SPECTRAL EFFICIENCY OF VARIABLE-RATE CODED QAM FOR FLAT FADEING CHANNELS

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ABSTRACT
We investigate a variable-rate coding scheme for Nakagami multipath fading channels. The coding scheme can utilize any set of $2L$-dimensional trellis codes originally designed for additive white Gaussian noise. A feedback channel from the receiver provides channel state information (CSI) at the transmitter. Knowing the CSI makes it possible to transmit at high spectral efficiencies under good channel conditions and respond to channel degradation through a smooth reduction of spectral efficiency. We develop a general technique to approximate the average spectral efficiency of the variable-rate coding scheme. The average spectral efficiency of the variable-rate scheme compares favorably to the spectral efficiency of more traditional fixed-rate coding schemes.

1. INTRODUCTION

In [1], Goldsmith and Varaiya determined a general expression for the Shannon capacity of an arbitrary single-user flat-fading channel with channel state information at the transmitter and receiver, given the probability density function (PDF) of the fading amplitude. Alouini and Goldsmith [2] then obtained closed-form expressions for the capacity of Nakagami multipath fading (NMF) channels.

Goldsmith and Varaiya [1] suggested obtaining the channel capacity by utilizing a variable-rate coding scheme, such that the code rate, or spectral efficiency measured in bits/s/Hz, is small when the instantaneous received carrier-to-noise ratio (CNR) is low. Correspondingly, when the instantaneous CNR is high, a high rate (high spectral efficiency) is used. Goldsmith and Chua [3, 4] studied a practical variable-rate scheme based on multilevel quadrature amplitude modulation (QAM) and Ungerboeck’s trellis codes [5].

Our goal is to obtain an expression for the average spectral efficiency of a general variable-rate coding scheme for NMF channels. Let $L$ be some positive integer. The coding scheme can utilize any set of $2L$-dimensional (2L-D) trellis codes originally designed for additive white Gaussian noise (AWGN) channels.

The paper is organized as follows: In Section 2, we give a brief overview of the system model. We then show, in Section 3, that there exists a simple expression characterizing the relationship between the CNR and the bit-error-rate (BER) for trellis codes on AWGN channels. In Section 4, the BER/CNR relationship is employed to approximate the average spectral efficiency of the general variable-rate coding scheme. As an example, we use our general technique to approximate the average spectral efficiency of a specific variable-rate encoder and decoder (codec) in Section 5. A discussion follows in Section 6.

2. SYSTEM OVERVIEW

We model the NMF as a multiplication of the transmitted signal by a time-varying factor. The signal is then disturbed by AWGN. Thus, the receiver reads a signal with a time-varying CNR. The PDF of the instantaneous received CNR, $\gamma$, is found to be [2]:

$$p_{\gamma}(\gamma) = \left( \frac{m}{\gamma} \right)^m \gamma^{m-1} \frac{\exp\left(-m\frac{\gamma}{\bar{\gamma}}\right)}{\Gamma(m)}, \quad \gamma \geq 0 \quad (1)$$

where $\bar{\gamma}$ is the expected value of the CNR and $m$ is the Nakagami fading parameter, a positive number $m \geq 1/2$.

It is known that a variable-rate codec based on trellis codes originally designed for AWGN may be used on a NMF channel [1, 2, 3, 4, 6]. We consider a general codec that utilizes a set of $N$ different 2L-D trellis codes for AWGN. The codes are based on 2-D QAM signal constellations with different number of symbols $M_n = 2^{k_n}$, where $k_n$ is some positive integer and $n = 1, 2, \ldots, N$. For $n < N$, $M_n < M_{n+1}$, thus the codes have different robustness against noise. The 2L-D code symbols are transmitted as $L$ consecutive 2-D modulation symbols, each drawn from one of the QAM constellations. An illustration of such a constellation of size $M = 256$ is shown in Fig. 1, where the minimum Euclidean distance between the modulation symbols is denoted by $d_0$.

If the $N$ constellations are nested within each other, as indicated in Fig. 1, then it is possible to design an encoder/decoder-structure which is able to encode and decode all $N$ codes [4]. Consequently, the hardware complexity is reduced considerably compared to a system where a separate encoder and decoder are needed for each of the codes.
As an example, we consider a variable-rate encoder and a variable-rate Viterbi decoder based on the International Telecommunications Union ITU-T V.34 modem standard. The codec utilizes \( N = 8 \) nested QAM signal constellations containing 4, 8, 16, 32, 64, 128, 256, and 512 symbols to encode and decode eight 4-D trellis codes [6]. The seven smallest constellations are the ones shown in Fig. 1.\(^1\)

To determine a good variable-rate codec, the relationship between the CNR and BER on an AWGN channel must be known for the different codes. The BER performance of the codes used in the example codec have been simulated, and the BER for different CNRs are plotted in Fig. 2, represented by boxes. It is shown in Section 3 that a simple exponential function approximates the BER/CNR relationship closely, as indicated by the solid lines in Fig. 2. The function is invertible, so the least CNR required to achieve a given target BER, denoted by BER\(_0\), can be found.

Assume that \( N \) quantization regions (or fading regions) are used to represent the instantaneous received CNR on the NMF channel. For each code \( n \), the smallest CNR required to guarantee the target BER\(_0\) is denoted by \( \gamma_n \). The set \( \{\gamma_n\}_{n=1}^N \) constitutes the lower thresholds for the \( N \) fading regions.

Let \( T \) be the time between the transmission of two consecutive 2-D modulation symbols. Every \( L \cdot T \) seconds, the receiver estimates the value of the instantaneous CNR to determine which code \( n \in \{1, 2, \ldots, N\} \) to use. The receiver then utilizes a feedback channel to inform the transmitter of its decision. We assume here that the feedback channel is both delay-free and error-free. The encoder requests \( p = L \cdot k_n - 1 \) information bits, and generates \( p + 1 = L \cdot k_n \) coded bits. The coded bits determine \( L \) transmittable 2-D modulation symbols from the signal constellation with \( \alpha_n = 2^{k_n} \) symbols.

The information rate for code \( n \), measured in information bits per second, is equal to \( R_n = p/(L \cdot T) = (k_n - 1/L)/T \). Assuming ideal Nyquist pulses, the bandwidth used is \( W = 1/T \), and the spectral efficiency of code \( n \) is equal to

\[
\frac{R_n}{W} = k_n - 1/L
\]  

measured in bits/s/Hz. The codes should have spectral efficiencies which increase with \( \gamma \), i.e. \( k_n < k_{n+1} \) for \( n = 1, 2, \ldots, N - 1 \). This makes it possible to transmit at high spectral efficiency when the instantaneous received CNR is high and to reduce the spectral efficiency as the CNR decreases.

### 3. APPROXIMATION OF BER/CNR RELATIONSHIP

In this section, we show that the BER for a \( 2L \)-D trellis code on an AWGN channel can be approximated by the expression

\[
\text{BER} \approx a \cdot \exp\left(-\frac{b \gamma^*}{M}\right)
\]  

where \( \gamma^* \) is the fixed CNR, the constants \( a \) and \( b \) are determined by the weight distribution of the trellis code, and \( M = 2^k \) is the size of the signal constellation for some integer \( k \).

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\(^1\)The 512-QAM constellation is omitted from the figure due to space and visibility requirements.
An approximation of the BER for a trellis code in AWGN is [7, eq. (14–21)]:

\[
\text{BER} \approx \frac{w_{d_{\text{e}}} p}{2} Q\left(\sqrt{\frac{d_{\text{e}}^2}{2} \gamma^*}\right)
\]

(4)

where \(d_{\text{e}}^2\) is the minimum Euclidean distance (MSED) between any two code words and \(w_{d_{\text{e}}}\) is the average number of information bits associated with the code-words in squared distance \(d_{\text{e}}^2\) from the correct one. The number of information bits per 2L-D code symbol is given by \(p\). Using the upper bound \(Q(y) \leq 1/2 \exp(-y^2/2)\), we find that

\[
\text{BER} \approx \frac{w_{d_{\text{e}}}}{2p} \exp\left(-\frac{d_{\text{e}}^2}{4} \gamma^*\right).
\]

(5)

Forney et al. [8] have calculated approximations for the average energy of QAM constellations with \(d_0 = 2\). In the Appendix, eq. (27), we determine exact expressions for the average energy of QAM constellations with arbitrary \(d_0\) and size \(M = 2^k\):

\[
E_{\text{avg}} = \begin{cases} 
\frac{6}{k^2} (M - 1) & \text{for } k \geq 2 \\
\frac{4}{6k^2} (M - 1) & \text{for } k = 3 \\
\frac{6}{5k^2} (M - 1) & \text{for odd } k \geq 5
\end{cases}
\]

(6)

(Sterian [9] obtained the expression in (6) for the special case \(d_0 = 2\) and \(k \geq 4\).) For all \(k\), an approximation for \(E_{\text{avg}}\) is \((1/6)E_{\text{avg}}\), and we therefore have that

\[
d_{\text{e}}^2 \approx \frac{6E_{\text{avg}}}{M}.
\]

(7)

The MSED is linearly dependent on \(d_{\text{e}}^2\), and it may thus be written as \(d_{\text{e}}^2 = \beta d_{\text{e}}^2\) for some positive real \(\beta\). Substituting (7) for \(d_{\text{e}}^2\), we find

\[
d_{\text{e}}^2 \approx \frac{6\beta E_{\text{avg}}}{M}.
\]

(8)

Substituting (8) in (5), we obtain the following approximation for BER,

\[
\text{BER} \approx \frac{w_{d_{\text{e}}}}{2p} \exp\left(-\frac{3\beta E_{\text{avg}} \gamma^*}{2}\right)
\]

(9)

which has the desired form (3) for \(a = w_{d_{\text{e}}}/2p\) and \(b = 3\beta E_{\text{avg}}/2\).

The approximation in (4) is only valid for large CNR, and hence so is the approximation in (9). However, plots of BER found in the literature [10] (see also Fig. 2) strongly indicate that we can use curve fitting techniques to determine values for \(a\) and \(b\) such that the expression can be used with good accuracy also for medium and low CNRs.

The proposed variable-rate coding scheme utilizes \(N\) trellis codes originally designed for AWGN channels. As we shall see in the next section, good values of the \(a\) and \(b\) parameters in (3) are needed for each code \(n\) in order to approximate the average spectral efficiency of the proposed coding scheme. We assume that a simulation of the BER performance and a curve fitting technique are used to obtain good parameter values, denoted by \(a_n\) and \(b_n\), for each code.

### 4. APPROXIMATION OF AVERAGE SPECTRAL EFFICIENCY

The variable-rate coding scheme for NMF channels uses a set of \(N\) different trellis codes with spectral efficiencies \(k_n - 1/L\). The average information rate, \(\langle R \rangle\), is the sum of the information rates \(R_n = (k_n - 1/L) \cdot W\) for the individual codes, each weighted by the probability \(P_n\) that the code \(n\) is used, i.e. the probability that the CNR falls in the fading region \(n\). The average spectral efficiency is found by dividing by the bandwidth \(W\):

\[
\frac{\langle R \rangle}{W} = \sum_{n=1}^{N} \frac{R_n}{W} \cdot P_n = \sum_{n=1}^{N} (k_n - 1/L) P_n.
\]

(10)

In order to calculate the probabilities \(P_n\) in (10), the thresholds \(\gamma_n\) for the fading regions must be known. Each code has a BER/CNR relationship given by (3) with known \(a_n\) and \(b_n\) parameters. Thus, it is possible to calculate the smallest CNR value \(\gamma_n\) which guarantees that the target BER \(\gamma_0\) is achieved for code \(n\). Substituting \(\gamma_n, M_n, a_n, b_n\) for \(\gamma^*, M, a, b\) in (3) and solving for \(\gamma_n\), we obtain

\[
\gamma_n = \frac{M_n \cdot k_n}{b_n}, \quad n = 1, 2, \ldots, N
\]

(11)

where \(K_n = -\ln(\text{BER}_0/a_n)\) and \(M_n = 2^{k_n}\).

For Nakagami fading, the probability \(P_n = \int_{\gamma_n}^{\gamma_{n+1}} p_\gamma(\gamma) d\gamma\) that the CNR falls in fading region \(n\) is given by [2, eq. (34)]:

\[
P_n = \frac{\Gamma(m, \frac{m}{m + \frac{K_n}{\mu}}) - \Gamma(m, \frac{m}{m + \frac{K_{n+1}}{\mu}})}{\Gamma(m)}
\]

(12)

where \(m\) is the Nakagami fading parameter and \(\Gamma(\cdot, \cdot)\) is the complementary incomplete gamma function defined by [11, eq. (11.2)]. The upper threshold for fading region \(N\) is \(\gamma_{N+1} = \infty\).

For the special case of \(m\) a positive integer, \(\Gamma(m) = (m-1)!\). Furthermore, there exists a closed-form expression for \(\Gamma(\cdot, \cdot)\) [11, eq. (11.6)]:

\[
\Gamma(m, \mu) = (m-1)! e^{-\mu} \sum_{i=0}^{m-1} \frac{\mu^i}{i!}
\]

(13)

where \(m = 1, 2, \ldots \) and \(\mu \geq 0\).

### 5. EXAMPLE CODEC

We assume that the example codec described in Section 2 is operating on a NMF channel. Values of the \(a_n\) and \(b_n\) parameters for this codec are tabulated in Table 1. The thresholds \(\gamma_n\) calculated from (11) are also listed in the table for target BER \(\gamma_0 = 10^{-3}\). The average spectral efficiency \(\langle R \rangle/W\) of the example codec may now be calculated from (10) and (12). The spectral efficiency for Nakagami fading parameter \(m \in \{1, 2, 4\}\) is plotted in Fig. 3 as a function of the average received CNR \(\gamma\) in dB.
Table 1: Parameters \(a_n\) and \(b_n\) for example codec and calculated thresholds \(\gamma_n\) for BER = 10\(^{-3}\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(M_n)</th>
<th>(a_n)</th>
<th>(b_n)</th>
<th>(\gamma_n) [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>896.0704</td>
<td>10.7367</td>
<td>7.1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>404.4353</td>
<td>6.8043</td>
<td>11.8</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>996.5492</td>
<td>8.7345</td>
<td>14.0</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>443.1272</td>
<td>8.2282</td>
<td>17.0</td>
</tr>
<tr>
<td>5</td>
<td>64</td>
<td>296.6007</td>
<td>7.9270</td>
<td>20.1</td>
</tr>
<tr>
<td>6</td>
<td>128</td>
<td>327.4874</td>
<td>8.2036</td>
<td>23.0</td>
</tr>
<tr>
<td>7</td>
<td>256</td>
<td>404.2837</td>
<td>7.8824</td>
<td>26.2</td>
</tr>
<tr>
<td>8</td>
<td>512</td>
<td>310.5283</td>
<td>8.2425</td>
<td>29.0</td>
</tr>
</tbody>
</table>

Figure 3: Approximation of average spectral efficiency for example codec (BER\(_0\) = 10\(^{-3}\)).

Using the closed-form expression for the maximum average spectral efficiency (MASE) obtained by Alouini and Goldsmith [2, eq. (20)], Hole et al. [6] showed that the average spectral efficiency of the example codec lies about 1.8 bits/s/Hz from the MASE.

6. DISCUSSION

We see from Fig. 3 that the variable-rate codec obtains both a large average spectral efficiency and a small target BER because it is able to exploit the time-varying nature of the instantaneous received CNR. To design a fixed-rate coding scheme that guarantees the same BER, it is necessary to design a code that achieves the target BER for the minimum observed CNR. This conservative design choice results in a small spectral efficiency. This was first observed by Goldsmith and Chua [4]. They compared the average spectral efficiency of variable-rate trellis-coded QAM with that of fixed-rate trellis codes designed for Rayleigh-fading channels and showed that rate adaption may save up to 20 dB in average received CNR, dependent on the BER requirements.

For satisfactory operation of the variable-rate coding scheme, both the variable-rate encoder and the variable-rate decoder must use the same code at any instant. A fast and error-free feedback channel is therefore essential to ensure error-free signaling between the encoder and decoder. The effect of time delay in the feedback channel is explored in [2], [6].

During periods of small CNR, the throughput may be low. A buffer is therefore required at the transmitter. The appropriate size of this buffer is a subject for further research.

The investigations in this paper are done for single-user NMF channels only. A natural extension is to examine variable-rate coding schemes for cellular systems.

APPENDIX—AVERAGE ENERGY OF QAM CONSTellATIONS

We calculate the average energy of QAM signal constellations with size \(M = 2^k\). For even \(k\), the constellation is square, while for odd \(k\), the constellation is cross-shaped. Examples of such constellations are shown in Fig. 1, where the distance between two adjacent symbols is denoted by \(d_0\).

Sterian [9] has determined exact expressions for the average energy for \(d_0 = 2\). We provide an alternative derivation of the expressions for arbitrary value of \(d_0\).

The first quadrant of a general constellation is as shown in Fig. 4. A square constellation consists of the crosses in the middle, while the cross constellation also contains the circles. The number \(P\) is associated with the number of symbols along each quadrature component. It is defined slightly differently for square and cross constellations,

\[
P = \begin{cases} \sqrt{M/4} & \text{for } k \text{ even} \\ \sqrt{M'/4} = \sqrt{M/8} & \text{for } k \text{ odd} \end{cases}
\]

where we have defined \(M' = M/2\) as the size of the QAM constellation.
Square constellations

We first determine the average energy for even \( k \geq 2 \), i.e., for square constellations. Due to the symmetry, the average energy of the whole constellation is equal to the average energy of one of the quadrants.

The number of symbols in each quadrant is \( M/4 \). For each symbol \( i \), the energy \( E_i \) is given in terms of its first and second coordinates \((x_i, y_i)\), \( E_i = x_i^2 + y_i^2 \). The average energy \( E_{sq} \) is given by

\[
E_{sq} = \frac{1}{M/4} \sum_{i=1}^{M/4} E_i = \frac{1}{M/4} \sum_{i=1}^{M/4} (x_i^2 + y_i^2) \tag{15}
\]

As indicated in (14), the number of symbols in each of the quadrature components is \( P = \sqrt{M/4} \). The symbols have coordinates \((x_i, y_i)\) where \( \{x_i\} = \{y_i\} = \{\frac{1}{2}d_0, \frac{3}{2}d_0, \frac{5}{2}d_0, \ldots, \frac{2P-1}{2}d_0\} \), and \( E_{sq} \) is thus given by

\[
E_{sq} = \frac{1}{(2P)^2} \sum_{n=1}^{P} \sum_{m=1}^{P} \left\{ \left( \frac{\sqrt{M}}{2} (2n-1) \right)^2 + \left( \frac{\sqrt{M}}{2} (2m-1) \right)^2 \right\} \\
= \frac{d_0^2}{4P^2} \left[ P \sum_{n=1}^{P} (2n-1)^2 + P \sum_{m=1}^{P} (2m-1)^2 \right] \\
= \frac{d_0^2}{2P} \sum_{n=1}^{P} (2n-1)^2. \tag{16}
\]

There exist a closed-form expression for the sum in (16), and to calculate it, we use the following formulas [12, eq. (0.121)]:

\[
\sum_{n=1}^{P} n = \frac{P(P+1)}{2}, \\
\sum_{n=1}^{P} n^2 = \frac{P(P+1)(2P+1)}{6}. \tag{17}
\]

Thus,

\[
\sum_{n=1}^{P} (2n-1)^2 = \sum_{n=1}^{P} (4n^2 - 4n + 1) \\
= 4 \sum_{n=1}^{P} n^2 - 4 \sum_{n=1}^{P} n + \sum_{n=1}^{P} 1 \\
= 4 \frac{P(P+1)(2P+1)}{6} - \frac{P(P+1)}{2} + P \tag{18}
\]

Using (18) in (16), and inserting \( P = \sqrt{M/4} \), we obtain

\[
E_{sq} = \frac{1}{4} d_0^2 (4P^2 - 1) = \frac{1}{4} d_0^2 (M - 1). \tag{19}
\]

Cross constellations

For the cross constellations, when \( k \geq 5 \) is odd, each of the quadrants consists of 3 regions denoted by \( A, B_1, \) and \( B_2 \), as indicated in Fig. 4, with average energies \( E_A, E_{B_1}, \) and \( E_{B_2} \), respectively. Regions \( B_1 \) and \( B_2 \) are mirror images of each other, thus \( E_{B_1} = E_{B_2} \).

When calculating the total average energy, the contribution from both \( B_1 \) and \( B_2 \) must be counted. But each of them consists of half as many symbols as \( A \), hence

\[
E_{cr} = \frac{1}{4} E_A + \frac{1}{8} E_{B_1} + \frac{1}{8} E_{B_2} = \frac{1}{4} E_A + \frac{1}{8} (E_A + E_{B_1}). \tag{20}
\]

Region A

The square region \( A \) consists of \( P \times P = M'/8 \) symbols, and is actually the first quadrant of the squared constellation of size \( M' = M/2 \). Thus, the average energy is

\[
E_A = E_{sq} |_{M' = M/2} = \frac{1}{4} d_0^2 (M/2 - 1). \tag{21}
\]

Regions \( B_1 \) and \( B_2 \)

The \( B_1 \) region consists of \( P/2 \times P = M'/8 = M/16 \) symbols (see Fig. 4). As for the square constellation, we calculate the average energy

\[
E_{B_1} = \frac{1}{M/16} \sum_{i=1}^{M/16} (x_i^2 + y_i^2). \tag{22}
\]

The coordinates are \( \{x_i\} = \{\frac{1}{2}d_0, \frac{3}{2}d_0, \ldots, \frac{2P-1}{2}d_0\} \), and \( \{y_i\} = \{\frac{1}{2}d_0, \frac{3}{2}d_0, \ldots, \frac{2P-1}{2}d_0\} \). Thus,

\[
E_{B_1} = \frac{2}{(2P)^2} \sum_{m=1}^{P} \sum_{n=1}^{P} \left( \frac{\sqrt{M}}{2} (2m-1) \right)^2 + \left( \frac{\sqrt{M}}{2} (2n-1) \right)^2 \\
= \frac{d_0^2}{2P} \left[ P \sum_{n=1}^{P} (2n-1)^2 + P \frac{1}{2} \sum_{m=1}^{P} (2m-1)^2 \right] \\
= \frac{d_0^2}{2P} \left[ \frac{P}{2} \sum_{n=1}^{P} (2n-1)^2 - \frac{1}{2} \sum_{m=1}^{P} (2m-1)^2 \right] \\
= \frac{d_0^2}{2P} \left\{ \frac{1}{8} (\frac{3}{2}P)^2 (4P^2 - 1) - \frac{1}{2} \frac{1}{8} P(4P^2 - 1) \right\} \\
= \frac{d_0^2}{2P} \left\{ \frac{1}{8} (\frac{3}{2}P)^2 (4P^2 - 1) - \frac{1}{8} \frac{1}{8} P(4P^2 - 1) \right\}. \tag{23}
\]
Gathering the last terms, we obtain
\[
E_{e_1} = \frac{1}{\pi} \varphi_0^2 \left( \frac{23}{7} M - 1 \right).
\] (24)

We insert (21) and (24) into (20) to obtain
\[
E_{cr} = \frac{1}{2} (E_A + E_{e_1}) = \frac{1}{6} \varphi_0^2 \left( \frac{31}{32} M - 1 \right).
\] (25)

8-cross constellation

For \( k = 3 \), we cannot obtain a constellation of the form in Fig. 4. Instead, a constellation formed as indicated in Fig. 1 is employed. This constellation is called 8-cross, and an enlarged view of this is shown in Fig. 5.

![Figure 5: Illustration of the 8-cross constellation](image)

Again, due to symmetry, the average energy will be equal to the average energy in the first quadrant. It is readily seen that this is
\[
E_{8\text{-cross}} = \frac{1}{4} \left( (\frac{4}{3} q)^2 + (\frac{4}{3} r)^2 + (\frac{4}{3} u)^2 + (\frac{4}{3} v)^2 \right)
= \frac{3}{2} \varphi_0^2
= \frac{1}{6} \varphi_0^2 \left( \frac{40}{32} M - 1 \right).
\] (26)

The average energy is written on the form (26) to correspond with the expressions for average energy of the larger constellations.

To summarize, we have found that the energy of QAM-constellations are,
\[
E_{avg} = \begin{cases} 
\frac{1}{6} \varphi_0^2 (M - 1) & \text{for even } k \geq 2 \\
\frac{3}{2} \varphi_0^2 \left( \frac{40}{32} M - 1 \right) & \text{for } k = 3 \\
\frac{1}{6} \varphi_0^2 \left( \frac{31}{32} M - 1 \right) & \text{for odd } k \geq 5 
\end{cases}
\] (27)

where the size of the constellation is \( M = 2^k \). Even \( k \) corresponds to square constellations, \( k = 3 \) corresponds to the special 8-cross constellation while odd \( k \geq 5 \) corresponds to the larger cross constellations.

REFERENCES


