Appendix A

Mathematical details

In this appendix, the mathematical details for some of the derivations presented earlier in this dissertation are given. In Appendix A.1 a validation of the existence of the eigen-solutions found earlier is given, in Appendix A.2 the details of the feature variance model are developed, and in Appendix A.3 an approximation of the variance expression useful in the closed form solutions is developed. Finally in Appendix A.4 a compact representation of the variance model is discussed along with an approach to find an upper bound of the Fisher criterion and some ideas for future optimization research.

A.1 Validation of existence of optimal solutions

Are we guaranteed that all solutions to the eigenproblems, Equations (5.23), (5.28), and (5.31), are valid solutions? That is, are we guaranteed that the solution to

$$\left(\mathbf{R}_{x_2x_2}{}^{-1}\mathbf{R}_{x_1x_1}\right)\mathbf{h} = \lambda\mathbf{h} \tag{A.1}$$

will yield a solution vector $\mathbf{h}$ such that

$$\frac{(\mathbf{h}^T\mathbf{R}_{x_1x_1}\mathbf{h})}{(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})} = \lambda, \tag{A.2}$$

where λ is the corresponding eigenvalue? Since $(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})$ is a scalar, λ is unmodified by dividing and multiplying by $(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})$, i.e.,

$$\lambda = \frac{(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})\lambda}{(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})} = \frac{\mathbf{h}^T\mathbf{R}_{x_2x_2}\lambda\mathbf{h}}{(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})}. \tag{A.3}$$

Combining this with Equation (A.1), we get

$$\lambda = \frac{\mathbf{h}^T\mathbf{R}_{x_2x_2}\left(\mathbf{R}_{x_2x_2}{}^{-1}\mathbf{R}_{x_1x_1}\right)\mathbf{h}}{(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})} = \frac{(\mathbf{h}^T\mathbf{R}_{x_1x_1}\mathbf{h})}{(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h})}, \tag{A.4}$$

which is identical to the requirement of Equation (A.2). Hence, all eigensolutions are valid solutions, provided that $\mathbf{R}_{x_2x_2}$ is non-singular and $(\mathbf{h}^T\mathbf{R}_{x_2x_2}\mathbf{h}) \neq 0$.

A.2 Variance expression

For the notations of this appendix, refer to Figure 5.1. The variance of the feature image, $v_i(m,n)$, is

$$\begin{aligned}\sigma_{v_i}^2 &= E\left\{(v_i(m,n)-\mu_{v_i})^2\right\}\\ &= E\left\{v_i^2(m,n)\right\} - 2E\left\{v_i(m,n)\right\}\mu_{v_i} + {\mu_{v_i}}^2\\ &= E\left\{\mathbf{w}^T\mathbf{z}_i(m,n)\mathbf{z}_i^T(m,n)\mathbf{w}\right\} - {\mu_{v_i}}^2\\ &= (\mathbf{w}^T\mathbf{R}_{z_iz_i}\mathbf{w}) - {\mu_{v_i}}^2,\end{aligned} \tag{A.5}$$

where

$$\mathbf{R}_{z_iz_i} = E\left\{\mathbf{z}_i(m,n)\mathbf{z}_i^T(m,n)\right\}. \tag{A.6}$$

The matrix $\mathbf{R}_{z_iz_i}$ is readily constructed from the autocorrelation function of the image $z_i(m,n)$, given by

$$\begin{aligned}r_{z_iz_i}(k,l) &= E\left\{z_i(m,n)z_i(m+k,n+l)\right\}\\ &= E\left\{y_i^2(m,n)y_i^2(m+k,n+l)\right\}.\end{aligned} \tag{A.7}$$

With this fourth order moment, no closed form solution for criteria incorporating the variance has yet been found. However, if we assume that $y_i(m,n)$ is Gaussian, we may write this as [83]

$$r_{z_iz_i}(k,l) = 2{c_{y_iy_i}}^2(k,l) + 4c_{y_iy_i}(k,l){\mu_{y_i}}^2 + {r_{y_iy_i}}^2(0,0), \tag{A.8}$$

where $c_{y_iy_i}(k,l)$ is the covariance function [83] of $y_i(m,n)$. Further

$$\begin{aligned}c_{y_iy_i}(k,l) &= E\left\{(y_i(m,n)-\mu_{y_i})\left(y_i(m+k,n+l)-\mu_{y_i}\right)\right\}\\ &= r_{y_iy_i}(k,l) - {\mu_{y_i}}^2,\end{aligned} \tag{A.9}$$

where $r_{y_iy_i}(k,l)$ is the autocorrelation function of $y_i(m,n)$. Hence, we have

$$\begin{aligned}r_{z_iz_i}(k,l) &= 2\left(r_{y_iy_i}(k,l)-{\mu_{y_i}}^2\right)^2 + 4\left(r_{y_iy_i}(k,l)-{\mu_{y_i}}^2\right){\mu_{y_i}}^2 + {r_{y_iy_i}}^2(0,0)\\ &= 2{r_{y_iy_i}}^2(k,l) - 2{\mu_{y_i}}^4 + {\mu_{v_i}}^2.\end{aligned} \tag{A.10}$$

Furthermore, we have

$$\begin{aligned}r_{y_iy_i}(k,l) &= E\left\{y_i(m,n)y_i(m+k,n+l)\right\}\\ &= E\left\{\mathbf{h}^T\mathbf{x}_i(m,n)\mathbf{x}_i^T(m+k,n+l)\mathbf{h}\right\}\\ &= (\mathbf{h}^T\mathbf{R}_{x_ix_i}(k,l)\mathbf{h}),\end{aligned} \tag{A.11}$$

where

$$\mathbf{R}_{x_ix_i}(k,l) = E\left\{\mathbf{x}_i(m,n)\mathbf{x}_i^T(m+k,n+l)\right\}. \tag{A.12}$$

Note that from Equation (5.13), we have $\mathbf{R}_{x_i x_i} = \mathbf{R}_{x_i x_i}(0,0)$. Combining Equations (A.5), (A.6), (A.10), and (A.11) we are able to estimate the variance of the extracted features, $\sigma_{v_i}^2$. This estimate is based on the autocorrelation function of the texture, $r_{x_i x_i}(m,n)$, the filter, $h(m,n)$, and the smoothing filter $w(m,n)$. Summarizing the derivations of this section, we may write the expression for $\sigma_{v_i}^2$ explicitly as

$$\begin{aligned}\sigma_{v_i}^2 &= (\mathbf{w}^T \mathbf{R}_{z_i z_i} \mathbf{w}) - {\mu_{v_i}}^2 \\ &= \sum_{k_1=0}^{M_w-1} \sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1} \sum_{l_2=0}^{N_w-1} w(k_2,l_2) r_{z_i z_i}(k_1-k_2, l_1-l_2) - {\mu_{v_i}}^2\end{aligned} \tag{A.13}$$

and

$$\sigma_{v_i}^2 = \sum_{k_1=0}^{M_w-1} \sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1} \sum_{l_2=0}^{N_w-1} w(k_2,l_2) 2{r_{y_i y_i}}^2(k_1-k_2, l_1-l_2) - 2{\mu_{y_i}}^4 + {\mu_{v_i}}^2 - {\mu_{v_i}}^2 \tag{A.14}$$

yielding

$$\sigma_{v_i}^2 = 2 \sum_{k_1=0}^{M_w-1} \sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1} \sum_{l_2=0}^{N_w-1} w(k_2,l_2) (\mathbf{h}^T \mathbf{R}_{x_i x_i}(k_1-k_2, l_1-l_2)\mathbf{h})^2 - 2\left(\mu_{x_i} \mathbf{1}^T \mathbf{h}\right)^4, \tag{A.15}$$

where $\mathbf{1} = [1\ 1 \cdots 1]^T$ and $w(m,n)$ is the smoothing filter, which is of size $M_w \times N_w$. A discussion of compact representations for this variance expression is found in Appendix A.4. Using the matrix/vector differentiation rules of Searle [118] we get the partial derivative

$$\begin{aligned}\frac{\partial \sigma_{v_i}^2}{\partial \mathbf{h}} = 8 \sum_{k_1=0}^{M_w-1} \sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1} \sum_{l_2=0}^{N_w-1} w(k_2,l_2) (\mathbf{h}^T \mathbf{R}_{x_i x_i}(k,l)\mathbf{h}) \left(\mathbf{R}_{x_i x_i}(k,l)\mathbf{h} + \mathbf{R}_{x_i x_i}(k,l)^T \mathbf{h}\right) \\ - 8\left(\mu_{x_i} \mathbf{1}^T \mathbf{h}\right)^3 \mu_{x_i} \mathbf{1}.\end{aligned} \tag{A.16}$$

A.3 Approximate variance expression

From Appendix A.2, Equation (A.13), we know that it is possible to write the variance as

$$\sigma_{v_i}^2 = \sum_{k_1=0}^{M_w-1} \sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1} \sum_{l_2=0}^{N_w-1} w(k_2,l_2) r_{z_i z_i}(k_1-k_2, l_1-l_2) - {\mu_{v_i}}^2, \tag{A.17}$$

but as described in Section 5.3.2, this is not found to yield a closed form solution. However, if we approximate $r_{z_i z_i}(k,l)$ by $\hat{r}_{z_i z_i}(k,l)$, where

$$\hat{r}_{z_i z_i}(k,l) = \rho_{i1} \hat{r}_{z_i z_i}(k-1,l) \quad \forall\ k > 0 \tag{A.18}$$

and

$$\hat{r}_{z_i z_i}(k,l) = \rho_{i2}\hat{r}_{z_i z_i}(k,l-1) \;\; \forall\; l > 0, \tag{A.19}$$

where ρ_{i1} and ρ_{i2} are scalar parameters and $\hat{r}_{z_i z_i}(0,0) = r_{z_i z_i}(0,0)$, we may write

$$r_{z_i z_i}(k,l) \approx \hat{r}_{z_i z_i}(k,l) = \rho_{i1}^{|k|}\rho_{i2}^{|l|} r_{z_i z_i}(0,0). \tag{A.20}$$

Note that the approximations in Equation (A.18) and (A.19) correspond to modeling the image as separable and autoregressive (AR) [83] of order one.[1] With this approximation, we may write

$$\begin{aligned}\sigma_{v_i}^2 &\approx \sum_{k_1=0}^{M_w-1}\sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1}\sum_{l_2=0}^{N_w-1} w(k_2,l_2)\rho_{i1}^{|k_1-k_2|}\rho_{i2}^{|l_1-l_2|} r_{z_i z_i}(0,0) - {\mu_{v_i}}^2 \\ &= r_{z_i z_i}(0,0)\sum_{k_1=0}^{M_w-1}\sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1}\sum_{l_2=0}^{N_w-1} w(k_2,l_2)\rho_{i1}^{|k_1-k_2|}\rho_{i2}^{|l_1-l_2|} - {\mu_{v_i}}^2.\end{aligned} \tag{A.21}$$

To keep expressions manageable the following is defined

$$\rho_i = \sum_{k_1=0}^{M_w-1}\sum_{l_1=0}^{N_w-1} w(k_1,l_1) \sum_{k_2=0}^{M_w-1}\sum_{l_2=0}^{N_w-1} w(k_2,l_2)\rho_{i1}^{|k_1-k_2|}\rho_{i2}^{|l_1-l_2|}, \tag{A.22}$$

thus $\sigma_{v_i}^2 \approx \rho_i r_{z_i z_i}(0,0) - {\mu_{v_i}}^2$. Applying (A.10) and assuming zero mean input, we get

$$\begin{aligned}\sigma_{v_i}^2 &\approx 2\rho_i {r_{y_i y_i}}^2(0,0) + {r_{y_i y_i}}^2(0,0) - {\mu_{v_i}}^2 \\ &= 2\rho_i(\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})^2 + (\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})^2 - (\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})^2 \\ &= 2\rho_i(\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})^2.\end{aligned} \tag{A.23}$$

To find an optimal solution, we need the partial derivative of this approximation of $\sigma_{v_i}^2$ with respect to $\mathbf{h}$. As we see, the approximation of $\sigma_{v_i}^2$ consists of two elements, $(\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})^2$ and ρ_i, that are both functions of $\mathbf{h}$. The relative perturbations of $(\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})$ and ρ_i as functions of small perturbations of $\mathbf{h}$ were tested. The ρ_i's were determined to yield minimum deviation between Equations (A.15) and (A.23). These tests indicated that the relative perturbations of $(\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})$ were generally much larger than the relative perturbations of ρ_i. Hence, approximating ρ_i as constant will only perturbate the derivative of $\sigma_{v_i}^2$ minimally and we get the approximation

$$\frac{\partial\sigma_{v_i}^2}{\partial\mathbf{h}} \approx 8\rho_i(\mathbf{h}^T\mathbf{R}_{x_i x_i}\mathbf{h})\mathbf{R}_{x_i x_i}\mathbf{h}. \tag{A.24}$$

When substituted into $\partial J_F/\partial\mathbf{h} = 0$, these approximate expressions yield the approximate optimal closed form solution given in Section 5.3.2.

[1]Generally, making the approximations in Equation (A.20) makes the estimate of $r_{z_i z_i}(k,l)$ less accurate. The approximations are, however, justified by the useful results they give.

A.4 Reformulation of the feature extraction model

For realistically sized filters, the expression for $\sigma_{v_i}^2$, Equation (A.15), contains four large summations. In Appendix A.4.1 an alternate and more compact expression for the feature variance is developed. The development yields a technique for finding an upper bound on the Fisher criterion, detailed in Appendix A.4.2. In Appendix A.4.3, a discussion on the application of this approach for one-dimensional optimal filter design is given.

A.4.1 Alternate expressions for feature mean and variance

In the development of the feature variance estimator, Section A.2, it was assumed that

$$y_i(m,n) = h(m,n) * x_i(m,n), \tag{A.25}$$

where $*$ represents two-dimensional convolution. Furthermore

$$\begin{aligned} r_{y_iy_i}(k,l) &= E\left\{y_i(m,n)y_i(m+k,n+l)\right\} \\ &\approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} y_i(m,n)y_i(m+k,n+l) \\ &= y_i(k,l) * y_i(-k,-l) \\ &= h(k,l) * x_i(k,l) * h(-k,-l) * x(-k,-l) \\ &= \left(h\left(k,l\right) * h(-k,-l)\right) * \left(x_i(k,l) * x(-k,-l)\right) \\ &= r_{hh}(k,l) * r_{x_ix_i}(k,l), \end{aligned} \tag{A.26}$$

known as the Wiener-Lee theorem [65, page 85]. As has been illustrated many times, it is possible to write a 2D convolution as a vector multiplication. Consequently, the above expression may be rewritten

$$r_{y_iy_i}(k,l) = \mathbf{r}_{hh}{}^T\mathbf{r}_{x_ix_i}(k,l). \tag{A.27}$$

Combining this with Equations (5.12) and (A.11), we get

$$\mu_{v_i} = r_{y_iy_i}(0,0) = \mathbf{r}_{hh}{}^T\mathbf{r}_{x_ix_i}(0,0). \tag{A.28}$$

Using Equation (A.10) we get

$$\begin{aligned} r_{z_iz_i}(k,l) &= 2r_{y_iy_i}{}^2(k,l) - 2\mu_{y_i}{}^4 + \mu_{v_i}{}^2 \\ &= 2\left(\mathbf{r}_{hh}{}^T\mathbf{r}_{x_ix_i}(k,l)\right)^2 - 2\mu_{y_i}{}^4 + \mu_{v_i}{}^2 \\ &= 2\mathbf{r}_{hh}{}^T\mathbf{r}_{x_ix_i}(k,l)\mathbf{r}_{x_ix_i}{}^T(k,l)\mathbf{r}_{hh} - 2\mu_{y_i}{}^4 + \mu_{v_i}{}^2 \\ &= 2\mathbf{r}_{hh}{}^T\mathbf{R}_{r_ir_i}(k,l)\mathbf{r}_{hh} - 2\mu_{y_i}{}^4 + \mu_{v_i}{}^2. \end{aligned} \tag{A.29}$$

Hence, using Equation (A.17) and remembering that it is assumed that $\sum w(k,l) = 1$, we get

$$\begin{aligned}
\sigma_{v_i}^2 &= \sum_{k_1=0}^{M_w-1}\sum_{l_1=0}^{N_w-1} w(k_1,l_1)\sum_{k_2=0}^{M_w-1}\sum_{l_2=0}^{N_w-1} w(k_2,l_2) r_{z_i z_i}(k_1-k_2,l_1-l_2) - {\mu_{v_i}}^2 \\
&= \sum_{k_1=0}^{M_w-1}\sum_{l_1=0}^{N_w-1} w(k_1,l_1)\sum_{k_2=0}^{M_w-1}\sum_{l_2=0}^{N_w-1} w(k_2,l_2) 2{\mathbf{r}_{hh}}^T \mathbf{R}_{r_i r_i}(k_1-k_2,l_1-l_2)\mathbf{r}_{hh} - 2{\mu_{y_i}}^4 + {\mu_{v_i}}^2 - {\mu_{v_i}}^2 \\
&= {\mathbf{r}_{hh}}^T \left(\sum_{k_1=0}^{M_w-1}\sum_{l_1=0}^{N_w-1} w(k_1,l_1)\sum_{k_2=0}^{M_w-1}\sum_{l_2=0}^{N_w-1} w(k_2,l_2) 2\mathbf{R}_{r_i r_i}(k_1-k_2,l_1-l_2)\right)\mathbf{r}_{hh} - 2{\mu_{y_i}}^4.
\end{aligned} \tag{A.30}$$

The huge matrix summation in the expression above is purely input signal dependent. Hence, if we denote the matrix sum by $\mathbf{A}_{x_i}$, we may write

$$\sigma_{v_i}^2 = {\mathbf{r}_{hh}}^T \mathbf{A}_{x_i}\mathbf{r}_{hh} - 2{\mu_{y_i}}^4 \tag{A.31}$$

and if we assume zero mean input, we get

$$\sigma_{v_i}^2 = {\mathbf{r}_{hh}}^T \mathbf{A}_{x_i}\mathbf{r}_{hh}. \tag{A.32}$$

A.4.2 Upper bound on the Fisher criterion

The derivatives of the mean and the variance with respect to $\mathbf{r}_{hh}$ are

$$\frac{\partial \mu_{v_i}}{\partial \mathbf{r}_{hh}} = \mathbf{r}_{x_i x_i}(0,0) \tag{A.33}$$

and

$$\frac{\partial \sigma_{v_i}^2}{\partial \mathbf{r}_{hh}} = {\mathbf{A}_{x_i}}^T \mathbf{r}_{hh} + \mathbf{A}_{x_i}\mathbf{r}_{hh} = 2\mathbf{A}_{x_i}\mathbf{r}_{hh}, \tag{A.34}$$

respectively. Then the derivative of the Fisher criterion becomes

$$\begin{aligned}
\frac{\partial J(\mathbf{r}_{hh})}{\partial \mathbf{r}_{hh}} &= \frac{2\left(\mu_{v_1}-\mu_{v_2}\right)\left(\frac{\partial \mu_{v_1}}{\partial \mathbf{r}_{hh}} - \frac{\partial \mu_{v_2}}{\partial \mathbf{r}_{hh}}\right)\left(\sigma_{v_1}^2+\sigma_{v_2}^2\right)}{\left(\sigma_{v_1}^2+\sigma_{v_2}^2\right)^2} - \frac{\left(\mu_{v_1}-\mu_{v_2}\right)^2\left(\frac{\partial \sigma_{v_1}^2}{\partial \mathbf{r}_{hh}} + \frac{\partial \sigma_{v_2}^2}{\partial \mathbf{r}_{hh}}\right)}{\left(\sigma_{v_1}^2+\sigma_{v_2}^2\right)^2} \\
&= \frac{2\left(\mu_{v_1}-\mu_{v_2}\right)\left(\mathbf{r}_{x_1x_1}(0,0)-\mathbf{r}_{x_2x_2}(0,0)\right)\left(\sigma_{v_1}^2+\sigma_{v_2}^2\right)}{\left(\sigma_{v_1}^2+\sigma_{v_2}^2\right)^2} \\
&\quad - \frac{\left(\mu_{v_1}-\mu_{v_2}\right)^2\left(2\mathbf{A}_{x_1}\mathbf{r}_{hh}+2\mathbf{A}_{x_2}\mathbf{r}_{hh}\right)}{\left(\sigma_{v_1}^2+\sigma_{v_2}^2\right)^2}.
\end{aligned} \tag{A.35}$$

Equating to zero yields

$$\left(\mathbf{r}_{x_1x_1}(0,0)-\mathbf{r}_{x_2x_2}(0,0)\right)\left(\sigma_{v_1}^2+\sigma_{v_2}^2\right) = \left(\mu_{v_1}-\mu_{v_2}\right)\left(\mathbf{A}_{x_1}+\mathbf{A}_{x_2}\right)\mathbf{r}_{hh} \tag{A.36}$$

and consequently the optimal $\mathbf{r}_{hh}$ is

$$\mathbf{r}_{hh} = \frac{\sigma_{v_1}^2 + \sigma_{v_2}^2}{\mu_{v_1} - \mu_{v_2}} \left(\mathbf{A}_{x_1} + \mathbf{A}_{x_2}\right)^{-1} \left(\mathbf{r}_{x_1 x_1}(0,0) - \mathbf{r}_{x_2 x_2}(0,0)\right). \tag{A.37}$$

The factor $\left(\sigma_{v_1}^2 + \sigma_{v_2}^2\right) / \left(\mu_{v_1} - \mu_{v_2}\right)$ is a scalar and multiplying $\mathbf{r}_{hh}$ by a scalar is equivalent to multiplying $h(m,n)$ by a scalar. Since it may easily be shown that scalar multiplication with $h(m,n)$ do not change the Fisher criterion value, the scalar may be neglected in the optimization.

Note that given a *valid* $\mathbf{r}_{hh}$, finding the corresponding filter $h(m,n)$ is a question of solving an equation set with the coefficients of $h(m,n)$ as the unknowns. However, not every $\mathbf{r}_{hh}$ corresponds to a filter $h(m,n)$. The computation of $\mathbf{r}_{hh}$, which for filters of quadratic size has almost four times the number of coefficients of $h(m,n)$, imposes complex symmetry properties. These properties imply that only particular $\mathbf{r}_{hh}$'s do correspond to any filter $h(m,n)$.

Although an optimal filter design approach using these derivations has not been found, it must still be noted that the criterion value corresponding to $\mathbf{r}_{hh}$ in Equation (A.37) is the maximally attainable criterion value for any $\mathbf{r}_{hh}$. Hence it is an *upper bound* for the Fisher criterion for any filter $h(m,n)$.

A.4.3 One-dimensional or separable optimal filter design

In the case of a one-dimensional filter $h(m,n)$, i.e., a filter where the spatial extent either vertically or horizontally is one, the symmetry relation on $\mathbf{r}_{hh}$ is much simpler. In this case, an optimal $\mathbf{r}_{hh}$ corresponding to to a filter $h(m,n)$ may be found. This *may* lead to an optimal solution for one-dimensional filters (applicable for one-dimensional textures or as separable filters for two-dimensional textures). However, this approach involves the non-trivial problem of solving a large quadratic equation set. This problem is not solved in this dissertation, but still the formulation is given for completeness and as an introduction for further research.

For a real valued, one-dimensional filter $h(m),\ m = 0 \cdots M-1$, the correlation function is defined to be

$$r_{hh}(k) = E\left\{h(m)h(m+k)\right\}. \tag{A.38}$$

Then it follows that $r_{hh}(k) = r_{hh}(-k)$, hence $r_{hh}(k)$ is symmetric around zero. Since, in the one-dimensional case, $\mathbf{r}_{hh}$ is defined by

$$\mathbf{r}_{hh} = \left[r_{hh}\left(-(M-1)\right) \cdots r_{hh}(M-1)\right]^T, \tag{A.39}$$

it then follows that $\mathbf{r}_{hh}$ is symmetric. If we define

$$\mathbf{r}_{hh}' = \left[r_{hh}\left(-(M-1)\right) \cdots r_{hh}(0)\right]^T, \tag{A.40}$$

we may rewrite the expressions for the mean and variance as

$$\mu_{v_i} = r_{y_i y_i}(0,0) = \mathbf{r}_{hh}'^T \mathbf{r}_{x_i x_i}'(0,0) \tag{A.41}$$

and

$$\sigma_{v_i}^2 = \mathbf{r}_{hh}'^T \mathbf{A}_{x_i}' \mathbf{r}_{hh}, \tag{A.42}$$

The transformations from $\mathbf{r}_{x_i x_i}(0,0)$ to $\mathbf{r}_{x_i x_i}'(0,0)$ and from $\mathbf{A}_{x_i}$ to $\mathbf{A}_{x_i}'$ are trivial. Note that we now have a vector $\mathbf{r}_{hh}'$ with the same number of elements as there are coefficients, M, in the filter we want to optimize.

Inserting the new expressions for the mean and variance into the Fisher criterion expression, differentiating with respect to $\mathbf{r}_{hh}'$ and equating to zero, yields

$$\mathbf{r}_{hh}' = \frac{\sigma_{v_1}^2 + \sigma_{v_2}^2}{\mu_{v_1} - \mu_{v_2}} \left(\mathbf{A}_{x_1}' + \mathbf{A}_{x_2}'\right)^{-1} \left(\mathbf{r}_{x_1 x_1}'(0,0) - \mathbf{r}_{x_2 x_2}'(0,0)\right). \tag{A.43}$$

This, as we see, is on the same form as the previous solution, Equation (A.37). We now have a scheme for optimization of one-dimensional texture filters according to the Fisher criterion. The only assumptions made in this solution are that the filter response, y, is Gaussian and that the input textures are wide sense stationary. The issue of finding $h(m,n)$ from $\mathbf{r}_{hh}'$ still remains to be solved, a problem involving the solution of the large quadratic equation set given by $r_{hh}(k,l) = h(k,l) * h(-k,-l)$.