')\, dt'$, whenever $0 \leq s \leq t < +\infty$.

The case of 1-param. semigroups

The unit disk can contain at most one fixed point. If exists, it is the DW-point. However, on the unit circle $\mathbb{T}$ there can be even infinitely many RBFPs. *Does there exists an analogue of Theorem D.2 for common RBFPs of evolution families?*
The answer "YES" was known before for one-parameter semigroups.

Theorem D.3 (Contreras, Díaz-Madrigal, Pommerenke 2006)

Let $(\phi_t) \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ be a 1-parameter semigroup, G its infinitesimal generator, and $\sigma \in \mathbb{T}$. Then the following conditions are "$\Longleftrightarrow$":

(A) σ is a RBFP of (ϕ_t);

(B) $\exists\angle G(\sigma) = 0$ and $\exists\angle G'(\sigma) \neq \infty$ (i.e. G has a RBNP at σ).

Moreover, if (A) or (B) holds, then

(C) $\angle G'(\sigma) \in \mathbb{R}$;

(D) $\angle\phi_t'(\sigma) = \exp\big(t\angle G'(\sigma)\big)$.

Regular boundary null-points

We are able to generalize Theorem D.3 to the non-autonomous case.

Definition

A point $\sigma \in \mathbb{T}$ is a *regular boundary null-point* (RBNP) of an infinit. generator G if condition (B) holds, i.e. if $\exists\angle G(\sigma) = 0$ and $\exists\angle G'(\sigma) \neq \infty$.

Theorem D.4 (Bracci, Contreras, Díaz-Madrigal 2008)

A holomorphic function $G : \mathbb{D} \to \mathbb{C}$ is an infinitesimal generator with a RBNP $\sigma \in \mathbb{T}$ if and only if it admits the following representation

$$G(z) = (\sigma - z)(1 - \overline{\sigma}z)\Big(p(z) - \lambda p_0(\overline{\sigma}z)\Big), \quad z \in \mathbb{D}, \tag{29}$$

where $p_0(z) := (1 + z)/(1 - z)$, $\lambda := \angle G'(\sigma) \in \mathbb{R}$, and $p : \mathbb{D} \to \mathbb{C}$ is a holomorphic function such that

$$\forall z \in \mathbb{D}\ \mathrm{Re}\, p(z) \geq 0 \quad \text{and} \quad \angle \lim_{z\to\sigma}(z - \sigma)p(z) = 0. \tag{30}$$

RBFP $\longleftrightarrow$ RBNP $+ L^1_{\mathrm{loc}}$

Theorem D.5 (Bracci, Contreras, Díaz-Madrigal, P.G. 2012)

Let $(\varphi_{s,t})$ be an evolution family, G its Herglotz vector field and $\sigma \in \mathbb{T}$. Then the following two assertions are "$\Longleftrightarrow$":

(A) σ is a RBFP of $\varphi_{s,t}$ for each $s \ge 0$ and $t \ge s$;

(B) the following two conditions hold:

  (B.1) for a.e. $t \ge 0$, $G(\cdot, t)$ has a RBNP at σ;

  (B.2) $\lambda(t) := \angle G'(\sigma, t)$ is of class L^1_{loc} on $[0, +\infty)$.

Moreover, if (A) or (B) holds, then

$$\angle \varphi'_{s,t}(\sigma_0) = \exp \int_s^t \lambda(t')\, dt' \quad \text{whenever } 0 \le s \le t < +\infty. \tag{31}$$

Remarks

- Asymmetry in Theorem D.5:

σ is a RBFP of all $\varphi_{s,t}$'s $\Rightarrow$ $\angle\varphi'_{s,t}(\sigma)$ is loc. abs-ly continuous in s and t

σ is a RBNP of $G(\cdot, t)$ for a.e. $t \geq 0$ $\not\Rightarrow$ $t \mapsto \angle G'(\sigma, t)$ is loc. integrable

- Comparison with Theorem D.2:

if σ is the DW-point of every $\varphi_{s,t}$, then $t \mapsto \angle G'(\sigma, t)$ is of class L^d_{loc}, while for the common RBFP σ, we only have L^1_{loc}.

Remarks — continued

- If only condition (B.1) holds, then $(\varphi_{s,t})$ does not need to have common BFP at σ (even *non-regular* one).
- Assume that $\varphi_{s,t}$'s share common BFP (not necessarily regular) at $\sigma \in \mathbb{T}$. What can be said about G? Open question even for one parameter semigroups?

Assume that a one-parameter semigroup (ϕ_t) has
a boundary fixed point σ.
Does this imply that
the infinitesimal generator G of (ϕ_t) satisfies $\exists\angle G(\sigma) = 0$?

One element of the proof

Let $(\psi_t)_{t\geq 0} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ be a one-parameter family (not necessary a semigroup!) with $\psi_0 = \mathrm{id}_{\mathbb{D}}$.
Suppose it is differentiable at $t = 0$ in the following sense:

$$\exists \lim_{t\to+0} \frac{\psi_t(z) - z}{t} =: G(z) \in \mathbb{C} \quad \text{for any } z \in \mathbb{D}. \tag{32}$$

It is known that in this case G is an infinitesimal generator.

Theorem (S. Reich, D. Shoikhet 1998)

Under the above conditions the one-parameter semigroup (ϕ_t) generated by G is given by

$$\phi_t = \lim_{n\to+\infty} \left(\psi_{t/n}\right)^{\circ n} \quad \text{for any } t \geq 0. \tag{33}$$

Last frame!!!

THANK YOU!!!