

Appendix C

Some thermodynamical relations

The extensive thermodynamical quantities ϵ , V , h and s for internal energy, volume, enthalpy and entropy, respectively, with

$$h = \epsilon + pV \quad (\text{C.1})$$

will be defined per unit mass,¹ so that $\rho = 1/V$. The first law of thermodynamics can then be written in the following two ways, among others, for a reversible process [Huang 1963]:²

$$\begin{aligned} d\epsilon &= T ds - p dV \\ &= T ds + \frac{p}{\rho^2} d\rho \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} dh &= T ds + V dp \\ &= T ds + \frac{1}{\rho} dp \end{aligned} \quad (\text{C.3})$$

By expressing the differentials in Eq. (C.3) by the differential of a spatial variable, we get

$$\nabla p = \rho \nabla h - \rho T \nabla s \quad (\text{C.4})$$

For an adiabatical process, $Ds/Dt = 0$, so at

$$\partial_t s + (\mathbf{u} \cdot \nabla) s = 0 \quad (\text{C.5})$$

In these lecture notes the relations above are used in Chapter 2 in the derivation of the expression for energy flow density in a compressible ideal fluid.

The velocity of sound, c , in a fluid is given by the adiabatic compressibility β_s , or equivalently by the isothermal compressibility β_T and the relation γ between the specific heats c_p and c_v at constant pressure and constant volume, respectively:

$$c^2 = \frac{1}{\rho \beta_s} \quad (\text{C.6})$$

$$\beta_{s,T} = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_{s,T} \quad (\text{C.7})$$

$$\beta_s = \gamma \beta_T \quad (\text{C.8})$$

$$\gamma = \frac{c_p}{c_v} \quad (\text{C.9})$$

¹Molar quantities are preferable in general. However, the usage per unit mass chosen here are useful for some derivations in Chapter 2.

²The sign of the dV term is chosen such that positive work is done *by* the system *on* its surroundings.

These relations have been used in Chapter 2 in the derivation of the range of validity of the incompressible approximation. For an ideal one-atomic gas, $\gamma = 5/3$. For real gases in the ideal gas limit ($p \rightarrow 0$), γ is generally a function of T ; for air (a two-atomic gas) at standard conditions, $\gamma \approx 7/5$.

The equation of state for an adiabatic ideal gas is an example of a *polytropic* equation:

$$p \rho^{-\kappa} = \text{konstant} \quad (\text{C.10})$$

The quantity κ is the *polytropic exponent*, and for an ideal gas $\kappa = \gamma$. Such equations of state are also used as an approximation for real (in the thermodynamical sense!) gases, where adiabatical (isentropic) processes may be described by effective constant values for κ for given intervals in p and T .³ This type of equation is mentioned in connection with the Bernoulli equation (??) applied to a gas.

³Notice that for a real gas, $\kappa \neq \gamma$ [Finjord 1988], not to mention that Eq. (C.10) with a constant κ is not exact for adiabatical processes in real gases!