Chapter 10

Instability and transition to turbulence in shear flow

In later chapters we will aim at developing a mathematical description of turbulence. First, in the present chapter, we therefore include a description of some stages in the transition from laminar to turbulent flow.[1] On one hand we will consider linear stabilit theory, applied to

- shear flow stability (in general)
- boundary layer stability (in particular)

On the other hand we will also consider the further development of early instabilities, at stages where linear stability theory is useless and leads to irrelevant predictions, i.e., essentially an experimental topic.[2] We will follow the classification

- flow with med 'patchwise' turbulence (in shear flow at a wall: boundary layers, pipe flow, ...)
- flow with the same rate everywhere for generation of turbulence (in 'free' shear flow: jets, wakes, ...)

and we will mostly consider the first group.

10.1 Linear hydrodynamic stability theory

Linear stability makes predictions about when a flow becomes unstable with respect to *infinitesimal* disturbances. It cannot predict whether an infinitesimally stable system is stable with respect to *large* disturbances.

The basic steps in a linear stability analysis are:

- Start with an (eventually approximate) solution of the equations of motion

[1]Within the scope of this course it will not, unfortunately, be possible to treat newer developments where some transitions to turbulence have been identified experimentally as transitions to chaos. Most spectacularly this is the case for Bénard convection and rotating Couette flow; we may refer to for instance [Acheson 1990], [Landau and Lifshitz 1987] or [Tritton 1988]. Popularized presentations like [Gleick 1987] may serve as a first introcuction to these matters.

[2]The presentation of empirical results in this chapter essentially follows [Tritton 1988].

- Superpose a small perturbation on it
- Disregard all term of second and higher orders in the perturbation amplitude
- Check whether the perturbation increases or decreases with time

A Fourier decomposition of the perturbation in the spatial coordinates can be done. (A survey of the use of Fourier integrals is included in Appendix G.) The linear approximation provides for no coupling between the various Fourier components. Therefore, each individual component's influence on the stability of the solution can be studied independently of the others. Since the Fourier integral can be interpreted as a superposition (for a given t) of plane waves $\exp(i\boldsymbol{kr})$ with a weight function in $\boldsymbol{k}$ space, we can treat all Fourier components in one mathematical operation by studying a perturbation on the form

$$\Delta Q \propto \exp\{i(\boldsymbol{kr} - \omega t)\} \tag{10.1}$$

where ΔQ refers to a perturbation of a quantity Q associated with the flow.[3] The emergence of a factor $\exp(-i\omega t)$ follows since we are considering flow stability: A sinusoidal disturbance will be transported with the flow, and for a given $\boldsymbol{r}$ it will be interpreted as a periodic oscillation in time. For a given $\boldsymbol{k}$, ω follows from the equations of motion. We use the notation

$$\omega = \omega_r + i\omega_i \tag{10.2}$$

where the sign of ω_i decides whether the disturbance increases or decreases with time:

- Stability if $\omega_i < 0$ for all $\boldsymbol{k}$ (necessary condition)
- Instability if $\omega_i > 0$ for some $\boldsymbol{k}$ (sufficient condition)

Somme comments on the results are in order:

- The growth of eventual unstable modes is *exponential* according to linear stability theory. However, higher orders in the perturbation will then become significant after some time, and the assumption about small perturbations breaks down. Linearized theory can usually not decide what type of deviation from exponential growth (see Figure 10.1) one will then have.
- Both $\omega_r = 0$ and $\omega_r \neq 0$ may occur in hydrodynamical stability theory. For $\omega_r = 0\,,\ \omega_i > 0$ the perturbation is predicted to grow continuously with time. For $\omega_r \neq 0\,,\ \omega_i > 0$ a periodic oscillation with a growing amplitude (see Figure 10.2) is predicted instead; this is called *overstability*[4]. [5]

Nonlinear stability theory takes over where the linear gives up. This is an extensive field in Fluid dynamics. We cannot include any of it in this course, and refer instead to for instance [Drazin and Reid 1981].

[3] We use a notation for the time dependence different from that of [Tritton 1988] as well as a considerable part of the classical hydrodynamical literature. There, $\exp(\sigma t)$ is used instead of $\exp(-i\omega t)$. When a sinusoidal perturbation is to be used, using our notation which is the usual one for a plane wave, is an aestehetical improvement.

[4] Which may be an untimely name.

[5] For boundary layer instability, there is the previously mentioned situation for $\omega_r \neq 0$ where a periodic disturbance increasing in *space* passes by an observation point by convection (more about that later on). This is usually not classified as overstability.

Figure 10.1: Deviations (broken lines) from exponential growth: (A) levelling off, (B) overexponential growth

Figure 10.2: Overstability

How valid are the predictions of the linearized theory? In this course we will treat boundary layer stability, where good verification has been found, but also pipe flow, where the theory does not predict the observed transition to turbulence.[6]

10.1.1 Example: The Kelvin-Helmholz instability

Let us consider a case where two fluids slide smoothly parallel to each other, see Figure 10.3. We will assume:

- 2D flow of ideal fluids ($\mathcal{R} \to \infty$)
- Different velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ (without loss of general applicability: $\boldsymbol{u}_1 \to \boldsymbol{u}_0$ and $\boldsymbol{u}_2 \to 0$, with $\boldsymbol{u}_0$ in the x direction along the common surface, where the pressure is p_0)
- Different densities (although not an essential condition)
- No gravity effects (except eventually as an argument that the most dense fluid flows below the other)

[6] For rotating Couette flow and Bénard convection one also finds good verification, see [Tritton 1988].

Figure 10.3: 2D parallel flow of two fluids with a surface of discontinuity

- At the outset: Plane surface of discontinuity between the fluids

We will show that the plane surface of discontinuity is always unstable. Let us superpose a weak, and in principle infinitesimal, disturbance of surface coordinates, velocities and pressure in the surface of discontinuity, with the perturbations ζ, $\boldsymbol{u}'$ and p' being proportional to

$$e^{i(kx-\omega t)}$$

I.e., a spatial perturbation with a given $\boldsymbol{k}$, where ω is to be found. The Euler and continuity equations for fluid 1 with the perturbation included are

$$\begin{aligned}\partial_t(\boldsymbol{u}_0+\boldsymbol{u}')+((\boldsymbol{u}_0+\boldsymbol{u}')\cdot\nabla)(\boldsymbol{u}_0+\boldsymbol{u}') &= -\frac{1}{\rho}\nabla(p_0+p') \\ \nabla\cdot(\boldsymbol{u}_0+\boldsymbol{u}') &= 0\end{aligned}$$

However, the euqations must also be satisfied for the unperturbed state, with $\boldsymbol{u}'=0$ and $p'=0$. The terms containing $\boldsymbol{u}_0$ and p_0 only will therefore cancel. If in addition we neglect the term $(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}'$, which is second order in $\boldsymbol{u}'$, we get a linearized set of equations for the perturbation. Using also that $\boldsymbol{u}_0$ points in the x direction, we get

$$\partial_t\boldsymbol{u}'+u_0\partial_x\boldsymbol{u}' = -\frac{1}{\rho}\nabla p' \tag{10.3}$$

$$\nabla\cdot\boldsymbol{u}' = 0 \tag{10.4}$$

Taking the divergence on both sides of the Euler equation, and using the equation of continuity, we get the Laplace equation

$$\nabla^2 p' = 0 \tag{10.5}$$

By inserting a solution on the assumed form[7]

$$p' = f(y)e^{i(kx-\omega t)}$$

we get

$$\frac{d^2f}{dy^2}-k^2f=0\,,\quad f(y)=\text{constant}\cdot e^{\pm ky}$$

[7]We will consider y values corresponding to the transversal displacement of the surface of discontinuity.

where only the lower sign can be used when the y axis points in the direction of fluid 1:

$$p' \propto e^{i(kx-\omega t)}e^{-ky} \tag{10.6}$$

By inserting this expression into the y component of Eq. (10.3), we find (using index 1 for fluid 1):

$$u'_y = \frac{kp'_1}{i\rho_1(ku_0 - \omega)} \tag{10.7}$$

The displacement of the surface of discontinuity in the y direction is $\zeta = \zeta(x,t)$, and its velocity of displacement at constant x is $\partial_t \zeta$. This must be equal to the y component of the flow velocity at the surface. To the lowest order, with u'_y equal to the value at the surface:

$$\partial_t \zeta = u'_y - u_0 \partial_x \zeta$$

Inserting $\zeta \propto e^{i(kx-\omega t)}$ we get:

$$u'_y = i\zeta(ku_0 - \omega) \tag{10.8}$$

By elimination of u'_y between Eqs. (10.7) and (10.8) we get

$$p'_1 = -\frac{\zeta\rho_1(ku_0 - \omega)^2}{k} \tag{10.9}$$

A similar calculation would have given the pressure p'_2 on the other side of the surface, the only differences being $u_0 \to 0$ and change of sign since $y < 0$ and thus $p'_2 \propto e^{ky}$:

$$p'_2 = \frac{\zeta\rho_2\omega^2}{k} \tag{10.10}$$

The pressures p'_1 and p'_2, infinitesimally close to the surface on each side of the surface of discontinuity, must be equal:

$$\rho_1(ku_0 - \omega)^2 = -\rho_2\omega^2$$

Solving for ω, we find that the given perturbation corresponds to two modes which both have an imaginary frequency component:

$$\omega = ku_0\frac{\rho_1}{\rho_1 + \rho_2}(1 \pm i\sqrt{\frac{\rho_2}{\rho_1}}) \tag{10.11}$$

We notice that the flow will be unstable even with regard to infinitesimally small disturbances, because of the mode with $\omega_i > 0$.[8]

In the case with a finite viscosity (finite $\mathcal{R}$) one has no longer a sharp tangential discontinuity. The velocity will vary from one value to another across a layer with a finite thickness, see Figure 10.4. Both experimental and theoretical results indicate that the instability sets in after a short time for large $\mathcal{R}$ also in this case. For later reference to this type of instability, we notice that for finite $\mathcal{R}$ the velocity profile has an *inflection point*.

A wellknown experimental observation of the Kelvin-Helmholz instability can be made with layered fluids, where a lighter fluid flows on top of a heavier in a thin vessel. When the

[8] One may get a more intuitive understanding of the reason for the instability by considering Figure 10.4d/e. Once a perturbed flow has developed, the fluid flowing past a 'top' formed by the fluid on the other side will have to move faster than the fluid in a 'trough'. Bernoulli's equation will then predict a pressure distribution which will make the perturbation grow.

Figure 10.4: Plane parallel flow at (a) finite and (b/c) infinite $\mathcal{R}$, which (d/e) is unstable

Figure 10.5: Kelvin-Helmholz instability with layered fluids

vessel is made to slant, the gravity will introduce a motion tangential to the surface. If the destabilizing effect due to it is strong enough to overcome the stabilizing effect of the layering, one may observe patterns as in Figure 10.5.[9]

10.2 Shear flow instability

As indicated by Figure **??** antyder, many of the velocity profiles we have treated in this course are of the shear flow type. We classify profiles without layering as follows:

[9] A even more wellknown observation, in particular in Stavanger, is the wavy motion of the polar front (the borderline between polar air and warmer air). With additional help from the Coriolis acceleration the instabilities develop further into low pressures.