Sample Problem 3.1 Compute the atmospheric pressure at elevation 20,000 ft, considering the atmosphere as a static fluid. Assume standard atmosphere at sea level. Use four methods; (*a*) air of constant density; (*b*) constant temperature between sea level and 20,000 ft; (*c*) isentropic conditions; (*d*) air temperature decreasing linearly with elevation at the standard lapse rate of 0.00356°F/ft.

Solution. From Appendix A, Table A.3 the conditions of the standard atmosphere at sea level are $T = 59.0°F$, $p = 14.70$ psia, $\gamma = 0.07648\ \text{lb/ft}^3$.

(a) Constant density

From Sec. 3.2: $\dfrac{dp}{dz} = -\gamma; \quad dp = -\gamma\, dz; \quad \displaystyle\int_{p_1}^{p} dp = -\gamma \int_{z_1}^{z} dz$

where subscript 1 indicates conditions at a reference elevation, sea level in this case,

so $$p - p_1 = -\gamma(z - z_1)$$

and $$p = 14.70 \times 144 - 0.07648(20{,}000) = 587\ \text{lb/ft}^2\ \text{abs} = 4.08\ \text{psia} \qquad \textit{ANS}$$

$= 0.28\, p_1$

(b) Isothermal

From Sec. 2.7: $pv = \text{constant}; \quad \text{hence } \dfrac{p}{\gamma} = \dfrac{p_1}{\gamma_1}$ if g is constant

Eq. (3.2): $$\frac{dp}{dz} = -\gamma, \quad \text{where } \gamma = \frac{p\gamma_1}{p_1}$$

so $$\frac{dp}{p} = -\frac{\gamma_1}{p_1} dz$$

Integrating, $$\int_{p_1}^{p} \frac{dp}{p} = \ln\frac{p}{p_1} = -\frac{\gamma_1}{p_1}\int_{z_1}^{z} dz = -\frac{\gamma_1}{p_1}(z - z_1)$$

and $$\frac{p}{p_1} = \exp\left[-\left(\frac{\gamma_1}{p_1}\right)(z - z_1)\right]$$

Thus $$p = 14.70 \exp\left[-\frac{0.07648}{14.70 \times 144}(20{,}000)\right] = 7.14\ \text{psia} \qquad \textit{ANS}$$

$= 0.49\, p_1$

(c) Isentropic

From Sec. 2.7: $pv^{1.4} = \dfrac{p}{\rho^{1.4}} = \text{constant}; \quad \text{hence } \dfrac{p}{\gamma^{1.4}} = \text{constant} = \dfrac{p}{\gamma_1^{1.4}}$

Eq. (3.2): $$\frac{dp}{dz} = -\gamma, \quad \text{where } \gamma = \gamma_1\left(\frac{p}{p_1}\right)^{1/1.4} = \gamma_1\left(\frac{p}{p_1}\right)^{0.714}$$

so $$dp = -\gamma_1\left(\frac{p}{p_1}\right)^{0.714} dz$$

Integrating:

$$\int_{p_1}^{p} p^{-0.714}\, dp = -\gamma_1 p_1^{-0.714}\int_{z_1}^{z} dz$$

$$p^{0.286} - p_1^{0.286} = -0.286\gamma_1 p_1^{-0.714}(z - z_1)$$

$$p^{0.286} = (14.70 \times 144)^{0.286} - 0.286(0.07648)(14.70 \times 144)^{-0.714}(20{,}000)$$

$$p = 942\ \text{lb/ft}^2\ \text{abs} = 6.54\ \text{psia} \qquad \textit{ANS}$$

$= 0.44\, p_1$

(d) Temperature decreasing linearly with elevation

For the standard lapse rate (Fig. 2.2): $T = a + bz$,

where $a = 59.00 + 459.67 = 518.67°R$ and $b = -0.003560°R/ft$

Eqs. (3.2) and (2.4): $$\frac{dp}{dz} = -\rho g; \quad \rho = \frac{p}{RT}$$

Combining to eliminate ρ, which varies, rearranging, and substituting for T,

$$\frac{dp}{p} = -\frac{g\, dz}{R(a + bz)}$$

Integrating: $$\int_1^2 \frac{dp}{p} = -\frac{g}{R}\int_1^2 \frac{dz}{a + bz}$$

$$\ln\left(\frac{p_2}{p_1}\right) = -\frac{g}{Rb}\ln\left(\frac{a + bz_2}{a + bz_1}\right) = \ln\left(\frac{a + bz_2}{a + bz_1}\right)^{-g/Rb}$$

i.e. $$\frac{p_2}{p_1} = \left(\frac{a + bz_2}{a + bz_1}\right)^{-g/Rb}$$

Here $$\frac{-g}{Rb} = \frac{-32.174}{1716(-0.003560)} = 5.27$$

and, from Table A.3: $p_1 = 14.696$ psia when $z_1 = 0$.

Thus $$\frac{p_2}{14.696} = \left(\frac{518.67 - 0.003560 \times 20{,}000}{518.67 + 0}\right)^{5.27} = 0.459$$

$$p_2 = 14.696(0.459) = 6.75\ \text{psia} \qquad \textit{ANS}$$

$= 0.46\, p_1$