GLOBAL WEAK SOLUTIONS FOR A VISCOUS LIQUID-GAS MODEL WITH TRANSITION TO SINGLE-PHASE GAS FLOW AND VACUUM

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Abstract. This work deals with a viscous two-phase liquid-gas model relevant for flow in wells and pipelines. The liquid is treated as an incompressible fluid whereas the gas is assumed to be polytropic. The model is rewritten in terms of Lagrangian coordinates and is studied in a free boundary setting where the liquid and gas masses are of compact support initially, and continuous at the boundary. Consequently, the initial masses involve transition to single phase gas flow and vacuum at the boundary. An appropriate balance between pressure and viscous forces is identified which allows to obtain pointwise upper and lower estimates of masses. These estimates rely on the assumption of a certain relation between rate of degeneracy of the viscosity coefficient and the rate that determines how fast the initial masses are vanishing at the boundary. By combining these estimates with basic energy type of estimates, higher order regularity estimates are obtained. Existence of global weak solutions is then proved by showing compactness for a class of semi-discrete approximations.

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1. Introduction

We are interested in a one-dimensional two-phase liquid-gas model of the drift-flux type [1, 15, 26]. This model is frequently used to simulate unsteady, compressible flow of liquid and gas in pipes and wells [27, 9, 21, 12]. The model consists of two mass conservation equations corresponding to each of the two phases, and one equation for the conservation of the mixture momentum. More precisely, it is given in the following form:

$$
\begin{gathered}
\partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] = 0 \\
\partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] = 0 \\
\partial_t[\alpha_l\rho_l u_l + \alpha_g\rho_g u_g] + \partial_x[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + P] = -q + \partial_x[\varepsilon\partial_x u_{mix}], \quad u_{mix} = \alpha_g u_g + \alpha_l u_l,
\end{gathered}
\tag{1}
$$

where $P, \varepsilon \geq 0$. Unknown variables are liquid and gas densities ρ_l, ρ_g and volume fractions $\alpha_l, \alpha_g \in [0,1]$ satisfying the fundamental relation

$$\alpha_g + \alpha_l = 1. \tag{2}$$

Furthermore, velocities of liquid and gas are represented by u_l, u_g whereas P is common pressure for both phases. Finally, q is representing external forces like gravity and friction. The momentum is given only for the mixture, therefore an additional closure law is needed, a so-called hydrodynamical closure law, which connects the two phase velocities. In addition, we need a thermodynamical equilibrium model which specifies the fluid properties.

Few results concerning existence, uniqueness, and stability seem to exist for two-phase liquid-gas models of the form (1). Compared to the single-phase Navier-Stokes model, several new and challenging problems occur when two phases of totally different character are introduced in one

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and the same model. Thus, in the works [7, 8] we have focused on a simplified model obtained by assuming that fluid velocities are equal $u_g = u_l = u$ and by neglecting the external forces, i.e., $q = 0$. More precisely, we considered a model in the form

$$\begin{aligned} \partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u] &= 0 \\ \partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u] &= 0 \\ \partial_t[\alpha_l\rho_l u] + \partial_x[\alpha_l\rho_l u^2] + \partial_x P &= \partial_x[\varepsilon\partial_x u], \qquad P, \varepsilon \geq 0. \end{aligned} \tag{3}$$

Here certain small gas effects have been neglected since we employ a simplified momentum equation where acceleration terms depend solely on the liquid phase. This is motivated by the fact that liquid phase density typically is much higher than gas phase density, i.e. $\rho_l/\rho_g = O(10^3)$.

The purpose of this paper is to continue the work with the model (3). Assuming that the liquid is incompressible, i.e. $\rho_l = \text{Const}$, whereas the gas is represented by a polytropic gas law relation $P = C\rho_g^\gamma$ with $\gamma > 1$ and C a positive constant, we get a pressure law of the form

$$P(n, m) = C\rho_l^\gamma \Big(\frac{n}{\rho_l - m}\Big)^\gamma. \tag{4}$$

This follows by observing that $\rho_g = \frac{\rho_l n}{\rho_l - m}$, where we use the notation $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$ as well as the relation (2). In particular, we see that pressure becomes singular at transition to pure liquid phase $\alpha_l = 1$ which yields $m = \rho_l$. In order to treat this difficulty we shall in this work consider (3) in a free boundary problem setting where the masses m and n initially occupy only a finite interval $[a, b] \subset \mathbb{R}$. That is,

$$n(x, 0) = n_0(x) \geq 0, \qquad m(x, 0) = m_0(x) \geq 0, \qquad u(x, 0) = u_0(x), \qquad x \in [a, b],$$

and $n_0 = m_0 = 0$ outside $[a, b]$. In particular, the initial masses n_0, m_0 are assumed to be positive in (a, b) and vanish at the boundary $x = a, b$. The viscosity coefficient ε is assumed to be a functional of the masses n and m, i.e. $\varepsilon = \varepsilon(n, m)$.

We rewrite the model (3) in terms of Lagrangian variables. An advantage is that the free boundaries are then converted into fixed and we get a model in the form (see next section for more details)

$$\begin{aligned} &\partial_t n + (nm)\partial_x u = 0 \\ &\partial_t m + m^2\partial_x u = 0 \\ &\partial_t u + \partial_x P(n, m) = \partial_x(\varepsilon(n, m)m\partial_x u), \qquad x \in (0, 1), \end{aligned} \tag{5}$$

with boundary conditions

$$n = m = 0, \qquad \text{at } x = 0, 1, \quad t \geq 0, \tag{6}$$

and initial data

$$n(x, 0) = n_0(x), \quad m(x, 0) = m_0(x), \quad u(x, 0) = u_0(x), \qquad x \in [0, 1]. \tag{7}$$

In [8], the initial masses n_0 and m_0 were assumed to be connected to vacuum at the boundary with a discontinuity. Then, it was shown that the masses n, m remained strictly positive in time up to the vacuum interface $x = 0, 1$. This estimate then played a crucial role in the study of global existence of weak solutions. By contrast, in this work initial masses n, m connect to vacuum continuously through the boundary condition (6), in other words, they vanish at the boundary where the vacuum interface is located. Consequently, more refined arguments are required to obtain a priori estimates that ensure existence of weak solutions. In particular, the analysis of this paper reveals that a certain relation must exist between rate of degeneracy associated with the viscosity coefficient $\varepsilon(n, m)$ and the rate of degeneracy of the initial masses n_0, m_0 at the boundary $x = 0, 1$, i.e. the rate that determines how fast the initial data is approaching zero at the boundary.

The main result of this paper is that we obtain an existence result for the model (5)–(7) for a class of weak solutions and for a flow regime where the viscosity coefficient is of the form

$$\varepsilon = \varepsilon(n, m) = \frac{n^\beta}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0, 1/3). \tag{8}$$

This form is quite natural in view of the pressure law (4) and ensures a certain balance between the pressure and viscous forces, represented respectively by (4) and (8), as m is approaching the critical limit ρ_l. This balance is sufficient to guarantee that the liquid mass m can be controlled pointwise from below and from above by means of a weight function $\phi(x) = x(1-x)$ when initial data is subject to a similar behavior. This pointwise control is then transferred to the gas mass n through the common fluid velocity u and the two mass conservation equations of (5). More precisely, by assuming initially that the gas and liquid masses n and m do not disappear or blow up on $(0,1)$, but degenerate at the boundary points at a certain rate $\alpha \in (0,1)$, given by

$$C^{-1}\phi(x)^{\alpha} \leq n_0(x) \leq C\phi(x)^{\alpha}, \qquad C^{-1}\phi(x)^{\alpha/2} \leq m_0(x) \leq C\phi(x)^{\alpha/2} < \rho_l, \tag{9}$$

for a suitable constant $C > 0$, then a similar behavior will be true for the masses n and m for all $t \in [0,T]$ for any specified time $T > 0$. We refer to Theorem 2.1 for a precise statement. This in turn allows us to obtain various estimates which ensure convergence to a class of weak solutions. The main tool in this analysis is the introduction of a suitable variable transformation allowing for application of ideas and techniques similar to those used in [23, 17, 19, 29, 24, 31, 28, 16] in previous studies of the single-phase Navier-Stokes equation.

The novelty of this paper compared to [8] can be summarized as follows:

- The model studied in this paper is different from the one studied in [8] in the sense that the ε coefficient now depends on both n and m together with the assumption of continuous masses n, m at the boundaries. More precisely, in [8] we considered a flow regime where viscosity was governed by the presence of the liquid phase only by using

$$\varepsilon(m) = \frac{m^{\beta}}{(\rho_l - m)^{\beta+1}}. \tag{10}$$

  For this case, the liquid mass m and gas mass n were shown to be non-vanishing at all points in the domain $[0,1]$. The current work focus on a flow regime where the viscosity to a large extent is governed by the presence of the gas phase as expressed by the functional

$$\varepsilon(n,m) = \frac{n^{\beta}}{(\rho_l - m)^{\beta+1}}. \tag{11}$$

  In particular, the boundary condition (6) implies that the viscosity coefficient (11) vanishes at the boundary $x = 0, 1$.
- The model (5)–(7) involves transition to single-phase gas flow at the boundary since $\alpha_l|_{x=0,1} = 0$. This situation is not included in the analysis of [8].
- The analysis of Section 3 shows that there is a fine balance between (i) the rate of degeneracy at the boundary associated with initial data n_0, m_0 and represented by a function of the form $\phi(x) = (x(1-x))^{\alpha}$; and (ii) the rate of degeneracy associated with the viscosity coefficient (11). More precisely, a specific relation between α and β is identified for $\beta \in (0, 1/3)$ and $\alpha \in (0,1)$. This may reflect some of the additional difficulty associated with two-phase flow compared to single-phase flow. It also represents a clear difference between the present work and the two-phase model studied in [8].

To motivate for further studies in the context of two-phase liquid-gas flow, we would like to emphasize some of the restrictions that are used in this work:

- Relatively strong smoothness assumptions are made on the initial data and weaker conditions would clearly be desirable, i.e. analysis that allows for discontinuous initial data. For some results in this direction in the single-phase gas flow setting, we refer to [13, 14] and references therein.
- A relative strong restriction on the rate of degeneracy associated with the viscous coefficient $\varepsilon(n,m)$ is assumed since $\beta \in (0, 1/3)$. It would be interesting to explore the model used in this paper in a framework where the upper limit of β can be relaxed.
- The assumptions on the initial data n_0, m_0 are rather restrictive concerning the rate of degeneracy at the boundary represented by the parameter $\alpha \in (0,1)$. This is due to the fact that we make use of the assumption that $c_0(x) = \frac{n_0(x)}{m_0(x)}$, in view of (9), can be

bounded as

$$C_1\phi(x)^{\alpha/2} \le c_0(x) \le C_2\phi(x)^{\alpha/2}.$$

Weaker conditions are clearly of interest.

- Apparently, a consequence of the techniques we rely on in this work is that we do not cover the mixed situation where the liquid phase does not vanish at the boundaries whereas the gas mass does vanish. In other words, the transition to single-phase gas-flow at the boundary seems inevitable, see Remark 3.1 for more details.
- The existence result presented in Section 2 relies on the fact that the α parameter is related to β in a certain manner. More precisely, for a choice of $\beta \in (0, 1/3)$, the α parameter cannot be chosen freely in $(0, 1)$ but must obey the relation (34). An interesting question is whether this is only an artifact of the techniques used in this work, or reflects a more intimate connection between degeneracy of the viscosity coefficient and behavior of initial masses at the boundary.
- The form of the viscous term represented by (11) is not based on experimental data or some deeper physical considerations, but represents a rather straightforward generalization in view of the relation between the functional form of pressure and viscosity as used for single-phase gas flow [23, 17, 19, 29, 24, 31, 28, 16]. Clearly, it is of interest to address this aspect more carefully in a context where experimental data is considered.
- It would be desirable to consider a more complete two-phase model where unequal fluid velocities are involved. An example of such a two-phase drift-flux model written in terms of Lagrangian variables is given in [12]. Various numerical methods for solving the model (1) in such a more general context can be found in [3, 4, 5, 9, 10, 11, 18, 20, 22, 27].

The rest of this paper is organized as follows. In Section 2 we give more details relevant for the model (5) obtained from (3), and we state various assumptions and the main theorem. In Section 3 we describe a priori estimates for an auxiliary model obtained from (5) by employing an appropriate variable transformation. Finally, in Section 4 we consider a family of approximate solutions obtained by defining a semi-discrete approximation to (5). The estimates of Section 3 are shown to hold for these approximate solutions, which in turn imply compactness and convergence to a global weak solution, as stated in Theorem 2.1.

2. A global existence result for a simplified viscous two-phase model

In the following we shall work with the compressible gas-incompressible liquid two-phase model

$$\begin{aligned} \partial_t n + \partial_x[nu] &= 0 \\ \partial_t m + \partial_x[mu] &= 0 \\ \partial_t[mu] + \partial_x[mu^2] + \partial_x P(n,m) &= \partial_x[\varepsilon(n,m)\partial_x u], \end{aligned} \tag{12}$$

where

$$P(n,m) = A\Big(\frac{n}{\rho_l - m}\Big)^{\gamma}, \qquad \gamma > 1, \tag{13}$$

$$\varepsilon(n,m) = B\frac{n^{\beta}}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0, 1/3), \tag{14}$$

where A and B are appropriate constants. One special feature of the above two-phase model (12)–(14) is that the pressure law becomes singular for pure liquid flow, that is, when $m = \rho_l\alpha_l = \rho_l$. To compensate for this, it is assumed that the viscosity coefficient $\varepsilon(n,m)$ reflects a similar behavior such that a proper balance between pressure and viscous forces takes place.

2.1. **Main idea.** The idea of this paper is to study the model (12)–(14) in a setting where sufficient pointwise control on the masses n and m can be ensured through a careful balance between pressure and viscous forces. Motivated by previous studies of the single-phase Navier-Stokes model [19, 31, 28, 25], we propose to study (12) in a free-boundary setting where the gas and liquid masses n and m are of compact support initially. More precisely, we study the model (12) with initial data

$$(n,m,mu)(x,0)=\begin{cases}(n_0,m_0,m_0u_0) & x\in[a,b],\\ (0,0,0) & \text{otherwise,}\end{cases}$$

with $n_0(x)>0$ and $m_0(x)>0$ for $x\in(a,b)$. In other words, we study the two-phase model in a setting where an initial true two-phase mixture region (a,b) is surrounded by vacuums states $n=m=0$ on both sides. Letting $a(t)$ and $b(t)$ denote the particle paths initiating from $(a,0)$ and $(b,0)$ respectively, these paths represent free boundaries, i.e., the interface of the gas-liquid mixture and the vacuum. They are given by

$$\frac{d}{dt}a(t)=u(a(t),t),\qquad \frac{d}{dt}b(t)=u(b(t),t),\tag{15}$$

together with an appropriate boundary condition. In [8] we studied (15) together with the boundary condition

$$\left(-P(n,m)+\varepsilon(m)u_x\right)(a(t)^+,t)=0,\qquad \left(-P(n,m)+\varepsilon(m)u_x\right)(b(t)^-,t)=0,\tag{16}$$

where $\varepsilon(m)$ was given by (10). In particular, it was assumed that the initial masses $n_0(x),m_0(x)$ connect to vacuum *discontinuously*, i.e., $\inf_{[0,1]}n_0(x),\inf_{[0,1]}m_0(x)\geq C_0>0$ for a positive constant C_0.

A main purpose of this work is to study the case where $n_0(x),m_0(x)$ connect to vacuum *continuously*, as described in [19, 31] in the context of single-phase Navier-Stokes equations. This means that the boundary condition (16) is replaced by

$$n(a(t),t)=n(b(t),t)=0,\qquad m(a(t),t)=m(b(t),t)=0.\tag{17}$$

Following along the line of previous studies for the single-phase Navier-Stokes equations [23, 17, 19, 31], it is convenient to replace the free boundaries $a(t)$ and $b(t)$ (which are unknown in Eulerian coordinates) by fixed boundaries using Lagrangian coordinates. First, in view of the particle paths $X_t(x)$ given by

$$\frac{dX_t(x)}{dt}=u(X_t(x),t),\qquad X_0(x)=x,$$

the system (12) takes the form

$$\begin{aligned}\frac{dn}{dt}+nu_x&=0\\ \frac{dm}{dt}+mu_x&=0\\ m\frac{du}{dt}+P(n,m)_x&=(\varepsilon(n,m)u_x)_x.\end{aligned}\tag{18}$$

Next, we introduce the coordinate transformation

$$\xi=\int_{a(t)}^{x}m(y,t)\,dy,\qquad \tau=t,\tag{19}$$

such that the free boundary $x=a(t)$ and $x=b(t)$, in terms of the (ξ,τ) coordinate system, are given by

$$\xi_a(\tau)=0,\qquad \xi_b(\tau)=\int_{a(t)}^{b(t)}m(y,t)\,dy=\int_a^b m_0(y)\,dy=\text{const},\tag{20}$$

where $\int_a^b m_0(y)\,dy$ is the total liquid mass initially, which we normalize to 1. Applying (19) to shift from (x,t) to (ξ,τ) in (18), we get

$$\begin{aligned}n_\tau+(nm)u_\xi&=0\\ m_\tau+(m^2)u_\xi&=0\\ u_\tau+P(n,m)_\xi&=(\varepsilon(n,m)mu_\xi)_\xi,\qquad \xi\in I:=(0,1),\quad \tau\geq 0,\end{aligned}$$

where boundary conditions, in light of (17), are given by

$$n(0,\tau)=n(1,\tau)=0,\qquad m(0,\tau)=m(1,\tau)=0.$$

In addition, we have the initial data

$$n(\xi,0)=n_0(\xi),\quad m(\xi,0)=m_0(\xi),\quad u(\xi,0)=u_0(\xi),\qquad \xi\in\bar{I}:=[0,1].$$

In the following, we find it convenient to replace the coordinates (ξ,τ) by (x,t) such that the model we shall work with in the rest of this paper is given in the form

$$\begin{aligned}
&\partial_t n+(nm)\partial_x u=0\\
&\partial_t m+m^2\partial_x u=0\\
&\partial_t u+\partial_x P(n,m)=\partial_x(E(n,m)\partial_x u),\qquad x\in(0,1),
\end{aligned}\tag{21}$$

with

$$P(n,m)=\Big(\frac{n}{\rho_l-m}\Big)^{\gamma},\qquad \gamma>1,\tag{22}$$

and

$$E(n,m)=\frac{mn^{\beta}}{(\rho_l-m)^{\beta+1}},\qquad 0<\beta<1/3,\tag{23}$$

where we, for simplicity, have set the constants $A=B=1$. Moreover, boundary conditions are given by

$$n(0,t)=n(1,t)=0,\qquad m(0,t)=m(1,t)=0,\tag{24}$$

whereas initial data are

$$n(x,0)=n_0(x),\quad m(x,0)=m_0(x),\quad u(x,0)=u_0(x),\qquad x=\bar{I}=[0,1].\tag{25}$$

Both initial masses $n_0(x),m_0(x)$ become zero at the boundary $x=0,1$. But it is also essential, as stated in a precise manner in assumption (A1) below, that the initial gas mass $n_0(x)$ is approaching zero *faster* than the initial liquid mass $m_0(x)$.

2.2. **Main result.** Before we state the main result for the model (21)–(25), we describe the notation we apply throughout the paper. $H^1(I)$ represents the usual Sobolev space defined over $I=(0,1)$ with norm $\|\cdot\|_{H^1(I)}$. Moreover, $L^p(K,B)$ with norm $\|\cdot\|_{L^p(K,B)}$ denotes the space of all strongly measurable, pth-power integrable functions from K to B where K typically is a subset of $\mathbb{R}$ and B is a Banach space. In addition, let $\alpha\in(0,1)$, $C^{\alpha}[0,1]$ denotes the Banach space of functions on $[0,1]$ which are uniformly Hölder continuous with exponent α and $C^{\alpha,\alpha/2}(D_T)$ for the Banach space of functions on $D_T=[0,1]\times[0,T]$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t.

In the following, we first state some main assumptions for the initial data n_0,m_0, and u_0 and the constants γ and β relevant for $P(n,m)$ and $E(n,m)$. Then we present the global existence result.

Main assumptions.

(A1) We assume that there are constants K_1,K_2,K_3, and K_4 such that for $\phi(x)=x(1-x)$ we have

$$\begin{aligned}
&K_1\phi(x)^{\alpha/2}\le m_0(x)\le K_2\phi(x)^{\alpha/2}<\rho_l,\qquad 0<\alpha<1,\\
&\text{such that }\sup_{x\in[0,1]} m_0(x)<\rho_l,\\
&K_3\phi(x)^{\alpha}\le n_0(x)\le K_4\phi(x)^{\alpha}.
\end{aligned}\tag{26}$$

In particular, this implies that for suitable constants C_1 and C_2

$$C_1\phi(x)^{\alpha/2}\le c_0(x):=\frac{n_0}{m_0}(x)\le C_2\phi(x)^{\alpha/2}.\tag{27}$$

Moreover, in view of (26), it also follows that

$$\frac{K_1}{\rho_l}\phi(x)^{\alpha/2}\le\frac{m_0(x)}{\rho_l}\le Q_0(x):=Q(m_0(x))\le\frac{K_2}{\inf(\rho_l-m_0(x))}\phi(x)^{\alpha/2},\tag{28}$$

for $Q(x)=\frac{x}{\rho_l-x}$. We also note that the following estimate holds, in view of (26)

$$\phi(x)^{k_1}\Big(\frac{\rho_l-m_0(x)}{m_0(x)}\Big)\le\phi(x)^{k_1-\alpha/2}\frac{\sup(\rho_l-m_0(x))}{K_1}\in L^1([0,1]),\qquad \text{for } k_1>\frac{1}{2k},\tag{29}$$

for all positive integer k since $k_1 - \alpha/2 > -1$;

(A2) More assumptions that are required for the analysis are:

$$\Big(\Big[\frac{n_0}{\rho_l - m_0}\Big]^{\beta}\Big)_x \in L^{2k}([0,1]), \qquad \Big(\Big[\frac{n_0}{\rho_l - m_0}\Big]^{\gamma}\Big)_x \in L^{2}([0,1]), \tag{30}$$

for sufficiently large positive integer k. In addition, we require that

$$c_{0,x} \in L^{\infty}([0,1]); \tag{31}$$

(A3) We assume that

$$u_0(x) \in L^{\infty}([0,1]) \quad \text{and} \quad (E(n_0,m_0)u_{0,x})_x \in L^2([0,1]); \tag{32}$$

(A4) We assume that

$$0 < \beta < \frac{1}{3}, \quad \gamma > 1; \tag{33}$$

(A5) We assume that the choice of α and β, for sufficiently large k, satisfy the relation

$$\frac{2k\beta + 1}{(2k-1) - 2k\beta} < \frac{1}{2\beta} - 1 - \frac{\alpha}{2} - \frac{1}{(2k-1)\beta}. \tag{34}$$

Some comments are in order here before we present the main result.

Remark 2.1. *We observe that $m_0|_{x=0,1} = 0$ implies that $\alpha_g|_{x=0,1} = 1$, i.e., transition to single-phase gas flow occur at the left and right boundary point. However, vacuum states exist at the boundaries since $n_0|_{x=0,1} = \rho_g|_{x=0,1} = 0$. In addition, we observe that $\sup_{x\in[0,1]} m_0(x) < \rho_l$ implies that the gas phase is present at all points in the domain (no transition to single-phase liquid flow).*

Remark 2.2. *Sharper lower estimates are required for the masses n_0 and m_0, as described by* (26) *of assumption (A1), compared to previous studies of a single-phase gas flow model* [31, 28]. *For single-phase flow it is typically only required that $0 \le \rho_0(x) \le C\phi(x)^{\alpha}$ for a suitable choice of C and α where ρ represents the gas density. The two-phase analysis requires an estimate of the rate of degeneracy for the lower bounds of n_0 and m_0, as described by* (26)*, such that the estimate* (27) *is obtained. Estimate* (27) *is used throughout the whole analysis.*

Remark 2.3. *Concerning the relation* (34)*, we see that letting k go to infinity we get the relation*

$$\frac{\beta}{1-\beta} < \frac{1}{2\beta} - 1 - \frac{\alpha}{2},$$

that is,

$$1 + \frac{\alpha}{2} < \frac{(1-2\beta)(1+\beta)}{2\beta(1-\beta)} := f(\beta). \tag{35}$$

Clearly, $f(\beta)$ goes to infinity as $\beta \to 0^+$ and $f(\beta)$ approaches 0 from above as $\beta \to \frac{1}{2}^-$. In particular, $f(\frac{1}{3}) = 1$ which implies that β must satisfy $\beta \in (0, 1/3)$ in order to allow α to become positive. In other words, for β close to $\frac{1}{3}$ the rate of degeneracy of n_0, m_0 must be low (α must be close to zero). As β becomes smaller (i.e., the rate of degeneracy of the viscous coefficient $\varepsilon(n,m)$ becomes lower), the rate of degeneracy α associated with n_0, m_0 can be higher. The coupling between α and β as described by (34) *is a "two-phase phenomenon" in the sense that it is not seen for the single-phase gas analysis as described in* [31, 28].

Theorem 2.1 (Main Result)**.** *Under the assumptions (A1)–(A5), the initial-boundary value problem* (21)–(25) *possesses a global weak solution (n, m, u) in the sense that:*

(A) *For any $T > 0$, we have the following regularity:*

$$\begin{aligned} &n, m, u \in L^{\infty}([0,1]\times[0,T]) \cap C^1([0,T]; H^1([0,1])),\\ &E(n,m)u_x \in L^{\infty}([0,1]\times[0,T]) \cap C^{\frac{1}{2}}([0,T]; L^2([0,1])). \end{aligned} \tag{36}$$

In particular, we have for a small constant $\mu > 0$ the estimate

$$\begin{aligned} &C(T)\phi(x)^{1+k_2} \leq m(x,t) \leq \min\{\rho_l - \mu, C(T)\phi(x)^{\frac{\alpha}{2}}\}, \\ &C_1 C(T)\phi(x)^{1+k_2+\frac{\alpha}{2}} \leq n(x,t) \leq C_2 \min\{\rho_l - \mu, C(T)\phi(x)^{\frac{\alpha}{2}}\}\phi(x)^{\frac{\alpha}{2}}, \end{aligned} \tag{37}$$

where $(x,t) \in [0,1] \times [0,T]$. The assumptions on k_2 is that for all positive integer k, sufficiently large,

$$\frac{2k\beta + 1}{(2k-1) - 2k\beta} < k_2 < \frac{1}{2\beta} - 1 - \frac{\alpha}{2} - \frac{1}{(2k-1)\beta}. \tag{38}$$

(B) *Moreover, the following equations hold,*

$$\begin{aligned} &n_t + nmu_x = 0, \quad m_t + m^2 u_x = 0, \\ &(n,m)(x,0) = (n_0(x), m_0(x)), \ \textit{for a.e. } x \in (0,1) \textit{ and any } t \geq 0, \\ &\int_0^\infty \int_0^1 \Big[u\phi_t + (P(n,m) - E(n,m)u_x)\phi_x\Big]\,dx\,dt + \int_0^1 u_0(x)\phi(x,0)\,dx = 0 \end{aligned} \tag{39}$$

for any test function $\phi(x,t) \in C_0^\infty(D)$, with $D := \{(x,t)\,|\,0 \leq x \leq 1,\ t \geq 0\}$.

The proof of Theorem 2.1 is based on a priori estimates for a class of approximate solutions of (21)–(25) and a corresponding limit procedure. These results are obtained by adopting techniques similar to those used in [31, 28] for the single-phase Navier-Stokes equation. A crucial part of this analysis is to obtain sufficient point-wise upper and lower limits for m and n, as described by Corollary 3.4. A main tool in this analysis is to focus, not on the mass m but instead the related quantity $Q(m) = m/(\rho_l - m)$ which connects pressure $P(n,m)$ and viscosity coefficient $E(n,m)$. It turns out that we naturally can reformulate the model (21) in terms of the variables (c,Q,u) instead of (n,m,u) where $c = n/m$. Together with higher order regularity of u and $(Q^\beta)_x$ as well as energy-conservation, certain pointwise upper and lower limits for $Q(m)$ can be derived. This, in turn, gives the required boundedness on m and n from below and above, together with the L^1 estimate of m_x and n_x. Finally, L^2 continuity in time is obtained for m and n. Armed with these estimates we can rely on standard compactness arguments to prove Theorem 2.1. This is done in Section 4.

3. Basic estimates

Below we derive a priori estimates for (n,m,u) assumed to be a smooth solution of (21)–(25). We then construct the approximate solutions of (21) in Section 4 by considering a semi-discrete approximation to (21)–(25).

More precisely, first we assume that (n,m,u) is a solution of (21)–(25) on $[0,T]$ satisfying

$$\begin{aligned} &n, n_t, n_x, n_{tx}, m, m_x, m_t, m_{tx}, u, u_x, u_t, u_{xx} \in C^{\alpha,\alpha/2}(D_T) \qquad \text{for some } \alpha \in (0,1), \\ &n(x,t) > 0, \qquad m(x,t) > 0, \qquad m(x,t) < \rho_l \quad \text{on } (0,1) \times [0,T]. \end{aligned} \tag{40}$$

In the following we will frequently take advantage of the fact that the model (21) can be rewritten in a form more amenable for deriving various estimates. We first describe this reformulation, and then present a number of a priori estimates, obtained mainly by relying on suitable modifications of techniques used in [28], see also references therein.

3.1. **A reformulation of the model** (21)**.**

We introduce the variable

$$c = \frac{n}{m}, \tag{41}$$

implicitly using that $m > 0$, and see from the first two equations of (21) that

$$c_t = \frac{1}{m}n_t - \frac{n}{m^2}m_t = -\frac{nm}{m}u_x + \frac{nm^2}{m^2}u_x = 0.$$

Consequently, the model (21)–(25) then can be written in terms of the variables (c, m, u) in the form

$$\begin{aligned} \partial_t c &= 0 \\ \partial_t m + m^2 \partial_x u &= 0 \\ \partial_t u + \partial_x P(c, m) &= \partial_x (E(c, m)\partial_x u), \qquad x \in (0, 1), \end{aligned} \tag{42}$$

with

$$P(c, m) = \Big(\frac{mc}{\rho_l - m}\Big)^{\gamma}, \qquad \gamma > 1, \tag{43}$$

and

$$E(c, m) = c^{\beta}\Big(\frac{m}{\rho_l - m}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{44}$$

Moreover, boundary conditions are given by

$$c(0, t) = c(1, t) = 0, \qquad m(0, t) = m(1, t) = 0, \tag{45}$$

whereas initial data are

$$c(x, 0) = c_0(x), \quad m(x, 0) = m_0(x), \quad u(x, 0) = u_0(x), \qquad x \in [0, 1]. \tag{46}$$

Furthermore, we introduce the variable

$$Q(m) = \frac{m}{\rho_l - m} = \frac{\alpha_l}{1 - \alpha_l} > 0, \tag{47}$$

since $m > 0$ and $m < \rho_l$ and observe that

$$\begin{aligned} Q(m)_t = \Big(\frac{m}{\rho_l - m}\Big)_t &= \Big(\frac{1}{\rho_l - m} + \frac{m}{(\rho_l - m)^2}\Big) m_t \\ &= \frac{\rho_l}{(\rho_l - m)^2} m_t = -\rho_l \frac{m^2}{(\rho_l - m)^2} u_x = -\rho_l Q(m)^2 u_x, \end{aligned}$$

in view of the second equation of (42). Consequently, we rewrite the model (42) in the form

$$\begin{aligned} \partial_t c &= 0 \\ \partial_t Q(m) + \rho_l Q(m)^2 \partial_x u &= 0 \\ \partial_t u + \partial_x P(c, Q(m)) &= \partial_x (E(c, Q(m))\partial_x u), \qquad x \in (0, 1), \qquad t > 0, \end{aligned} \tag{48}$$

with

$$P(c, Q(m)) = c^{\gamma} Q(m)^{\gamma}, \qquad \gamma > 1, \tag{49}$$

and

$$E(c, Q(m)) = c^{\beta} Q(m)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{50}$$

This model is then subject to the boundary conditions

$$c(0, t) = c(1, t) = 0, \qquad Q(0, t) = Q(1, t) = 0. \tag{51}$$

In addition, we have the initial data

$$c(x, 0) = c_0(x), \quad Q(x, 0) = Q_0(x), \quad u(x, 0) = u_0(x), \qquad x = [0, 1], \tag{52}$$

where $Q_0(x) = Q(m_0(x)) = \frac{m_0(x)}{\rho_l - m_0(x)}$. In particular, the first equation of (48) gives that

$$c(x, t) := c_0(x) := \frac{n_0}{m_0}(x) > 0, \qquad x \in (0, 1), \qquad t > 0, \tag{53}$$

for initial data as prescribed in assumption (A1).

3.2. **A priori estimates.** Now we derive a priori estimates for (n, m, u) by making use of the reformulated model (48)–(52). In Section 4 we consider corresponding semi-discrete versions. Note that in the following we will use C to denote a generic positive constant depending only on the initial data and $C(T)$ to indicate dependence on the given time T.

Lemma 3.1 (Energy estimate)**.** *We have the basic energy estimate*

$$\begin{aligned}\int_0^1 \Big(\frac{1}{2}u^2 + \frac{c^\gamma}{\rho_l(\gamma-1)}Q(m)^{\gamma-1}\Big)(x,t)\,dx + \int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}(u_x)^2\,dx\,ds \\ = \int_0^1 \Big(\frac{1}{2}u_0^2 + \frac{c_0^\gamma}{\rho_l(\gamma-1)}Q(m_0)^{\gamma-1}\Big)\,dx, \qquad \forall t\in[0,T] \\ \le C(T).\end{aligned} \tag{54}$$

Moreover,

$$cQ(m)(x,t) \le C(T), \qquad \forall (x,t)\in[0,1]\times[0,T], \tag{55}$$

and for any positive integer k,

$$\int_0^1 u^{2k}(x,t)\,dx + k(2k-1)\int_0^t\int_0^1 u^{2k-2}c^\beta Q(m)^{1+\beta}(u_x)^2\,dx\,ds \le C(T). \tag{56}$$

Proof. We multiply the third equation of (48) by u and integrate over $[0,1]$ in space. Applying the boundary condition (51) and the equation

$$\frac{c^\gamma}{\rho_l(\gamma-1)}(Q^{\gamma-1})_t + c^\gamma Q^\gamma u_x = 0, \tag{57}$$

obtained from the second equation of (48) by multiplying with $c^\gamma Q^{\gamma-2}$, we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{1}{2}u^2 + \frac{c^\gamma}{\rho_l(\gamma-1)}Q^{\gamma-1}\Big)(x,t)\,dx + \int_0^1 E(c,Q)(u_x)^2\,dx = 0.$$

From this, (54) follows, by application of (32), (27) and (28).

Next, we focus on (55). From the second equation of (48) we deduce the equation

$$\frac{1}{\rho_l}(Q^\beta)_t + \beta Q^{\beta+1}u_x = 0. \tag{58}$$

Multiplying with c^β and integrating over $[0,t]$, we get

$$(cQ)^\beta(x,t) = (cQ)^\beta(x,0) - \beta\rho_l\int_0^t c^\beta Q^{\beta+1}u_x\,ds. \tag{59}$$

Then, we integrate the third equation of (48) over $[0,x]$ and get

$$\int_0^x u_t(y,t)\,dy + P(c,Q) - P(c(0,t),Q(0,t)) + (E(c,Q)u_x)(0,t) = E(c,Q)u_x = c^\beta Q(m)^{\beta+1}u_x.$$

Using the boundary condition (51) and inserting the above relation into the right hand side of (59), we get

$$\begin{aligned}(cQ)^\beta(x,t) &= (cQ)^\beta(x,0) - \beta\rho_l\int_0^t\Big(\int_0^x u_s(y,s)\,dy + P(c,Q)\Big)\,ds \\ &= (cQ)^\beta(x,0) - \beta\rho_l\int_0^x (u(y,t)-u_0(y))\,dy - \beta\rho_l\int_0^t P(c,Q)\,ds\end{aligned} \tag{60}$$

Consequently, since $P(c,Q)\ge 0$

$$(cQ)^\beta(x,t) \le (cQ)^\beta(x,0) + \beta\rho_l\int_0^1 |u(y,t)|\,dy + \beta\rho_l\int_0^1 |u_0(y)|\,dy. \tag{61}$$

Applying Hölder's inequality and (54) we can bound $\int_0^1 |u|\,dy$, hence the upper bound (55) follows in view of (27) and (28).

Finally, we focus on estimate (56). Multiplying the third equation of (48) by $2ku^{2k-1}$, integrating over $[0,1]\times[0,t]$ and integration by parts together with application of the boundary conditions (51), we get

$$\begin{aligned}\int_0^1 u^{2k}\,dx + 2k(2k-1)\int_0^t\int_0^1 c_0^{\beta}Q(m)^{\beta+1}(u_x)^2u^{2k-2}\,dx\,ds\\ = \int_0^1 u_0^{2k}\,dx + 2k(2k-1)\int_0^t\int_0^1 c_0^{\gamma}Q(m)^{\gamma}u^{2k-2}u_x\,dx\,ds.\end{aligned}\tag{62}$$

For the last term we have by the Cauchy type inequality $ab\le(1/4\varepsilon)a^2+\varepsilon b^2$ where $\varepsilon>0$, that

$$\begin{aligned}&\int_0^t\int_0^1 c_0^{\gamma}Q(m)^{\gamma}u^{2k-2}u_x\,dx\,ds\\ &\quad\le \frac{1}{4\varepsilon}\int_0^t\int_0^1 c_0^{2(\gamma-\beta/2)}Q(m)^{2\gamma-\beta-1}u^{(2k-2)}\,dx\,ds + \varepsilon\int_0^t\int_0^1 c_0^{\beta}Q(m)^{\beta+1}u^{(2k-2)}(u_x)^2\,dx\,ds\\ &\quad\le \frac{1}{4\varepsilon}\sup_{[0,1]}(c_0)\int_0^t\int_0^1 (c_0Q(m))^{2\gamma-\beta-1}u^{(2k-2)}\,dx\,ds + \varepsilon\int_0^t\int_0^1 c_0^{\beta}Q(m)^{\beta+1}u^{(2k-2)}(u_x)^2\,dx\,ds.\end{aligned}$$

The last term clearly can be absorbed in the second term of the left-hand side of (62) by the choice $\varepsilon=1/2$. Finally, let us see how we can bound the term $\int_0^t\int_0^1 u^{(2k-2)}(c_0Q(m))^{2\gamma-1-\beta}\,dx\,ds$. In view of Young's inequality $ab\le(1/p)a^p+(1/q)b^q$ where $1/p+1/q=1$, we get for the choice $p=k$ and $q=k/(k-1)$

$$\begin{aligned}\int_0^t\int_0^1 u^{(2k-2)}(c_0Q(m))^{2\gamma-1-\beta}\,dx\,ds &\le \frac{1}{k}\int_0^t\int_0^1 (c_0Q(m))^{(2\gamma-1-\beta)k}\,dx\,ds + \frac{k-1}{k}\int_0^t\int_0^1 u^{2k}\,dx\,ds\\ &\le C(T) + \frac{k-1}{k}\int_0^t\int_0^1 u^{2k}\,dx\,ds,\end{aligned}$$

by using (55). To sum up, we get

$$\begin{aligned}\int_0^1 u^{2k}\,dx + k(2k-1)\int_0^t\int_0^1 c_0^{\beta}Q(m)^{\beta+1}(u_x)^2u^{2k-2}\,dx\,ds\\ \le \int_0^1 u_0^{2k}\,dx + k(2k-1)\sup_{[0,1]}(c_0)\Big[C(T)+\frac{k-1}{k}\int_0^t\int_0^1 u^{2k}\,dx\,ds\Big].\end{aligned}\tag{63}$$

In view of (63), application of (27) and Gronwall's inequality then gives the estimate (56). □

Taking advantage of a more refined use of the relation (60) together with the higher order integrability of u given by (56), we can obtain a sharper estimate for the upper bound of $cQ(m)$ as follows.

Corollary 3.1. *We have the estimates*

$$c(x)Q(m)\le C(T)\phi(x)^{\alpha},\tag{64}$$

and

$$Q(m)\le C(T)\phi(x)^{\alpha/2},\tag{65}$$

where $\phi(x)=x(1-x)$.

Proof. In addition to (60), we also have

$$(cQ)^{\beta}(x,t)=(cQ)^{\beta}(x,0)+\beta\rho_l\int_x^1(u(y,t)-u_0(y))\,dy-\beta\rho_l\int_0^t P(c,Q)\,ds.\tag{66}$$

Consequently, (60) and (66) give for a positive integer k, respectively,

$$(cQ)^{\beta}\le(c_0Q_0)^{\beta}+C\Big(\int_0^x u^{2k}\,dx\Big)^{1/2k}x^{(2k-1)/2k}+Cx,$$

and

$$(cQ)^\beta \le (c_0Q_0)^\beta + C\Big(\int_x^1 u^{2k}\,dx\Big)^{1/2k}(1-x)^{(2k-1)/2k} + C(1-x),$$

where we have used Hölder's inequality with $p = 2k$ and $q = \frac{2k}{2k-1}$. Together with the fact that $\min(x, 1-x) \le 2x(1-x)$ this implies that

$$\begin{aligned}(cQ)^\beta &\le (c_0Q_0)^\beta + C\Big(\int_0^1 u^{2k}\,dx\Big)^{1/2k}(x(1-x))^{(2k-1)/2k} + Cx(1-x)\\ &\le (c_0Q_0)^\beta + C(T)(x(1-x))^{(2k-1)/2k},\end{aligned}$$

where we have used (56). Thus, in light of (27) and (28), we conclude that

$$(cQ) \le C(x(1-x))^\alpha + C(T)(x(1-x))^{(2k-1)/2k\beta}.$$

Observing that $\alpha < \frac{2k-1}{2k\beta}$ for all positive integers k when $\beta \in (0, 1/3)$, we conclude that

$$(cQ) \le C(T)(x(1-x))^\alpha.$$

Then, from (27) of assumption (A1) it follows that

$$Q(m) \le C(T)(x(1-x))^{\alpha/2}.$$

□

Remark 3.1. *Note that the above refined arguments are necessary in order to obtain the estimate $Q(m) \le C(T)$ because (61) together with (27) and (28) imply that*

$$cQ(m) \le C(x(1-x))^\alpha + C,$$

which does not allow us to extract an upper bound for $Q(m)$ since $c(x)$ degenerates at the boundary. On the other hand, (65) implies that m has to become zero at the boundary $x = 0, 1$, i.e., transition to single-phase gas flow is inescapable. Apparently, the interesting case where the gas-phase n vanishes at the boundary whereas m does not, cannot be explored in the current framework.

Lemma 3.2 (Additional higher order regularity)**.** *We have for all integers k the estimate*

$$\int_0^1 (\partial_x[cQ(m)]^\beta)^{2k}(x,t)\,dx \le C(T), \tag{67}$$

for a suitable constant $C(T)$.

Proof. We have

$$(cQ)^\beta_t + \rho_l\beta c^\beta Q^{\beta+1}u_x = 0.$$

That is,

$$(cQ)^\beta_{tx} = -\rho_l\beta(u_t + P(c,Q)_x),$$

where we have used the third equation of (48). Integrating in t over $[0,t]$, and then multiplying by $[(cQ)^\beta_x]^{2k-1}$ and integrating in x over $[0,1]$ gives

$$\begin{aligned}\int_0^1 [(cQ)^\beta_x]^{2k}\,dx = \int_0^1 [(c_0Q_0)^\beta_x][(cQ)^\beta_x]^{2k-1}\,dx - \rho_l\beta\int_0^1 [u(t,x) - u_0(x)][(cQ)^\beta_x]^{2k-1}\,dx\\ - \rho_l\beta\int_0^1 [(cQ)^\beta_x]^{2k-1}\int_0^t P(c,Q)_x\,ds\,dx.\end{aligned}$$

Repeated use of Young's inequality $ab \le \frac{\varepsilon^p}{p}a^p + \frac{\varepsilon^{-q}}{q}b^q$ then yields with $p = 2k$ and $q = 2k/(2k-1)$ and suitable choices of $\varepsilon > 0$

$$\begin{aligned}\int_0^1 [(cQ)^\beta_x]^{2k}\,dx &\le C\int_0^1 [(c_0Q_0)^\beta_x]^{2k}\,dx + \frac{1}{2}\int_0^1 [(cQ)^\beta_x]^{2k}\,dx\\ &\quad + C\int_0^1 u^{2k}(x,t)\,dx + C\int_0^1 u_0^{2k}(x,t)\,dx + C\int_0^1\Big(\int_0^t P(c,Q)_x\,ds\Big)^{2k}\,dx.\end{aligned}$$

Consequently, applying (30), Jensen's inequality, and (56) it follows that

$$\begin{aligned}\int_0^1 [(cQ)^\beta_x]^{2k}\,dx &\le C(T) + C\int_0^1 \Big(\int_0^t P(c,Q)_x\,ds\Big)^{2k}\,dx\\ &\le C(T) + C(T)\int_0^t \max(cQ)^{(\gamma-\beta)2k}\int_0^1 [(cQ)^\beta_x]^{2k}\,dx\,ds.\end{aligned}$$

From this, in view of (55) and Gronwall's lemma, the result follows. □

Corollary 3.2 (Additional regularity with weight function)**.** *We have the estimate*

$$\int_0^1 \psi(x)(\partial_x[cQ(m)]^\beta)^{2k}(x,t)\,dx \le C(T), \tag{68}$$

for a constant $C(T)$ and a bounded function $\psi(x)$.

In the following lemmas the role played by the weighted function $\phi(x)$ is crucial. In particular, a proper balance between the weighted function $\phi(x)$ and $c(x)$ must be ensured in order to get the desired estimates. The assumption (26) (which implies (27)) is used here.

Lemma 3.3. *For any $k_1 > \frac{1}{2k}$, from (29) it follows that $\phi(x)^{k_1}Q(m_0(x))^{-1} \in L^1([0,1])$. Then*

$$\int_0^1 \frac{\phi(x)^{k_1}}{Q(m(x,t))}\,dx \le C(T). \tag{69}$$

Proof. We have

$$\Big(\frac{\phi(x)^{k_1}}{Q(m)}\Big)_t = \rho_l\phi(x)^{k_1}u_x = \rho_l\big(\phi(x)^{k_1}u\big)_x - \rho_l\big(\phi(x)^{k_1}\big)_x u. \tag{70}$$

Integrating over $[0,1]\times[0,t]$ and using Young's inequality with $p = 2k$ and $q = 2k/(2k-1)$ and the fact that $\phi(x)|_{x=0,1} = 0$, we get

$$\begin{aligned}\int_0^1 \frac{\phi(x)^{k_1}}{Q(m)}\,dx &= \int_0^1 \frac{\phi(x)^{k_1}}{Q(m_0)}\,dx - \rho_l\int_0^t\int_0^1 \Big(\phi(x)^{k_1}\Big)_x u\,dx\,ds\\ &\le C + C\int_0^t\int_0^1 u^{2k}(x,s)\,dx\,ds + C\int_0^t\int_0^1 \phi(x)^{\frac{2k(k_1-1)}{2k-1}}\,dx\,ds\\ &\le C(T).\end{aligned} \tag{71}$$

Here we have used (56) and the observation that $k_1 > \frac{1}{2k}$ implies that $\frac{2k(k_1-1)}{2k-1} > -1$ which ensures that $\int_0^1 \phi(x)^{\frac{2k(k_1-1)}{2k-1}}\,dx$ is finite. □

The following remark, taken from [28], is also relevant to us.

Remark 3.2. *Note that the finite propagation property implies the finiteness of the integral $\int_0^1 \frac{1}{Q(x,t)}\,dx$ which is stronger than Lemma 3.3. However, this boundedness cannot be obtained here without using the weight $\phi(x)^{k_1}$ where k_1 is a positive constant which can be arbitrary small. The boundedness of $\int_0^1 \frac{1}{Q(x,t)}\,dx$ holds once the L^∞ bound on the velocity u is given, see Lemma 3.7.*

Considering the special choice $k_1 = \frac{1}{2k-1}$ in Lemma 3.3 we get the following result.

Corollary 3.3. *The following estimate holds:*

$$\int_0^1 \frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m(x,t))}\,dx \le C(T). \tag{72}$$

This result is used to obtain a lower limit for $Q(m)$ as given by the next lemma.

Lemma 3.4. *Let β satisfy $0 < \beta < 1/3$. Then, for any positive k it follows that $0 < 2k - 1 - 2k\beta$. Moreover, let α and β satisfy the relation (34) of assumption (A5). This implies that for all positive integer k, sufficiently large, the following relation holds*

$$\frac{2k\beta+1}{2k-1-2k\beta} < \frac{1}{2\beta} - 1 - \frac{\alpha}{2}. \tag{73}$$

Let k_2 be chosen in the interval

$$\frac{2k\beta+1}{2k-1-2k\beta} < k_2 < \frac{1}{2\beta} - 1 - \frac{\alpha}{2}, \tag{74}$$

and observe that

$$\max\Big(\frac{\beta}{1-\beta}, \frac{1}{2k-1}\Big) < \frac{2k\beta+1}{2k-1-2k\beta} < k_2. \tag{75}$$

Then, the following estimate holds

$$Q(m(x,t)) \geq C(T)\phi(x)^{1+k_2}. \tag{76}$$

Proof. Sobolev's embedding theorem $W^{1,1}(I) \hookrightarrow L^\infty(I)$, together with estimate (72) and the fact that $c(x)^{-1} \leq C_1^{-1}\phi(x)^{-\alpha/2}$ and $c(x) \leq C_2\phi(x)^{\alpha/2}$, yields (for a sufficiently large k)

$$\begin{aligned}
&\frac{\phi(x)^{1+k_2}}{Q(m(x,t))}\\
&\leq \int_0^1 \frac{\phi(x)^{1+k_2}}{Q(m(x,t))}\,dx + \int_0^1 \Big|\Big(\frac{\phi(x)^{1+k_2}}{Q(m(x,t))}\Big)_x\Big|\,dx\\
&\leq (\max\phi(x))^{1+k_2-\frac{1}{2k-1}} \int_0^1 \frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx + \int_0^1 \Big|\Big(\frac{\phi(x)^{1+k_2}c(x)}{cQ(m)}\Big)_x\Big|\,dx\\
&\leq C(T) + (1+k_2)\int_0^1 \frac{\phi(x)^{k_2}|\phi'(x)|}{Q(m)}\,dx + \int_0^1 \frac{\phi(x)^{1+k_2}|c'(x)|}{cQ(m)}\,dx + \int_0^1 \frac{\phi(x)^{1+k_2}c(x)}{[cQ(m)]^2}\Big|[cQ(m)]_x\Big|\,dx\\
&\leq C(T) + C(\max\phi(x))^{k_2-\frac{1}{2k-1}}\int_0^1 \frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx\\
&\qquad + C(\max\phi(x))^{1+k_2-\alpha/2-\frac{1}{2k-1}}\int_0^1 \frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx + C\int_0^1 \frac{\phi(x)^{1+k_2+\alpha/2}}{[cQ(m)]^{\beta+1}}\Big|([cQ(m)]^\beta)_x\Big|\,dx\\
&\leq C(T) + C\int_0^1 \phi(x)^{1+k_2-\frac{\beta\alpha}{2}}\frac{([cQ(m)]^\beta)_x}{Q(m)^{\beta+1}}\,dx\\
&\leq C(T) + C\int_0^1 \phi(x)^{3/4+k_2}\frac{([cQ(m)]^\beta)_x}{Q(m)^{\beta+1}}\,dx,
\end{aligned} \tag{77}$$

where we have used that $\frac{\beta\alpha}{2} \leq \frac{1}{4}$, that is, $1-\frac{\beta\alpha}{2} \geq \frac{3}{4}$ which implies that $\phi(x)^{1-\frac{\beta\alpha}{2}} \leq \phi(x)^{\frac{3}{4}}$. Next,

$$\begin{aligned}
&\int_0^1 \frac{\phi(x)^{3/4+k_2}}{Q(m)^{\beta+1}}([cQ(m)]^\beta)_x\,dx\\
&= \int_0^1 \frac{\phi(x)^{1/4k+1/4+k_2}\phi(x)^{(2k-1)/4k}}{Q(m)^{\beta+1}}([cQ(m)]^\beta)_x\,dx\\
&\leq \Big(\int_0^1 \phi(x)^{(k-1/2)}([cQ(m)]^\beta)_x^{2k}\,dx\Big)^{1/p}\Big(\int_0^1 \frac{\phi(x)^{(1/4k+1/4+k_2)q}}{Q(m)^{(\beta+1)q}}\,dx\Big)^{1/q}\\
&\leq C(T)\Big(\int_0^1 \frac{\phi(x)^{(1/4k+1/4+k_2)q}}{Q(m)^{(\beta+1)q}}\,dx\Big)^{1/q},
\end{aligned} \tag{78}$$

where we have used Hölder's inequality with $p = 2k$ and $q = 2k/(2k-1)$ together with Corollary 3.2. Moreover, applying (72) we estimate as follows:

$$\begin{aligned}
&\Big(\int_0^1 \frac{\phi(x)^{(1/4k+1/4+k_2)q}}{Q(m)^{(\beta+1)q}}\,dx\Big)^{1/q} = \Big(\int_0^1 \frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)} \cdot \frac{\phi(x)^{(1/4k+1/4+k_2)q-\frac{1}{2k-1}}}{Q(m)^{(\beta+1)q-1}}\,dx\Big)^{1/q}\\
&\leq C(T)\max\Big(\frac{\phi(x)^{1+k_2}}{Q(m)}\Big)^{(\beta+1)-1/q}\phi(x)^{(1/4k+1/4+k_2)-\frac{1}{(2k-1)q}-(1+k_2)((\beta+1)-1/q)}.
\end{aligned} \tag{79}$$

We define the parameter k_3 and observe that

$$\begin{aligned}
k_3 &:= (1/4k + 1/4 + k_2) - \frac{1}{(2k-1)q} - (1+k_2)((\beta+1) - 1/q) \\
&= k_2 - \frac{1}{4k} + \frac{1}{4} - (1+k_2)(\beta + \frac{1}{2k}) \\
&= k_2(1-\beta) - \beta + \frac{1}{4} - \frac{1}{2k}\Big(\frac{3}{2} + k_2\Big).
\end{aligned}$$

Noting that

$$k_2 > \frac{2k\beta + 1}{2k - 1 - 2k\beta} > \frac{2k\beta}{2k - 2k\beta} = \frac{\beta}{1-\beta},$$

we conclude that

$$k_3 > \frac{1}{4} - \frac{1}{2k}\Big(\frac{3}{2} + k_2\Big) \geq 0,$$

for k large enough, in view of the upper bound for k_2 given by (74). Consequently,

$$\max\Big(\frac{\phi(x)^{1+k_2}}{Q(m)}\Big) \leq C(T) + C(T)\max\Big(\frac{\phi(x)^{1+k_2}}{Q(m)}\Big)^{(\beta+1)-1/q}.$$

Since $\beta+1-1/q = \beta+1/2k < 1$ for a given $\beta \in (0,1)$ by choosing k large enough, we can conclude that

$$\max\Big(\frac{\phi(x)^{1+k_2}}{Q(m)}\Big) \leq C(T),$$

which proves (76). □

Corollary 3.4. *We have the upper and lower bounds*

$$\begin{aligned}
C(T)\phi(x)^{1+k_2} &\leq Q(m(x,t)) \leq C(T)\phi(x)^{\frac{\alpha}{2}}, && (80)\\
C(T)\phi(x)^{1+k_2} &\leq m(x,t) \leq \min\{C(T)\phi(x)^{\alpha/2}, \rho_l - \mu\}, && (81)\\
C_1 C(T)\phi(x)^{1+k_2+\frac{\alpha}{2}} &\leq n(x,t) \leq C_2 \min\{C(T)\phi(x)^{\alpha/2}, \rho_l - \mu\}\phi(x)^{\frac{\alpha}{2}}, && (82)
\end{aligned}$$

where $\mu > 0$ is a small constant.

Proof. The first estimate (80) follows from Corollary 3.1 and Lemma 3.4. For the second estimate (81) we observe that for $Q(x) = \frac{x}{\rho_l - x}$, which is strictly increasing for $x \in [0, \rho_l)$, the inverse exists and is given by $Q^{-1}(y) = \frac{\rho_l y}{1+y}$. Thus, in view of (80) it follows

$$m \geq Q^{-1}\Big(C(T)\phi(x)^{1+k_2}\Big) = \frac{\rho_l C(T)\phi(x)^{1+k_2}}{1 + C(T)\phi(x)^{1+k_2}} \geq \frac{\rho_l C(T)}{1 + C(T)}\phi(x)^{1+k_2}.$$

and

$$m \leq Q^{-1}\Big(C(T)\phi(x)^{\alpha/2}\Big) = \frac{\rho_l C(T)\phi(x)^{\alpha/2}}{1 + C(T)\phi(x)^{\alpha/2}} \leq \min\{\rho_l C(T)\phi(x)^{\alpha/2}, \rho_l - \mu\},$$

for a suitable small $\mu > 0$. In view of the fact that $n(x,t) = m(x,t)c_0(x)$ and the estimate (27), the last estimate (82) follows. □

Corollary 3.5. *We have the estimates*

$$\int_0^1 |\partial_x m|\, dx \leq C(T), \qquad \int_0^1 |\partial_x n|\, dx \leq C(T), \tag{83}$$

for a suitable constant $C(T)$.

Proof. It follows that

$$
\begin{aligned}
\partial_x([cQ(m)]^\beta) &= c^\beta \partial_x(Q(m)^\beta) + \beta c^{\beta-1} Q(m)^\beta \partial_x c \\
&= \beta c^\beta Q(m)^{\beta-1} Q'(m)\partial_x m + \beta c^{\beta-1} Q(m)^\beta \partial_x c \\
&= \beta \rho_l c^\beta Q(m)^{\beta-1} \frac{Q(m)^2}{m^2} \partial_x m + \beta c^{\beta-1} Q(m)^\beta \partial_x c \\
&= \beta \rho_l c^\beta \frac{Q(m)^{\beta+1}}{m^2} \partial_x m + \beta c^{\beta-1} Q(m)^\beta \partial_x c \\
&= \frac{\beta}{\rho_l} c^\beta Q(m)^{\beta-1} [1+Q(m)]^2 \partial_x m + \beta c^{\beta-1} Q(m)^\beta \partial_x c \\
&= \frac{\beta c}{\rho_l} [cQ(m)]^{\beta-1} [1+Q(m)]^2 \partial_x m + \beta c^{\beta-1} Q(m)^\beta \partial_x c,
\end{aligned} \tag{84}
$$

since $Q'(m) = (\rho_l/m^2)Q(m)^2$ and $m = \rho_l Q(m)/(1+Q(m))$. For $x \in (0,1)$ where $cQ(m) > 0$ and $c > 0$ we can rewrite in the form

$$
\begin{aligned}
\frac{\beta}{\rho_l}[1+Q(m)]^2 \partial_x m &= \frac{[cQ(m)]^{1-\beta}}{c} \partial_x([cQ(m)]^\beta) - \frac{[cQ(m)]^{1-\beta}}{c} \beta c^{\beta-1} Q(m)^\beta \partial_x c \\
&= \frac{[cQ(m)]^{1-\beta}}{c} \partial_x([cQ(m)]^\beta) - \beta \frac{Q(m)}{c} \partial_x c.
\end{aligned}
$$

Consequently, using Young's inequality with coefficients $p = 2k$ and $q = \frac{2k}{2k-1}$ we get

$$
\begin{aligned}
&\int_0^1 |\partial_x m|\,dx \le C \int_0^1 \phi(x)^{\frac{2k-1}{2k}} |\partial_x([cQ(m)]^\beta)| \frac{[cQ(m)]^{1-\beta}}{c\phi(x)^{\frac{2k-1}{2k}}}\,dx + C\int_0^1 \frac{Q(m)}{c}|\partial_x c|\,dx \\
&\le \frac{C}{2k}\int_0^1 \phi(x)^{2k-1}(\partial_x[cQ(m)]^\beta)^{2k}\,dx + \frac{C(2k-1)}{2k}\int_0^1 \frac{[cQ(m)]^{\frac{2k(1-\beta)}{2k-1}}}{c^{\frac{2k}{2k-1}}\phi(x)}\,dx + C\int_0^1 \frac{Q(m)}{c}|\partial_x c|\,dx \\
&\le C(T) + C(T)\int_0^1 \phi(x)^{k_4}\,dx + C(T)\int_0^1 |\partial_x c|\,dx,
\end{aligned}
$$

with

$$
k_4 = \frac{2k(1-\beta)\alpha}{2k-1} - 1 - \frac{\alpha}{2}\frac{2k}{2k-1},
$$

and where we have used Lemma 3.2, Corollary 3.1, and (27) of assumption (A1). We observe that for $\beta \in (0, 1/3)$ we get

$$
k_4 > (1-\beta)\alpha - \frac{\alpha}{2}\frac{2k}{2k-1} - 1 > \alpha\Big(\frac{2}{3} - \frac{1}{2}\frac{2k}{2k-1}\Big) - 1 > -1,
$$

for sufficiently large k. Applying (31) of assumption (A2), the first estimate of (83) has been proved. Clearly,

$$
\int_0^1 |\partial_x n|\,dx \le \int_0^1 |m||\partial_x c|\,dx + \int_0^1 c|\partial_x m|\,dx \le C(T),
$$

in view of (31), Corollary 3.4, and the above estimate of $\int |m_x|\,dx$. □

Remark 3.3. *Note that the above estimate gives us upper bounds for the bounded variation of m and n in space. However, the bound $\int_0^T\int_0^1 m_t\,dx\,ds$ depends on the bound of $\int_0^T\int_0^1 u_x^2\,dx\,ds$ by applying the second equation of (21). This term, in view of the estimate of Lemma 3.1, requires a positive pointwise lower limit of $c^\beta Q(m)^{\beta+1}$. And this is not available due to the degeneracy of c as well as $Q(m)$. Consequently, more refined estimates must be obtained.*

Lemma 3.5. *Under the assumptions of Lemma 3.4 we can prove that*

$$
\int_0^1 u_t^2\,dx + \int_0^t\int_0^1 c^\beta Q(m)^{\beta+1} u_{xt}^2\,dx\,ds \le C(T). \tag{85}
$$

Proof. We differentiate the third equation of (48) with respect to time t, multiply the resulting equation by $2u_t$ and integrate over $[0,1]\times[0,t]$, and obtain

$$\begin{aligned}\int_0^1 u_t^2(x,t)\,dx + 2\int_0^t\int_0^1 (c^\gamma Q(m)^\gamma)_{xt}u_t\,dx\,ds = \int_0^1 u_t^2(x,0)\,dx\\ + 2\int_0^t\int_0^1 (c^\beta Q(m)^{\beta+1}u_x)_{xt}u_t\,dx\,ds.\end{aligned} \tag{86}$$

First, it follows that

$$\int_0^1 u_t^2(x,0)\,dx \le C(T), \tag{87}$$

by considering the momentum equation of (48) at time $t=0$

$$u_t(x,0) + (c(x)^\gamma Q(m(x,0))^\gamma)_x = (c^\beta(x)Q(m(x,0))^{\beta+1}u_x(x,0))_x,$$

together with assumptions (A2) and (A3), as given by (30) and (32). Moreover, using the second equation of (48) it follows that

$$\begin{aligned}\int_0^t\int_0^1 (c^\beta Q(m)^{\beta+1}u_x)_{xt}u_t\,dx\,ds &= -\int_0^t\int_0^1 (c^\beta Q(m)^{\beta+1}u_x)_t u_{xt}\,dx\,ds\\ &= -\int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}u_{xt}^2\,dx\,ds + (\beta+1)\rho_l\int_0^t\int_0^1 c^\beta Q(m)^{\beta+2}u_x^2 u_{xt}\,dx\,ds,\end{aligned} \tag{88}$$

and

$$\begin{aligned}\int_0^t\int_0^1 (c^\gamma Q(m)^\gamma)_{xt}u_t\,dx\,ds &= -\int_0^t\int_0^1 (c^\gamma Q(m)^\gamma)_t u_{xt}\,dx\,ds\\ &= \gamma\rho_l\int_0^t\int_0^1 c^\gamma Q(m)^{\gamma+1}u_x u_{xt}\,dx\,ds.\end{aligned} \tag{89}$$

Using this in (86) we get

$$\begin{aligned}&\int_0^1 u_t^2(x,t)\,dx + 2\int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}u_{xt}^2\,dx\,ds\\ &\le C + 2(\beta+1)\rho_l\int_0^t\int_0^1 c^\beta Q(m)^{\beta+2}u_x^2u_{xt}\,dx\,ds - 2\gamma\rho_l\int_0^t\int_0^1 c^\gamma Q(m)^{\gamma+1}u_xu_{xt}\,dx\,ds.\end{aligned} \tag{90}$$

For the two last terms on the right hand side we estimate as follows:

$$\begin{aligned}&2(\beta+1)\rho_l\int_0^t\int_0^1 c^\beta Q(m)^{\beta+2}u_x^2u_{xt}\,dx\,ds\\ &\le \frac{1}{2}\int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}u_{xt}^2\,dx\,ds + 2(\beta+1)^2\rho_l^2\int_0^t\int_0^1 c^\beta Q(m)^{\beta+3}u_x^4\,dx\,ds,\end{aligned} \tag{91}$$

where we have used $ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}$ with $\varepsilon = \frac{1}{2}$. Similarly,

$$\begin{aligned}&2\gamma\rho_l\int_0^t\int_0^1 c^\gamma Q(m)^{\gamma+1}u_xu_{xt}\,dx\,ds\\ &\le \frac{1}{2}\int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}u_{xt}^2\,dx\,ds + 2\gamma^2\rho_l^2\int_0^t\int_0^1 c^{2\gamma-\beta}Q(m)^{2\gamma-\beta+1}u_x^2\,dx\,ds.\end{aligned} \tag{92}$$

Inserting this in (90) we get

$$\begin{aligned}&\int_0^1 u_t^2(x,t)\,dx + \int_0^t\int_0^1 c^\beta Q(m)^{\beta+1}u_{xt}^2\,dx\,ds\\ &\le C + 2(\beta+1)^2\rho_l^2 I_1 + 2\gamma^2\rho_l^2 I_2,\end{aligned} \tag{93}$$

where $I_1 = \int_0^t \int_0^1 c^\beta Q(m)^{\beta+3} u_x^4\,dx\,ds$ and $I_2 = \int_0^t \int_0^1 c^{2\gamma-\beta} Q(m)^{2\gamma-\beta+1} u_x^2\,dx\,ds$. We then estimate as follows:

$$I_1 = \int_0^t \int_0^1 c^\beta Q(m)^{\beta+3} u_x^4\,dx\,ds \le \int_0^t \max(Q^2 u_x^2) V(s)\,ds \tag{94}$$

where $V(s) = \int_0^1 c^\beta Q(m)^{\beta+1} u_x^2\,dx$. We observe that the third equation of (48) gives

$$E(c,m)u_x = P(c,m) + \int_0^x u_t(y,t)\,dy = P(c,m) - \int_x^1 u_t(y,t)\,dy.$$

It follows, using Hölder's inequality, Lemma 3.4, Corollary 3.1, and (27), that

$$\begin{aligned}
Q^2 u_x^2 &= (c^\beta Q(m)^{\beta+1} u_x)^2 (cQ(m))^{-2\beta} \\
&= (cQ)^{-2\beta}\Big(\int_0^x u_t\,dx + P(c,m)\Big)^2 \\
&\le C(cQ)^{-2\beta}\Big(x(1-x)\int_0^1 u_t^2\,dx + P(c,m)^2\Big) \\
&\le C\phi(x)\phi(x)^{-\beta\alpha}\phi(x)^{-2\beta(1+k_2)}\int_0^1 u_t^2\,dx + C(cQ)^{2(\gamma-\beta)} \\
&\le C\phi(x)^{1-2\beta(1+k_2+\alpha/2)}\int_0^1 u_t^2\,dx + C(T) \le C(T)\int_0^1 u_t^2\,dx + C(T),
\end{aligned}$$

where we have used that, for sufficiently large k, $k_2 + 1 + \alpha/2 < \frac{1}{2\beta}$, see (74). Consequently, we have

$$I_1 \le C(T)\int_0^t V(s)\,ds + C(T)\int_0^t V(s)\int_0^1 u_t^2\,dx\,ds \le C(T) + C(T)\int_0^t V(s)\int_0^1 u_t^2\,dx\,ds, \tag{95}$$

where $V(s) \in L^1([0,T])$ in view of (54) of Lemma 3.1. Moreover, by (55) of Lemma 3.1 we have

$$I_2 = \int_0^t \int_0^1 c^{2\gamma-\beta} Q(m)^{2\gamma-\beta+1} u_x^2\,dx\,ds \le \int_0^t \max(cQ)^{2(\gamma-\beta)} V(s)\,ds \le C(T)\int_0^t V(s)\,ds \le C(T). \tag{96}$$

Using (95) and (96) in (93) we get

$$\begin{aligned}
\int_0^1 u_t^2(x,t)\,dx &+ \int_0^t \int_0^1 c^\beta Q(m)^{\beta+1} u_{xt}^2\,dx\,ds \\
&\le C(T) + C(T)\int_0^t V(s)\int_0^1 u_t^2\,dx\,ds,
\end{aligned} \tag{97}$$

which by application of Gronwall's lemma yields

$$\int_0^1 u_t^2\,dx \le C(T)\exp\Big(C(T)\int_0^t V(s)\,ds\Big) \le C(T).$$

□

We now can sum up the following "regularity in space" type of estimates:

Lemma 3.6. *Under the assumptions of Lemma 3.4, we have*

$$\int_0^1 |m_x|\,dx \le C(T), \qquad \int_0^1 |n_x|\,dx \le C(T), \tag{98}$$

$$\|c(x)^\beta Q(m(x,t))^{\beta+1} u_x(x,t)\|_{L^\infty(D_T)} \le C(T), \qquad D_T = [0,1]\times[0,T], \tag{99}$$

$$\int_0^1 |(c^\beta Q(m)^{\beta+1} u_x)_x(x,t)|\,dx \le C(T). \tag{100}$$

Proof. The bounds of (98) have already been proven, see Corollary 3.5. Moreover, we have that

$$\begin{aligned} c^{\beta}Q(m)^{\beta+1}u_x &= \int_0^x u_t\,dx + (cQ(m))^{\gamma}, \\ (c^{\beta}Q(m)^{\beta+1}u_x)_x &= u_t + ([cQ(m)]^{\gamma})_x. \end{aligned} \tag{101}$$

Estimate (99) follows directly from the first relation of (101), Corollary 3.1, Hölder's inequality, and Lemma 3.5. Estimate (100) follows from the second relation of (101), by application of Corollary 3.1, Lemma 3.2, and Lemma 3.5. □

Next, we focus on various regularity estimates for the fluid velocity u. We have the following estimates.

Lemma 3.7. *Under the assumptions of Lemma 3.4 where we choose k_2 such that for a sufficient large integer k*

$$k_2 \le \frac{1}{2\beta} - 1 - \frac{\alpha}{2} - \frac{1}{(2k-1)\beta}, \tag{102}$$

which clearly is possible in light of (34), it follows that

$$\int_0^1 |u_x(x,t)|\,dx \le C(T), \qquad \|u(x,t)\|_{L^\infty(D_T)} \le C(T). \tag{103}$$

Proof. From the momentum equation of (48) we get

$$\begin{aligned} u_x &= c^{\gamma-\beta}Q(m)^{\gamma-\beta-1} + c^{-\beta}Q(m)^{-\beta-1}\int_0^x u_t\,dy \\ &= c^{\gamma-\beta}Q(m)^{\gamma-\beta-1} - c^{-\beta}Q(m)^{-\beta-1}\int_x^1 u_t\,dy. \end{aligned}$$

From this we estimate as follows:

$$\begin{aligned} \int_0^1 |u_x|\,dx &\le \int_0^1 c^{\gamma-\beta}Q(m)^{\gamma-\beta-1}\,dx + \int_0^1 c^{-\beta}Q(m)^{-\beta-1}\min\Big(\int_0^x |u_t|\,dy, \int_x^1 |u_t|\,dy\Big)\,dx \\ &\le \int_0^1 c^{\gamma-\beta}Q(m)^{\gamma-\beta-1}\,dx + \sqrt{2}\Big(\int_0^1 c^{-\beta}Q(m)^{-\beta-1}\phi(x)^{1/2}\,dx\Big)\Big(\int_0^1 |u_t|^2\,dy\Big)^{1/2}, \end{aligned} \tag{104}$$

where we have used Hölder's inequality and $\min(x, 1-x) \le 2x(1-x) = 2\phi(x)$ for $x \in [0,1]$. For the first integral on the right hand side of (104) we have:

$$\begin{aligned} \int_0^1 c^{\gamma-\beta}Q(m)^{\gamma-\beta-1}\,dx &\le \int_0^1 (cQ(m))^{\gamma-\beta}Q(m)^{-1}\,dx \le C\int_0^1 \phi(x)^{(\gamma-\beta)\alpha-\frac{1}{2k-1}}\frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx \\ &\le C\max\phi(x)^{(\gamma-\beta)\alpha-\frac{1}{2k-1}}\int_0^1 \frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx \le C(T), \end{aligned}$$

by choosing k sufficiently large, and where we have applied Corollary 3.1 and Corollary 3.3. For the second integral on the right hand side of (104) we have

$$\begin{aligned} \int_0^1 c^{-\beta}Q(m)^{-\beta-1}\phi(x)^{1/2}\,dx &\le C\int_0^1 Q(m)^{-\beta}\phi(x)^{\frac{1}{2}-\frac{\beta\alpha}{2}-\frac{1}{2k-1}}\frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx \\ &\le C\int_0^1 \phi(x)^{\frac{1}{2}-\frac{\beta\alpha}{2}-\frac{1}{2k-1}-\beta(1+k_2)}\frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx \\ &\le C(T)\max\phi(x)^{\frac{1}{2}-\frac{\beta\alpha}{2}-\frac{1}{2k-1}-\beta(1+k_2)}\int_0^1 \frac{\phi(x)^{\frac{1}{2k-1}}}{Q(m)}\,dx \le C(T), \end{aligned}$$

in view of Corollary 3.3 and Lemma 3.4, see estimate (76), and by using (102) which implies that $\beta(k_2+1) \le \frac{1}{2} - \frac{\alpha\beta}{2} - \frac{1}{(2k-1)}$. Thus, the first estimate of (103) has been proved. For the second estimate it is sufficient to observe that $\int_0^1 |u|\,dx \le C(T)$, which follows from Lemma 3.1, and apply Sobolev's embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$. □

The final estimates that are required, are about L^2-continuity in time. More precisely, we have.

Lemma 3.8. *Under the assumptions of Lemma 3.4, we have for* $0 < s \le t \le T$ *that*

$$\int_0^1 |m(x,t) - m(x,s)|^2 \, dx \le C(T)|t-s|, \tag{105}$$

$$\int_0^1 |n(x,t) - n(x,s)|^2 \, dx \le C(T)|t-s|, \tag{106}$$

$$\int_0^1 |u(x,t) - u(x,s)|^2 \, dx \le C(T)|t-s|, \tag{107}$$

$$\int_0^1 c(x)^\beta |Q(m)^{\beta+1} u_x(x,t) - Q(m)^{\beta+1} u_x(x,s)|^2 \, dx \le C(T)|t-s|. \tag{108}$$

Proof. We have, by Hölder's inequality and the second equation of (48) where we tactically have assumed $s < t$,

$$\begin{aligned}
\int_0^1 |Q(m)(x,t) - Q(m)(x,s)|^2 \, dx &= \int_0^1 \Big| \int_s^t Q(m)_\xi(x,\xi) \, d\xi \Big|^2 \, dx \\
&= \int_0^1 \Big| \int_0^T \chi_{[s,t]}(\xi) Q(m)_\xi(x,\xi) \, d\xi \Big|^2 \, dx \\
&\le \int_0^1 (t-s) \Big(\int_0^T |Q(m)_t(x,t)|^2 \, dt \Big) \, dx = (t-s)\rho_l^2 \int_0^T \int_0^1 Q(m)^4 u_x^2 \, dx \, dt \\
&= (t-s)\rho_l^2 \int_0^T \int_0^1 (c^{-\beta} Q(m)^{3-\beta}) c^\beta Q(m)^{\beta+1} u_x^2 \, dx \, dt \\
&\le (t-s)\rho_l^2 \int_0^T \max\Big(\frac{Q(m)^{3-\beta}}{c^\beta} \Big) V(s) \, ds \\
&\le C(T)(t-s) \int_0^T V(s) \, ds \le C(T)(t-s),
\end{aligned}$$

by using Lemma 3.1 where $V(s)$ is as defined in (94), and the fact that

$$\frac{Q^{3-\beta}}{c^\beta} \le C \frac{\phi(x)^{(3-\beta)\alpha/2}}{\phi(x)^{\beta\alpha/2}} \le C\phi(x)^{(\frac{3}{2}-\beta)\alpha} \le C,$$

in view of the restrictions on the parameters α and β. The fact that

$$m_t = (\rho_l - m)^2 Q(m)_t,$$

implies that this L^2 continuity estimate for $Q(m)$ can be taken over to m as well, resulting in the estimate (105). Moreover, the relation $c(x)m(x,t) = n(x,t)$ also implies that (106) is obtained from (105). Next, we focus on (107). Clearly we have

$$\int_0^1 |u(x,t) - u(x,s)|^2 \, dx = \int_0^1 \Big| \int_s^t u_\xi(x,\xi) d\xi \Big|^2 \, dx \le |t-s| \int_0^T \int_0^1 u_\xi^2(x,\xi) \, dx \, d\xi \le C(T)|t-s|,$$

where we have used Lemma 3.5. What remains is to prove (108). First, we see that

$$\int_0^1 c^\beta |Q(m)^{\beta+1} u_x(x,t) - Q(m)^{\beta+1} u_x(x,s)| \, dx \le |t-s| \int_0^T \int_0^1 [c^\beta Q(m)^{\beta+1} u_x]_\xi^2(x,\xi) \, dx \, d\xi.$$

Furthermore, we observe that

$$(c^\beta Q(m)^{\beta+1} u_x)_t = c^\beta Q(m)^{\beta+1} u_{xt} - \rho_l(\beta+1) c^\beta Q(m)^{\beta+2} u_x^2.$$

Thus,

$$\begin{aligned}
&\int_0^T\int_0^1[(c^\beta Q(m)^{\beta+1}u_x)_t]^2\,dx\,dt\\
&\le 2\int_0^T\int_0^1[c^\beta Q(m)^{\beta+1}]c^\beta Q(m)^{\beta+1}u_{xt}^2\,dx\,dt+2\rho_l^2(\beta+1)^2\int_0^T\int_0^1 c^{2\beta}Q(m)^{2\beta+4}u_x^4\,dx\,dt\\
&\le C(T)+C\int_0^T\max(c^\beta Q(m)^{\beta+3}u_x^2)V(t)\,dt,
\end{aligned}$$

where we have used (85) in Lemma 3.5 and Corollary 3.1 and $V(t)=\int_0^1 c^\beta Q(m)^{\beta+1}u_x^2(x,t)\,dx$. To estimate the last term on the right hand side we observe that

$$\begin{aligned}
c^\beta Q(m)^{\beta+3}u_x^2&=\frac{(cQ(m))^{1-\beta}}{c}(c^\beta Q(m)^{\beta+1}u_x)^2\\
&\le C\phi(x)^{\alpha(1/2-\beta)}\Big(\int_0^x u_t\,dx+P(c,Q)\Big)^2\\
&\le C(T),
\end{aligned}$$

again by application of Lemma 3.5 and Corollary 3.1 and the fact that $\beta<1/2$. Consequently, (108) has been proved. □

Remark 3.4. *The use of* (34) *in Lemmas 3.4–3.8 implies that a certain balance must hold between the parameters α and β. In particular, this relates the rate of the degeneracy to zero for initial data n_0 and m_0 at $x=0,1$ represented by $\phi(x)^\alpha$ and the degeneracy of the viscous coefficient $\varepsilon(n,m)=\frac{n^\beta}{(\rho_l-m)^{\beta+1}}$. This is a "two-phase phenomenon" in the sense that such a condition is not required for the single-phase analysis presented in* [31, 28].

4. Proof of existence result

In order to construct weak solutions to the initial-boundary problem (IBVP) (21)–(25), we apply the line method which can be described as follows [24]. For any given positive integer N, let $h=\frac{1}{N}$. Discretizing the derivatives with respect to x in (21), a system of ODEs of the following form is obtained

$$\begin{aligned}
&\frac{d}{dt}n_{2i}^h(t)+[n_{2i}^h(t)m_{2i}^h(t)]D_xu_{2i}^h(t)=0, && i=1,\dots,N-1,\\
&\frac{d}{dt}m_{2i}^h(t)+[m_{2i}^h(t)]^2D_xu_{2i}^h(t)=0, && i=1,\dots,N-1,\\
&\frac{d}{dt}u_{2i-1}^h(t)+D_xP(n_{2i-1}^h(t),m_{2i-1}^h(t))\\
&\qquad=\frac{1}{h}\Big(E(n_{2i}^h(t),m_{2i}^h(t))D_xu_{2i}^h-E(n_{2i-2}^h(t),m_{2i-2}^h(t))D_xu_{2i-2}^h\Big), && i=1,\dots,N,
\end{aligned}\tag{109}$$

with the boundary conditions

$$n_0^h(t)=n_{2N}^h(t)=0,\qquad m_0^h(t)=m_{2N}^h(t)=0,\qquad t\ge 0,\tag{110}$$

and initial data

$$\begin{aligned}
&n_{2i}^h(0)=n_0\Big(2i\frac{h}{2}\Big)>0,\qquad \rho_l>m_{2i}^h(0)=m_0\Big(2i\frac{h}{2}\Big)>0, && i=1,\dots,N-1,\\
&u_{2i-1}^h(0)=u_0\Big((2i-1)\frac{h}{2}\Big), && i=1,\dots,N,
\end{aligned}\tag{111}$$

where we have used the assumption (26) for the initial data n_0 and m_0. Here $D_xa_i=(a_{i+1}-a_{i-1})/h$ and $D_xa_{i-1}=(a_i-a_{i-2})/h$. Moreover, for $i=1,N$ we set $u_{-1}^h=u_{2N+1}^h(t)=0$ when we apply the third equation of (109), without loss of generality since $E(n_0^h(t),m_0^h(t))=E(n_{2N}^h(t),m_{2N}^h(t))=0$.

Theory for ordinary differential equations ensure that the Cauchy problem (109)–(111) admits a temporarily local solution in the domain

$$\mathbb{R}^{3N-1} = \{(n_{2i}^h(t), m_{2i}^h(t), u_{2j-1}^h(t)) \mid i = 1, \dots, N-1,\ j = 1, \dots, N\}$$

in the regularity class $C(0,T) \times C(0,T) \times C(0,T)$. Let $[0, T_1)$ be the right maximal interval of existence of this solution. The two first equations of (109) and the initial condition (111) imply that

$$n_{2i}^h(t) > 0, \qquad \rho_l > m_{2i}^h(t) > 0, \qquad \text{for } 0 < t < T_1 \qquad \text{and} \qquad i = 1, \dots, N-1. \tag{112}$$

The below analysis will show that T_1 can be any constant.

We now consider the local solutions $(n_{2i}(t), m_{2i}(t), u_{2i-1}(t))$ which satisfy (112) and introduce an auxiliary semi-discrete model associated with (109) which is more amenable for analysis. More precisely, we introduce the variable $c_{2i}^h(t)$ and $Q_{2i}^h(t)$ given by

$$c_{2i}^h(t) = \frac{n_{2i}^h(t)}{m_{2i}^h(t)}, \qquad Q_{2i}^h(t) = Q(m_{2i}^h(t)), \qquad i = 1, \dots, N-1, \tag{113}$$

for $Q(m) = m/(\rho_l - m)$ and observe that $c_{2i}^h(t) > 0$ and $Q_{2i}^h(t) > 0$ are well-defined for $0 < t < T_1$ and $i = 1, \dots, N-1$ by application of (112). Moreover, by applying (109) we observe that the quantities $(c_{2i}^h(t), Q_{2i}^h(t), u_{2i-1}^h(t))$ are described by a system of ODEs of the form

$$\begin{aligned}
&\frac{d}{dt} c_{2i}^h(t) = 0, && i = 1, \dots, N-1,\\
&\frac{d}{dt} Q_{2i}^h(t) + \rho_l [Q_{2i}^h(t)]^2 D_x u_{2i}^h(t) = 0, && i = 1, \dots, N-1,\\
&\frac{d}{dt} u_{2i-1}^h(t) + D_x P(c_{2i-1}^h(t), Q_{2i-1}^h(t)) &&\\
&\qquad = \frac{1}{h}\Big(E(c_{2i}^h(t), Q_{2i}^h(t)) D_x u_{2i}^h - E(c_{2i-2}^h(t), Q_{2i-2}^h(t)) D_x u_{2i-2}^h\Big), && i = 1, \dots, N,
\end{aligned} \tag{114}$$

where $P(c, Q)$ and $E(c, Q)$ are given by (49) and (50). Boundary conditions are given by

$$c_0^h(t) = c_{2N}^h(t) = 0, \qquad Q_0^h(t) = Q_{2N}^h(t) = 0, \qquad t \geq 0, \tag{115}$$

and initial data

$$\begin{aligned}
&c_{2i}^h(0) = c_0\Big(2i\frac{h}{2}\Big), \qquad Q_{2i}^h(0) = Q_0\Big(2i\frac{h}{2}\Big), && i = 1, \dots, N-1,\\
&u_{2i-1}^h(0) = u_0\Big((2i-1)\frac{h}{2}\Big), && i = 1, \dots, N.
\end{aligned} \tag{116}$$

In the following we neglect the index h and use the notation $(c_{2i}(t), Q_{2i}(t), u_{2i-1}(t))$. First of all, we obtain the following basic energy estimate by mimicing the arguments used in Lemma 3.1 similar to the references [23, 24, 31, 28].

Lemma 4.1. *Let $(c_{2i}(t), Q_{2i}(t), u_{2i-1}(t))$ be the solution to* (114) *with boundary and initial data given by* (115) *and* (116)*, respectively. Then we have*

$$\begin{aligned}
&\sum_{i=1}^{N}\Big(\frac{1}{2}u_{2i-1}^2(t) + \frac{c_{2i}^\gamma}{\rho_l(\gamma-1)} Q_{2i}^{\gamma-1}(t)\Big) h + \int_0^t \sum_{i=1}^{N} c_{2i}^\beta Q_{2i}^{\beta+1}(s) (D_x u_{2i}(s))^2 \, h \, ds\\
&\qquad = \sum_{i=1}^{N}\Big(\frac{1}{2}u_{2i-1}^2(0) + \frac{c_{2i}^\gamma}{\rho_l(\gamma-1)} Q_{2i}^{\gamma-1}(0)\Big) h, \qquad \forall t \in [0, T].
\end{aligned} \tag{117}$$

From the above a priori estimates it follows, similar to [23, 24], that the solution of (114) with boundary and initial conditions consistent with (110) and (111), exists for $0 \leq t < \infty$ and $c_{2i}(t) > 0$ and $Q_{2i}(t) > 0$ for $0 \leq t < \infty$ for any given h and $i = 1, \dots, N-1$. As a consequence, by application of the variable transformation (113), the existence and uniqueness of global solutions $(n_{2i}(t), m_{2i}(t), u_{2i-1}(t))$ to (109)–(111) follows as well, where (112) holds for any $T_1 < \infty$.

In the following we list a number of estimates that hold for the approximate solutions $(n_{2i}(t), m_{2i}(t), u_{2i-1}(t))$ that are uniform with respect to the discretization parameter h. These estimates are obtained by combining discrete arguments similar to those used in [24] and the techniques used in Section 3 to derive estimates for the continuous model.

Lemma 4.2. *Let $(n_{2i}(t), m_{2i}(t), u_{2i-1}(t))$ be the solution to (109)-(111). Then we have*

$$\frac{n_{2i}(t)}{\rho_l - m_{2i}(t)} = c_{2i}Q_{2i}(t) \le C(T)\phi(ih)^{\alpha}, \qquad Q_{2i}(t) \le C(T)\phi(ih)^{\alpha/2}, \tag{118}$$

and

$$\sum_{i=1}^{N} D_x\Big([c_{2i}Q_{2i}(t)]^{\beta}\Big)^{2k} h \le C(T), \tag{119}$$

for any positive integer k.

Lemma 4.3. *Let $(n_{2i}(t), m_{2i}(t), u_{2i-1}(t))$ be the solution to (109)-(111). Then we have for any positive integer k*

$$\sum_{i=1}^{N} \frac{\phi(ih)^{\frac{1}{2k-1}}}{Q_{2i}(t)} h \le C(T), \tag{120}$$

and

$$\sum_{i=1}^{N} u_{2i-1}^{2k}(t)\, h + k(2k-1)\int_0^t \sum_{i=1}^{N} u_{2i-1}^{2k-2}(s) c_{2i}^{\beta} Q_{2i}^{1+\beta}(s)\Big(D_x u_{2i-2}(s)\Big)^2 h\, ds \le C(T), \tag{121}$$

and

$$\sum_{i=1}^{N}\Big[\frac{d}{dt}u_{2i-1}(t)\Big]^2 h + \int_0^t \sum_{i=1}^{N} c_{2i}^{\beta} Q_{2i}^{\beta+1}(s)\Big(D_x\Big[\frac{d}{ds}u_{2i-2}(s)\Big]\Big)^2 h\, ds \le C(T). \tag{122}$$

Finally, we have

$$Q_{2i}(t) \ge C(T)\phi(ih)^{1+k_2}, \qquad i = 0, \dots, N, \tag{123}$$

where k_2 satisfies (38).

Lemma 4.4. *Let $(n_{2i}(t), m_{2i}(t), u_{2i-1}(t))$ be the solution to (109)-(111). Then we have*

$$\sum_{i=1}^{N} |m_{2i}(t) - m_{2i-2}(t)| \le C(T), \qquad \sum_{i=1}^{N} |n_{2i}(t) - n_{2i-2}(t)| \le C(T), \tag{124}$$

and

$$\sum_{i=1}^{N} |u_{2i+1}(t) - u_{2i-1}(t)| \le C(T), \tag{125}$$

and

$$|u_{2i+1}(t)| \le C(T), \tag{126}$$

and

$$\Big|E(n_{2i}(t), m_{2i}(t))D_x u_{2i}(t)\Big| \le C(T), \tag{127}$$

and

$$\sum_{i=1}^{N}\Big|E(n_{2i+2}(t), m_{2i+2}(t))D_x u_{2i+2}(t) - E(n_{2i}(t), m_{2i}(t))D_x u_{2i}(t)\Big| \le C(T). \tag{128}$$

Moreover, we have the time continuity estimates

$$\sum_{i=1}^{N} |m_{2i}(t) - m_{2i}(s)|^2\, h \le C(T)|t-s|, \qquad \sum_{i=1}^{N} |n_{2i}(t) - n_{2i}(s)|^2\, h \le C(T)|t-s|, \tag{129}$$

and

$$\sum_{i=1}^{N} |u_{2i-1}(t) - u_{2i-1}(s)|^2 \, h \le C(T)|t-s|, \tag{130}$$

and

$$\sum_{i=1}^{N} |E(n_{2i}(t), m_{2i}(t)) D_x u_{2i}(t) - E(n_{2i}(s), m_{2i}(s)) D_x u_{2i}(s)|^2 \, h \le C(T)|t-s|. \tag{131}$$

Following along the line of previous works, see for example [24, 31, 28], approximate solutions $(n_h(x,t), m_h(x,t), u_h(x,t))$ for $(x,t) \in D_T = [0,1] \times [0,T]$ are defined as follows:

$$\begin{aligned} &n_h(x,t) = n_{2i}(t), \qquad m_h(x,t) = m_{2i}(t), \\ &u_h(x,t) = \frac{1}{h}\Big([x-(i-1/2)h]u_{2i+1}(t) + [(m+1/2)h - x]u_{2i-1}(t)\Big), \end{aligned} \tag{132}$$

for

$$x \in ((i-1/2)h, (i+1/2)h), \qquad i = 0, 1, \ldots, N.$$

In particular, we observe that for this approximate solution we have

$$\partial_x u_h(x,t) = \frac{1}{h}\Big(u_{2i+1}(t) - u_{2i-1}(t)\Big) = D_x u_{2i}(t).$$

From estimate (124) follows that n_h and m_h as functions of x have total variations uniformly bounded with respect to x for any fixed $t > 0$. Let $\{t_i \,|\, i = 1, 2, \ldots\}$ be a countable dense set in $[0,T]$. By Helly's theorem and the diagonal process we can select sequences m_{h_j} and n_{h_j} which converge a.e. for $x \in [0,1]$ and any t_j. Then m_{h_j} (and n_{h_j}) tend to a function m (and n) in $L^2(0,1)$ for any $t_i > 0$. This convergence in $L^2(0,1)$ is uniform with respect to time t due to the time continuity (129).

The same arguments apply to the approximations $u_h(x,t)$, thanks to the estimates (125) and (130) as well as $E(n_h, m_h)\partial_x u_h$ in view of (128) and (131)). Thus, we can conclude that u_h converge to a limit function u a.e. in x and for any $t \in [0,T]$ and the convergence in $L^2(0,1)$ is uniform with respect to t. Similarly, $E(n_h, m_h)\partial_x u_h$ convergence to a limit function $E(n,m)\partial_x u$ a.e. in x and for any $t \in [0,T]$ and the convergence in $L^2(0,1)$ is uniform with respect to t.

By standard arguments it can now be shown that the limit function (n, m, u) obtained from (n_h, m_h, u_h) is a weak solution in the sense of (39) of Theorem 2.1, see for example [24] for details. This completes the proof of Theorem 2.1.

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