Chapter 5

Potential flow: Surface waves on a liquid

In this chapter we will illustrate the use of potential theory with an example of flow with a *free surface*. For this type of flow the boundary condition at the surface is different from what one usually encounters in the traditional fluid dynamics courses.

5.1 Potential theory

A continuous and irrotational flow of an incompressible fluid is guided by the Laplace equation (**??**), one of the most well-studied in all of mathematics:

$$\nabla^2\Phi = 0$$

For a fluid flow it is most often solved with the Neumannn boundary condition (**??**), which is

$$\nabla_\perp\Phi = 0 \tag{5.1}$$

while the boundary condition in electrostatics usually will be of the Dirichlet type (see for instance [Papatzacos 2003]):

$$\Phi = \text{konstant} \tag{5.2}$$

The linearity of the Laplace equation may be employed to find solutions for quite complicated instances of flow by superposition of more simple solutions. In elementary textbooks like [Gerhart et al. 1992] it is for example shown how the velocity potential for flow past a cylinder (see Problem 3.1) emerges by superposition of uniform flow and a source-sink doublet, in the limit where the distance between the source and the sink approaches zero. In a later chapter we will return to a variant of that example, where a further superposition of the velocity field (**??**) for a line vortex allows us to calculate the lift on a rotating cylinder in a vorticityless fluid flow.

Notice that the Laplace equation is a *kinematical* condition imposed on the flow. It is not based on the Euler equation (eventually on an integrated form like the Bernoulli equation). Subsequent to the solution of the Laplace equation, the resulting velocity field may be inserted into the equation of motion, which then produces the pressure distribution.

Analytical solutions for fluid flow in reservoirs is to a large extent based on the Laplace equation with Neumann and/or Dirichlet boundary conditions imposed. We will therefore

Figure 5.1: Wave propagating in the x direction, with surface deviation in the z direction

assume that the most usual methods to find solutions of the equations are known, for instance from courses in reservoir technology or in mathematical modeling. In the following we will instead treat a case of flow with a free surface, where in fact the *energy equation* is used to find one of the boundary conditions.

5.2 Combined gravity and capillary waves

A liquid surface deviating from plane horizontal equilibrium will try to smoothen itself by two mechanisms:

- The force of gravity
- The surface tension

Inertia and conservation of energy imply that such surface disturbances may propagate as waves. We will study waves due to a combination of the two effects.

5.2.1 Assumptions and simplifications

We introduce a Cartesian coordinate system with the z axis pointing upwards and the xy plane in the liquid surface's equilibrium plane; see Figure 5.1. With a liquid[1] depth h ($h > 0$ by definition), the bottom corresponds to $z = -h$. The surface's local vertical deviation

[1]Think of it as water, the most readily available liquid medium!

from equilibrium is denoted by ζ. We will study 2D transversal surface waves propagating in the positive x direction, with surface amplitude a, wavelength λ and period τ. Appendix E provides a brief summary of the formalism for the case of plane waves. The pressure in the liquid in an arbitrary point below the surface is $p = p(z, x)$. The gas pressure infinitesimally above the surface is p_0 = constant. We will use the following simplifying assumptions:

- Nonviscous flow, negligible $\mu\nabla^2\boldsymbol{u}$ term

  (The assumption will not prevent us in calculating the dissipation rate perturbatively, based on the solution for a nonviscous flow)

- Linearized Euler equation, $|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| \ll |\partial_t\boldsymbol{u}|$

- Wave amplitude much less than the water depth:

$$a \ll h \tag{5.3}$$

- No interaction with the gas above the liquid, apart from the constant gas pressure at the surface

The linearization assumption gives, with typical scales introduced:

$$\frac{1}{\lambda}(\frac{a}{\tau})^2 \ll \frac{a}{\tau}\frac{1}{\tau}$$

or [2]

$$a \ll \lambda \tag{5.4}$$

Explicitly, the waves' surface amplitude must then be much less than their wavelength. A corollary: Even if the surface will be locally sloping when a wave is present, we may replace $\frac{\partial}{\partial n}$ by $\frac{\partial}{\partial z}$, where n denotes the local surface unit normal. If furthermore we take the curl on both sides of the Euler equation (**??**), with $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = 0$, that gives

$$\partial_t \operatorname{curl}\boldsymbol{u} = 0 \qquad \Rightarrow \qquad \operatorname{curl}\boldsymbol{u} = \text{constant}$$

However, since the time average of $\boldsymbol{u}$ for a harmonically oscillating motion equals zero, it follows that $\operatorname{curl}\boldsymbol{u} = 0$. The flow in a liquid due to oscillatory motion with a small amplitude can thus be considered a *potential flow* to lowest order, and we may introduce the velocity potential Φ from Eq. (**??**)

5.2.2 Equations and solution

The solution for the fluid flow must satisfy the Laplace equation (**??**), with impermeability as the bottom boundary condition:

$$u_z = -\frac{\partial\Phi}{\partial z} = 0 \qquad (z = -h) \tag{5.5}$$

At the surface the timedependent Bernoulli equation will provide a boundary condition. In Eq. (**??**) we neglect the quadratic term in $\boldsymbol{u}$ and make the replacement $f(t) = p_0/\rho g$:

$$p - p_0 = -\rho g z + \rho\partial_t\Phi \tag{5.6}$$

[2]Waves where the surface amplitude is much smaller than both the wavelength and the liquid depth, are called *Airy waves*.

We now apply this equation to the liquid infinitesimally below the surface, with $z = \zeta$. For a nonnegligible surface tension α, the difference $p - p_0$ will equal the *capillary pressure*. Using Appendix F we may express the capillary pressure by α and the surface curvature. Our assumptions imply no y dependence, so we get:

$$\rho g \zeta - \rho(\partial_t \Phi)_{z=\zeta} - \alpha \frac{\partial^2 \zeta}{\partial x^2} = 0 \tag{5.7}$$

Applying ∂_t to the equation we notice that the vertical velocity component at the surface can be expressed in two ways:

$$(u_z)_{z=\zeta} = \frac{\partial \zeta}{\partial t} = -(\frac{\partial \Phi}{\partial z})_{z=\zeta} \tag{5.8}$$

After permutation of the order of derivations in the surface tension term, and approximating $z = \zeta$ by $z = 0$ since we presuppose that the amplitude of the surface oscillations is small, we get the surface boundary condition:

$$\{\frac{\partial \Phi}{\partial z} + \frac{1}{g}\frac{\partial^2 \Phi}{\partial t^2} - \frac{\alpha}{\rho g}\frac{\partial}{\partial z}\frac{\partial^2 \Phi}{\partial x^2}\}_{z=0} = 0 \tag{5.9}$$

We consider a monochromatic plane-wave solution of the Laplace equation, with an amplitude depending on the distance from the surface:

$$\Phi = f(z)\cos(kx - \omega t) \tag{5.10}$$

Insertion gives

$$\frac{\partial^2 f}{\partial z^2} - k^2 f = 0 \tag{5.11}$$

which has a general solution[3]

$$f(z) = Ae^{kz} + Be^{-kz} \tag{5.12}$$

The integration constants A and B are determined by the boundary condition (5.5) and introduction of another constants C which determines the amplitude:

$$-kAe^{-kh} + kBe^{kh} = 0 \quad , \quad C = 2Ae^{-kh} = 2Be^{kh} \tag{5.13}$$

It follows that

$$f(z) = \frac{1}{2}C\{e^{k(z+h)} + e^{-k(z+h)}\} \tag{5.14}$$

and

$$\Phi = C\cosh(k(z+h))\cos(kx - \omega t) \tag{5.15}$$

The dispersion relation $\omega = \omega(k)$ and the phase velocity $V = \omega/k$ (see Appendix E) is obtained by inserting the solution into Eq. (5.9):

$$k \sinh kh - \frac{\omega^2}{g}\cosh kh + \frac{\alpha k^3}{\rho g}\sinh kh = 0 \tag{5.16}$$

[3]For the limiting case of a "deep" fluid one must choose $B = 0$, siden e^{-kz} diverges exponentially ($z < 0$). Later on we will consider this case for the final solution. Here we only notice that in the deep-fluid limit one gets a simpler derivation of the results, which corresponds to replacing $\cosh(k(z+h))$ by e^{kz} and $\tanh kh$ by 1 in the final expressions. In the problems section we will only consider this limit.

We then obtain the dispersion relation

$$\omega^2 = (gk + \frac{\alpha k^3}{\rho}) \tanh kh \tag{5.17}$$

as well as a *quadratic* superposition formula for the phase velocities V_g for pure *gravity waves* ($\alpha = 0$) V_α for pure *capillary waves* ($\alpha \gg \rho g/k^2$):

$$\begin{aligned} V^2 &= V_g^2 + V_\alpha^2 && (5.18) \\ V_g^2 &= \frac{g}{k} \tanh kh && (5.19) \\ V_\alpha^2 &= \frac{\alpha k}{\rho} \tanh kh && (5.20) \end{aligned}$$

Later on we will see that if a wave is a *superposition* of plane waves with different wavelengths instead of being monochromatic, the *group velocity* will be a more meningful velocity measure than the phase velocity.

Two limiting cases: Deep vs. shallow liquid

We will consider the solutions for the two limiting cases $\lambda \ll h$ (deep liquid) and $\lambda \gg h$ (shallow liquid). The hyperbolic tangent factor has different approximations in the two limits, see [Rottmann 1995]:

$$\begin{aligned} \tanh kh &= 1 - 2e^{-2kh} + \ldots && (kh \gg 1, \text{ deep liquid}) \\ \tanh kh &= kh\,(1 - \frac{1}{3}(kh)^2 + \ldots) && (kh \ll 1, \text{ shallow liquid}) \end{aligned}$$

To the lowest order the phase velocities reduce to

$$\begin{aligned} V^2 &= \frac{g\lambda}{2\pi}(1 + \frac{4\pi^2\alpha}{\rho g \lambda^2}) && \text{(deep liquid)} && (5.21) \\ V^2 &= gh(1 + \frac{4\pi^2\alpha}{\rho g \lambda^2}) && \text{(shallow liquid)} && (5.22) \end{aligned}$$

In a deep liquid, long-wavelength gravity waves will propagate faster than short-wavelength ones.[4] In the shallow-liquid limit the phase velocity will approach a constant. Figure 5.2 illustrates this for the case of pure gravity waves.

Based on the result for a shallow liquid we may get a qualitative understanding of the phenomenon of *surf formation* for ocean waves moving over a sloping bottom in the vicinity of the shore; se Figure 5.3. Since $h_T > h_B$ we have $V_T > V_B$: Small disturbances on a wave crest will propagate faster than those in the wave trough.[5]

In the limit of a deep liquid, for a constant z, the cosh factor in Eq. (5.15) can be approximated by $\frac{1}{2}e^{k(z+h)}$. Including the factor $\frac{1}{2}e^{kh}$ into the amplitude factor C, we get the solution for the velocity potential in this limit:

$$\Phi = Ce^{kz} \cos(kx - \omega t) \qquad \text{(deep liquid)} \tag{5.23}$$

[4]This is analogous to the *normal dispersion* phenomenon in optics.

[5]This is derived from the theory for small amplitudes, and it is questionable how legitimate the conclusion is for finite-amplitude waves.

Figure 5.2: Phase velocity as a function of wavelength for a given depth, for $\alpha = 0$

Figure 5.3: Surf formation

5.2.3 The minimum of the phase velocity

As a reminder, in the case of a deep fluid there is the relation

$$V^2 = \frac{g}{k} + \frac{\alpha k}{\rho}$$

For long-wavelength waves ($k^2 \ll \rho g/\alpha$) where obviously we may disregard the capillary force, the dispersion relation has a form different from that at short wavelengths ($k^2 \gg \rho g/\alpha$), however V is seen to grow in both cases when the limit is approached. At some value $\lambda = \lambda_{min}$ of the wavelength, there must then be a *minimal* value $V = V_{min}$ of the phase velocity[6], see Figure 5.4. By derivation:

$$2V\frac{dV}{d\lambda} = \frac{dk}{d\lambda}\frac{d}{dk}(\frac{g}{k} + \frac{\alpha k}{\rho}) = 0 \qquad \Rightarrow \qquad \frac{g}{k_{min}^2} = \frac{\alpha}{\rho}$$

The properties of this minimum of V are:

$$\lambda_{min} = 2\pi\sqrt{\frac{\alpha}{\rho g}} \tag{5.24}$$

$$V_{min}^2 = 2\sqrt{\frac{\alpha g}{\rho}} \tag{5.25}$$

[6]For $\lambda < \lambda_{min}$ the waves are said to have *anomalous dispersion*: The phase velocity *increases* when the wavelength *decreases*.

Figure 5.4: $V = V(\lambda)$ for $\lambda \ll h$

Waves with a different λ, both shorter and longer, will propagate with a velocity larger than given by Eq. (5.25). The minimum velocity corresponds to $V_g = V_\alpha = \frac{1}{2}V$.

For a water surface against air:

$$\begin{aligned} \alpha &= 0.0725\,\mathrm{J\,m^{-2}} \qquad (20\,^{\circ}\mathrm{C}) \\ \rho &\approx 10^3\,\mathrm{kg\,m^{-3}} \\ g &\approx 9.81\,\mathrm{m\,s^{-2}} \end{aligned}$$

This gives:

$$\lambda_{min} = 1.71\,\mathrm{cm} \tag{5.26}$$

$$V_{min} = 23.1\,\mathrm{cm\,s^{-1}} \tag{5.27}$$

One will often observe large-wavelength gravity waves in water with short-wavelength ripples riding on them. The latter are capillary waves.[7] The slowly moving ripples one may observe at sea from a small boat, will usually have wave lengths of order 1 cm.[8]

5.2.4 The motion of the liquid particles

The components of the local flow velocity can now be found from (5.15) and (**??**):

$$u_x = kC\cosh(k(z+h))\sin(kx-\omega t) \tag{5.28}$$

$$u_z = -kC\sinh(k(z+h))\cos(kx-\omega t) \tag{5.29}$$

These expressions can be integrated to find the deviation of the fluid particles from their equilibrium positions. We denote the instant positions of the moving particles by x and z, with x_0 and z_0 indicating their equilibrium positions. For small-amplitude waves we may approximate the instantaneous positions in the arguments of the hyperbolic functions by their equilibrium values. Integration then gives

$$x - x_0 = \frac{kC}{\omega}\cosh(k(z_0+h))\cos(kx_0-\omega t)$$

[7] The most exact way of determining α experimentally, involves measurement of the wavelength of standing capillary waves excited by a tuning fork or an oscillator, according to [Sommerfeld 1964].

[8] The author of these lecture notes believes he observed such waves himself at an age when he knew nothing about fluid dynamics.

Figure 5.5: Theoretical particle paths for particles in *a* moderately deep, *b* deep and *c* shallow liquid

$$= \tilde{a}\cos(kx_0 - \omega t) \tag{5.30}$$

$$\begin{aligned} z - z_0 &= \frac{kC}{\omega}\sinh(k(z_0 + h))\sin(kx_0 - \omega t) \\ &= \tilde{b}\sin(kx_0 - \omega t) \end{aligned} \tag{5.31}$$

This means that, to the accuracy implied in the assumptions, the particles in the liquid will move in *elliptical* orbits during the wave's period:

$$(\frac{x - x_0}{\tilde{a}})^2 + (\frac{z - z_0}{\tilde{b}})^2 = 1 \tag{5.32}$$

Explicitly, the surface wave motion has both transversal and longitudinal components. The ratio of the ellipse's principal axes has the two limiting values

$$\lim_{h\to\infty} \frac{\tilde{b}}{\tilde{a}} = 1 \qquad \text{(deep liquid)} \tag{5.33}$$

$$\lim_{h\to 0} \frac{\tilde{b}}{\tilde{a}} = 0 \qquad \text{(shallow liquid)} \tag{5.34}$$

For $h \to \infty$ the particle paths will be circles at a given λ, with radii decreasing exponentially with the distance from the surface. In more shallow water the paths become ellipses, the more elongated the closer to the bottom one comes, and in the limit $h \to 0$ the waves become purely *longitudinal*. See Figure 5.5.

The expressions for the instantaneous streamlines can be found from the stream function. From Eqs. (**??**) and (**??**) we get

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \Phi}{\partial z} \quad , \quad \frac{\partial \Psi}{\partial z} = -\frac{\partial \Phi}{\partial x}$$

which gives

$$\frac{\partial \Psi}{\partial x} = kC\sinh(k(z + h))\cos(kx - \omega t) \tag{5.35}$$

$$\frac{\partial \Psi}{\partial z} = kC\cosh(k(z + h))\sin(kx - \omega t) \tag{5.36}$$

Figure 5.6: Theoretical streamline pattern at $t = 0$

Figure 5.7: Photo of streamlines (short exposure)

Integrated:

$$\Psi = C \sinh(k(z+h)) \sin(kx - \omega t) \tag{5.37}$$

On a streamline, $\Psi =$ constant. At $t = 0$:

$$\Psi = C \sinh(k(z+h)) \sin kx$$

At the surface, according to (5.31), at $t = 0$:

$$\zeta \propto \sin kx$$

The bottom is a streamline: $\Psi = 0$ for $z = -h$. We also have $\Psi = 0$ for $kx = 0, \pm\pi, \pm 2\pi, \ldots$, and $\frac{\partial \Psi}{\partial x} = 0$ for $kx = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \ldots$. This indicates a streamline pattern as in Figure 5.6. The predictions for particle paths and streamlines have been confirmed experimentally. See Figures 5.7 and 5.8, which are photos of a particle suspension in a liquid. From the latter figure one may also observe from the initial and final points on the various curves how the *phase* of the waves propagate.

Figure 5.8: Photo of path curves (long exposure)

5.3 The group velocity of surface waves

In Appendix E we have given an argument that the group velocity

$$U = \frac{d\omega}{dk}$$

is a measure of the propagation velocity of the *energy* or the *information* in a wave. For the case of superposed frequencies it is more meaningful than the phase velocity. Since waves with a finite spatial extent may be expressed as a Fourier integral or sum over monochromatic waves with different frequencies, each of which fills the whole space from $-\infty$ to ∞, the group velocity will always be the quantity of practical interest.

The relation (**??**)

$$U = V - \lambda \frac{dV}{d\lambda}$$

shows that except if $\omega \propto k$, in general $U \neq V$ also for monochromatic plane waves, if V depends on λ. For the various special cases[9] of surface waves in a liquid we get:

- Pure gravity wave, deep liquid:

$$V = \sqrt{\frac{g\lambda}{2\pi}} \quad , \quad \frac{dV}{d\lambda} = \frac{1}{2\lambda} V \quad \Rightarrow \quad U = \frac{1}{2} V \tag{5.38}$$

- Pure gravity wave, shallow liquid:

$$V = \sqrt{gh} \quad , \quad \frac{dV}{d\lambda} = 0 \quad \Rightarrow \quad U = V \tag{5.39}$$

[9]The first two cases will be useful for us later on.

- Pure capillary wave, deep liquid:

$$V = \sqrt{\frac{2\pi\alpha}{\rho\lambda}} \quad , \quad \frac{dV}{d\lambda} = -\frac{1}{2\lambda}V \quad \Rightarrow \quad U = \frac{3}{2}V$$

- Capillary wave, shallow liquid:

$$V = 2\pi\sqrt{\frac{h\alpha}{\rho\lambda^2}} \quad , \quad \frac{dV}{d\lambda} = -\frac{1}{\lambda}V \quad \Rightarrow \quad U = 2V$$

Also the general case follows from Eqs. (**??**) and (5.17):

$$\begin{aligned} U &= V + k\frac{dV}{dk} \\ &= V(1 + \frac{k}{2V^2}\frac{d(V^2)}{dk}) \\ &= V\{1 + \frac{k}{2(\frac{g}{k} + \frac{\alpha k}{\rho})\tanh kh}[\frac{1}{k}(-\frac{g}{k} + \frac{\alpha k}{\rho})\tanh kh + (\frac{g}{k} + \frac{\alpha k}{\rho})\frac{h}{\cosh^2 kh}]\} \\ &= \frac{1}{2}V\{\frac{3\alpha k^2 + \rho g}{\alpha k^2 + \rho g} + \frac{2kh}{\sinh 2kh}\} \end{aligned} \tag{5.40}$$

The term dispersion means spreading. We have seen that if V depends on k, different wavelengths imply different phase velocities. A localized wave which is a superposition of Fourier components with dispersion, will thus *change shape* with time, because each component will move with its own velocity. Thus, dispersion has a quite

We will now show explicitly that for a monochromatic plane surface wave on a liquid, the energy propagation velocity equals the group velocity. Let us consider a control volume, a "box" with length λ in the wave's direction of propagation and horizontal width b perpendicular to that direction. The vertical "walls" reach from the surface to the bottom. The wave energy arrives and leaves through the two end walls with distance λ and width b. Denoting

- S energy flowing through an end wall per period
- E time average of the total energy in the box
- U_E the propagation velocity of the energy

we get

$$\frac{S}{\tau} = \frac{E}{bh\lambda}U_E\, bh \tag{5.41}$$

and thus

$$U_E = \frac{S}{E}\frac{\lambda}{\tau} = \frac{S}{E}V \tag{5.42}$$

In Problem 5.1 you will be asked to show that in the case $h \to \infty$, S and E are given by[10]

$$E = \frac{1}{2}a^2\lambda(\rho g + \alpha k^2) \tag{5.43}$$

$$S = \frac{1}{4}a^2\lambda(\rho g + \alpha k^2) + \pi\alpha a^2 k \tag{5.44}$$

[10] In the finite h case, the expression for E would be unchanged, while the λ term in S would be multiplied by a factor $(1 + \frac{2kh}{\sinh 2kh})$. See [Massey 1983].

where

$$a = \frac{kC}{\omega} e^{kh}$$

Inserting into Eq. likuev:se we get

$$U_E = \frac{1}{2} V \frac{3\alpha k^2 + \rho g}{\alpha k^2 + \rho g} \tag{5.45}$$

For monochromatic plane combined gravity and capillary waves in a deep liquid [11] (where $\sinh 2kh \gg 2kh$), we have thus shown that

$$U_E = U \tag{5.46}$$

A wave with anomalous dispersion ($U > V$) contains less energy per wavelength in the propagation direction than what passes through the end walls per period. To keep such a wave moving thus requires more energy than what travels with the wave. Energy transport in a medium with normal dispersion is easier in a medium with normal dispersion than in one with anomalous dispersion.

In this section we have used waves in a liquid to illustrate a relation which has a much wider validity in physics, for instance in electrodynamics and optics.[12] Only with energy absorption present the relation $U_E = U$ might be broken.

5.4 Dissipation of surface waves

If the viscous dissipation of energy in a wave for $a \ll \lambda$ is to be calculated, the most accurate result would emerge if a wave solution of Eq. (**??**) were found with the convective acceleration term neglected but the viscous term kept. However, finding such a solution (which might be done readily enough, since the equation would still be linear) one would renounce the simplicity of the expressions found in this chapter. We will therefore study the condition for our expressions to give acceptable results when the dissipation is to be calculated.

Consider first the quality of the potential flow approximation itself. Take the curl of the linearized Navier-Stokes equation, and what you get is a diffusion equation, a linearized version of the vorticity equation (**??**):

$$\partial_t \operatorname{curl} \boldsymbol{u} = \frac{\mu}{\rho} \nabla^2 \operatorname{curl} \boldsymbol{u} \tag{5.47}$$

If a periodically varying (in time) boundary condition at the surface is imposed, with frequency ω, one finds an attenuation factor which is exponential in the distance from the surface. [13] Over a distance equal to the *attenuation depth* δ, where

$$\delta = \sqrt{\frac{2\mu}{\rho\omega}} \tag{5.48}$$

[11] The result also holds for finite h, as we would obtain by employing the result stated in the preceeding footnote.

[12] Massey's otherwise excellent textbook [Massey 1983] claims, in fact, that the relation $U/V = S/E$ only holds in the case without surface tension! Lord Rayleigh, Reynolds, *et al.* might have rotated in their graves: Massey forgets to include the flow of *surface energy* when he calculates S. A breakdown of the relation (in a medium without energy absorption) would really be a counterexample with far-reaching consequences for physics, quite some "change of paradigm".

[13] Convince yourself that $\operatorname{curl} \boldsymbol{u} = K\hat{\mathbf{e}}_y e^{z/\delta} \cos(z/\delta + \omega t)$ is a solution, an attenuated wave propagating in the direction of negative z, with the correct tupe of boundary condition.

the amplitude of curl $\boldsymbol{u}$ will be reduced by a factor $\frac{1}{e}$.

Specialize now to waves in a deep fluid. If

$$\delta \ll \lambda \qquad \Rightarrow \qquad \lambda^2\omega \gg \frac{\mu}{\rho} \tag{5.49}$$

the vorticity would be localized in a thin layer close to the surface, and in the rest of the liquid there would be potential flow. Specialize further to pure gravity waves in the deep fluid, introduce the dispersion relation (5.17), and obtain:

$$\begin{aligned} \lambda^2\sqrt{\frac{2\pi g}{\lambda}} &\gg \frac{\mu}{\rho} \\ \lambda^{\frac{3}{2}} &\gg \frac{\mu}{\rho\sqrt{2\pi g}} \\ &\approx (0.025\,\mathrm{mm})^{\frac{3}{2}} \qquad (\mu \approx 10^{-3}\,\mathrm{kg\,m^{-1}\,s^{-1}},\ \mathrm{vann},\ 20\,^{\mathrm{o}}\mathrm{C}) \end{aligned} \tag{5.50}$$

For all gravity-dominated waves in a deep liquid (that is, with $\lambda \gg 1.71$ cm) the potential description is thus excellent. [14]

The tangential stresses in the surface must be zero for a viscous liquid, if it is assumed (as we do) that the viscous interaction with the gas is negligible. The non-viscous solutions for the velocity we have found, do not give zero if inserted into the viscous term in the Navier-Stokes equation. However, as for the vorticity, a closer examination would show that the deviation in the velocity decrease exponentially over a thin layer close to the surface.[15]

Therefore, a perturbative calculation of the dissipation of mechanical energy E_{mek} must be a good approximation. That is, the potential flow solution is inserted into Eq. (**??**) when the loss rate is to be calculated. In Problem 5.1 we rediscover the *general* result

$$E_{mek} = 2E_{kin} = 2E_{pot} \tag{5.51}$$

irrespective of whether the wave is of the gravity or the capillary type. As the energy loss is due to motion, we have [16]

$$\partial_t E_{mek} = \partial_t E_{kin} \tag{5.52}$$

This gives, by insertion from Eq. (**??**):

$$\begin{aligned} \partial_t E_{mek} &= -\frac{1}{2}\mu \int_V (\partial_i u_k + \partial_k u_i)^2\, dV \\ &= -2\mu \int_V (\partial_i \partial_k \Phi)^2\, dV \end{aligned} \tag{5.53}$$

$$\begin{aligned} E_{mek} &= \rho \int_V \boldsymbol{u}\cdot\boldsymbol{u}\, dV \\ &= \rho \int_V (\partial_i \Phi)^2\, dV \end{aligned} \tag{5.54}$$

[14] Other cases must be checked explicitly. For instance, for purely capillary waves in a deep fluid one gets $\lambda \gg \frac{1}{(2\pi)^3}\frac{\mu^2}{\rho\alpha} \approx 5.6\times10^{-8}$ mm.

[15] According to [Landau and Lifshitz 1987] these gradient deviations are not "anomalously large", as distinct from the velocity variation over a boundary layer.

[16] Implicitly, the energy loss rate must be small enough that Eq. (5.51) is a good approximation, by potential energy being transformed back to kinetic energy so that the balance is maintained.

If Eq. (5.15) is inserted into the integrands, it follows from simple derivations that

$$\begin{aligned}(\partial_i\Phi)(\partial_i\Phi) &= k^2C^2\left\{\sinh^2(k(z+h))\cos^2(kx-\omega t)+\cosh^2(k(z+h))\sin^2(kx-\omega t)\right\}\\(\partial_i\partial_k\Phi)(\partial_i\partial_k\Phi) &= 2k^4C^2\left\{\cosh^2(k(z+h))\cos^2(kx-\omega t)+\sinh^2(k(z+h))\sin^2(kx-\omega t)\right\}\end{aligned}$$

We are interested in *time averages* over one period. Since $\overline{\sin^2(kx-\omega t)} = \overline{\cos^2(kx-\omega t)}$, it follows that [17]

$$\overline{(\partial_i\partial_k\Phi)(\partial_i\partial_k\Phi)} = 2k^2\,\overline{(\partial_i\Phi)(\partial_i\Phi)} =$$

We notice that the energy loss is roughly exponentially damped over each step $\lambda/4\pi$ in distance from the surface, as was mentioned above. Furthermore, we find that the *loss factor*, defined as

$$\gamma = -\frac{\partial_t\overline{E}_{mek}}{2\overline{E}_{mek}} \tag{5.55}$$

is a constant:

$$\begin{aligned}\gamma &= 2\frac{\mu}{\rho}k^2\\&\approx (12700\,\mathrm{s})^{-1}\,(\frac{1\,\mathrm{m}}{\lambda})^2\end{aligned} \tag{5.56}$$

The result holds under the assumption that the potential flow approximation is valid. From the definition of the loss factor it follows that the energy decreases as $\propto e^{-2\gamma t}$, and thus the wave amplitude as $\propto e^{-\gamma t}$. If $\gamma\tau \ll 1$, the energy does not change much over one period. For waves in deep water we get:

$$\begin{aligned}(\gamma\tau)_g &= 2\frac{\mu}{\rho}k^2\frac{2\pi}{\sqrt{gk}}\\&\approx \frac{1}{15800}\,(\frac{1\,\mathrm{m}}{\lambda})^{\frac{3}{2}} \qquad \text{(gravity waves, deep wawter)}\end{aligned} \tag{5.57}$$

$$\begin{aligned}(\gamma\tau)_\alpha &= 2\frac{\mu}{\rho}k^2\frac{2\pi}{\sqrt{\frac{\alpha k^3}{\rho}}}\\&\approx \frac{1}{8.53}\,(\frac{1\,\mathrm{mm}}{\lambda})^{\frac{1}{2}} \qquad \text{(capillary waves, deep water)}\end{aligned} \tag{5.58}$$

We have thus shown that the mechanical energy of plane ocean waves with wavelength 1 m or above have a half-life to be counted in hours, while those with wavelength 1 mm or below will be damped over a few periods.

5.5 Ship waves: Kelvin's limit angle

An object traveling through the air (an airplane, an incoming missile, a bullet, ...) with supersonic velocity drags behind it a conical shock wave, where θ, the half apex angle, will depend on the velocity. The shock wave is created by interference between sound generated by the object in all points of its trajectory when it passed by: The object behaves as a 'point source', and the wave energy radiates from each points as *spherical waves*. The waves

[17]For waves in a deep liquid the time average becomes trivial. [Landau and Lifshitz 1987], where the calculation is done for a deep liquid, is imprecise on this point.

Figure 5.9: The Mach cone, with $\sin\theta = \frac{c}{u} = \frac{1}{\mathcal{M}}$

Figure 5.10: Wake waves in deep water, with enveloping lines drawn

propagating from the various points combine to create the stationary conical pattern moving together with the object, see Figure 5.9. An observer which is passed by the spherical surface, will hear a 'sonic boom'.[18]

A related phenomenon emerges behind a ship moving in deep water. The surface waves are observed to have a V-formed enveloping pattern as in Figure 5.10, where the half apex angle has a *fixed* value $\theta \approx 19.5^{\circ}$, *independent* of the ship's velocity provided it is large enough that significant waves are created. This considerable difference arises because while $V \sim \sqrt{\lambda}$ for gravity waves in a deep liquid, there is no appreciable dispersion for sound waves in air.

In the treatment of this phenomenon, called *Kelvin's limit angle* (which presumable most people have observed for themselves), we will in what follows use the treatment in [Dysthe 1980/81]. It is based on simple physical arguments.[19] We will differentiate between observers

[18] A popular saying is that the boom arises when a plane passes through the 'sonic wall'. Beside, there are two almost coinciding cones, with top points in the front and back ends of the object, respectively.

[19] One might go to the opposite extreme instead, and use the occasion for an even more thorough introduction

at rest with respect to the ship and to the ocean, and use that the propagation of the wave energy equals the group velocity.

We define:

- $\boldsymbol{k}$ wave number vector with direction indicating a plane wave's propagation direction
- $\boldsymbol{u}_s$ the ocean's flow velocity relative to an observer
- $\boldsymbol{u}$ the ship's velocity relative to the ocean
- $\boldsymbol{U}_g$ a wave's group velocity relative to the ocean
- $\boldsymbol{U}_E$ energy propagation velocity relative to the ship

The scalar value $k = \frac{2\pi}{\lambda}$ is $2\pi \times$ the number of periods of the wave per unit length in the propagation direction. Consider waves at a distance from the ship much larger than the wavelength, so that to a good approximation they may locally be considered to be plane waves. The number of wave crests passing an observer per unit time will be changed by

$$\frac{u_s}{\lambda}\cos\angle(\boldsymbol{k}, \boldsymbol{u}_s) = \frac{\boldsymbol{k}\cdot\boldsymbol{u}_s}{2\pi}$$

when the observer is moving relative to the medium of propagation of the waves. The wave's angular frequency relative to the observer, ϖ, is *Doppler shifted* relative to the value ω from Eq. (5.17):

$$\varpi = \sqrt{gk} + \boldsymbol{k}\cdot\boldsymbol{u}_s \tag{5.59}$$

An observer travelin on the ship will notice:

- $\varpi = 0$ (static wave pattern)
- $\boldsymbol{u}_s = -\boldsymbol{u}$
- The wave energy will radiate from the ship (however not isotropically), since the ship is the only wave source

Letting ψ denote the angle between $\boldsymbol{k}$ and $\boldsymbol{u}$, we get from Eqs. (5.38) and (5.59):

$$\begin{aligned} \sqrt{gk} &= \boldsymbol{k}\cdot\boldsymbol{u} \\ 2U_g k &= ku\cos\psi \\ U_g &= \frac{1}{2}u\cos\psi \end{aligned} \tag{5.60}$$

As seen from Figure 5.11: For a given point on the circular wave,

$$\boldsymbol{U}_E = \boldsymbol{U}_g - \boldsymbol{u} \tag{5.61}$$

For θ, the angle between the wake line and an arbitrary point in the wave pattern behind the ship, we find geometrically an expression which can be maximized by derivation with respect to $\sin\psi$:

$$\begin{aligned} \operatorname{tg}\theta &= \frac{u_g\sin\psi}{u - u_g\cos\psi} \\ &= \frac{\sin\psi\cos\psi}{1+\sin^2\psi} \end{aligned} \tag{5.62}$$

to mathematical physics, as in [Sommerfeld 1964]. One may then introduce *Bessel functions* for the interfering circular waves, and use the *saddle point method* for the calculation of integrals, etc.

Figure 5.11: Geometrical relation between velocity vectors

$$(\mathrm{tg}\,\theta)_{max} = \frac{1}{\sqrt{8}} \qquad\qquad (\sin\psi \;=\; \frac{1}{\sqrt{3}}) \tag{5.63}$$

Thus, we have found that the entire wave pattern (for pure gravity waves in deep water) in the wake of a moving ship is wedge-shaped, with a fixed value for half the apex angle irrespective of the velocity of the ship:

$$\theta_{max} \approx 19^{\mathrm{o}}\, 28' \tag{5.64}$$

Two enveloping lines for this apex angle have been drawn in Figure 5.10, and we notice an excellent agreement between theory and practice.

The essential point is that the value of θ_{max} is an *universal constant* for gravity waves in a deep liquid. It does not depend on the liquid density or the acceleration of gravity; its origin is purely geometrical. [20]

5.6 Problems

Problem 5.1 A plane monochromatic surface wave propagates in water with a depth much larger than the wavelength. Consider a width b perpendicular to the direction of propagation.

a) Find the potential (gravity) energy E_g over one wavelength in the direction of propagation.

b) Find the corresponding kinetic energy E_{kin}.

c) Find the corresponding energy E_σ due to surface tension.

d) Find the energy S which flows through an area from the surface to the bottom, with the direction of propagation as normal, during one period τ.

e) Show that

$$\frac{U}{V} = \frac{S}{E_g + E_{kin} + E_\sigma}$$

[20] The 'little green men' navigating on the liquid ammonia oceans of Triton (or wherever) would observe the same value!

also for the case with surface tension included.[21]

Problem 5.2 Plane ocean waves arrive perpendicularly to the shoreline at Sola beach. At a distance from the shore where the sea can still be considered deep, they have surface amplitude 0.5m and wavelength 2m. Calculate the energy flux per meter of the shoreline, using the linearized theory and assuming that capillarity can be neglected.

Comment: Wave power may be technically and economically interesting—some people think.

Problem 5.3 a) Find the loss factor for the waves in the last problem.

b) Find the loss factor for combined gravity/capillary waves with such a wavelength that the phase velocity equals its minimum value.

[21] I.e., correct the error in Massey's textbook.