WELL-POSEDNESS OF A COMPRESSIBLE GAS-LIQUID MODEL WITH A FRICTION TERM IMPORTANT FOR WELL CONTROL OPERATIONS

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ABSTRACT. In this work we continue our investigations of a compressible gas-liquid model with special focus on inclusion of external frictional forces in the momentum balance. The model is often used for multiphase well flow modeling important for different well control operations. The frictional forces have a major impact on the pressure gradient, which determines the pressure distribution along the wellbore. Compression and decompression of gas in turn strongly depend on the pressure level along the wellbore. A precise understanding of these mechanisms is important since gas-kick scenarios and blow-out behavior is strongly linked to decompression effects. This work is a continuation of the recent work ["Global weak solutions for a gas-liquid model with external forces and general pressure law", SIAM J. Appl. Math. **71** (2), pp. 409–442, 2011]. The novelty of the present work lies in the fact that: (i) we consider a full momentum equation whereas a simplified one was used in the first work; (ii) the gas and liquid masses vanish at the boundaries making the analysis more involved; (iii) special care must be given to the frictional term to make sure that it is balanced with other terms such that a well-defined model is obtained. The analysis ensures that global existence of weak solutions is obtained under suitable assumptions on initial data (e.g. decay rate at the boundaries for gas and liquid mass) and parameters that determine growth rate of mass terms associated with, respectively, the wall friction term and viscous term.

Subject classification. 76T10, 76N10, 65M12, 35L60

Key words. two-phase flow, well model, gas-kick, weak solutions, Lagrangian coordinates, free boundary problem, friction term

1. Introduction

This work is devoted to a study of a transient gas-liquid two-phase model which, in Lagrangian variables, takes the following form:

$$\begin{aligned}
\partial_t n + (n\zeta)\partial_x u &= 0 \\
\partial_t \zeta + \zeta^2 \partial_x u &= 0 \\
\partial_t u + \partial_x p(n,\zeta) &= -f\zeta^\beta u|u| + \partial_x(E(n,\zeta)\partial_x u), \qquad x \in (0,1),
\end{aligned} \tag{1}$$

with constants $f, \beta > 0$. Here n is the gas mass, ζ the total mass (sum of gas and liquid mass), whereas u is the common fluid velocity. The pressure law, when liquid is assumed to be incompressible (ρ_l=const) and gas is treated as an ideal gas, takes the form

$$p(n,\zeta) = \Big(\frac{n}{\rho_l - [\zeta - n]}\Big)^\gamma, \qquad \gamma > 1. \tag{2}$$

The first term on the right hand side of the momentum equation represents wall friction where the parameter $\beta > 0$ describes the mass growth rate, whereas the second term takes into account

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The research of Helmer A. Friis has been supported by the Research Council of Norway under grant number 197739/V30 ("DMPL"), whereas the research of Steinar Evje has been supported by A/S Norske Shell.

other viscous effects and is characterized by the coefficient

$$E(n,\zeta) := \Big(\frac{\zeta}{(\rho_l - [\zeta - n])}\Big)^{\theta+1}, \qquad 0 < \theta < 1/2. \tag{3}$$

Moreover, boundary conditions are given by

$$n(0,t) = \zeta(0,t) = 0, \qquad n(1,t) = \zeta(1,t) = 0, \tag{4}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad \zeta(x,0) = \zeta_0(x), \quad u(x,0) = u_0(x), \qquad x \in (0,1). \tag{5}$$

This model problem represents a natural continuation of the work [12] where an existence result for a similar model was established with main focus on external forces like gravity and friction. We refer to this work for further motivation concerning application of this type of model in the context of well flow modeling. This work, in turn builds upon the works [8, 9], see also [23, 24] as well as the recent work [5] for similar studies.

A main concern in the work [12], as well as the current work, is inclusion and analysis of effects related to wall friction. Such friction terms are important for realistic predictions of pressure distribution along the wellbore, which in turn is crucial for the study of gas compression and decompression effects relevant for gas-kick flow scenarios [1, 12, 4]. The purpose of this work is to extend the analysis of the model in the following manner:

- First, we consider a situation where masses n and ζ vanish at the boundary, consequently, we cannot obtain a positive lower limit. This makes the analysis leading to the a priori estimates more involved;
- Secondly, we consider a full momentum equation, in contrast to the model analyzed in [12] where a simplified version of the momentum equation was considered.

The heart of the matter in the analysis is the use of an appropriate variable transformation which allows writing the two-phase model (1)–(5) in a form which naturally opens up for exploiting single-phase techniques. It turns out that we naturally can reformulate the initial boundary value (IBV) problem (1)–(5) described in terms of the variables (n,ζ,u) into a corresponding IBV problem described in terms of the variables (c,Q,u) where $c = n/\zeta$ and $Q(c,\zeta) = \zeta/(\rho_l - [1-c]\zeta)$. In particular, this connection allows us to explore the role played by the frictional term. New challenges due to the decay of masses to zero at the boundaries and the presence of the wall friction term are handled as follows:

- Concerning the degeneracy at the boundaries we mainly follow the ideas of [10, 11] where a weighting function $\phi(x)$, which vanishes at the boundaries, is employed.
- The pointwise upper bound of masses as expressed by Lemma 3.2 strongly depends on the fact that the wall friction term, $-f\zeta^\beta u|u|$ takes the form $-h(c,Q)u|u|$ in terms of (c,Q,u) where $h(c,Q)$, given by (49), becomes bounded for $Q > 0$. For this estimate we require that the initial gas and liquid mass decay to zero at the same rate, as stated in assumption (26). Moreover, due to the fact that the friction term contains a higher order velocity term $u|u|$, we can not directly from the energy estimate of Lemma 3.1 obtain the refined upper bound on Q as described by (70). We need the higher order L^p-regularity of u as provided by Lemma 3.3 for that purpose.
- New arguments must be introduced to obtain the result of Lemma 3.8 due to the appearance of the frictional term. In particular, we must show that $W(t) = \int_0^1 |(h(c,Q)u)_x|dx$ is in $L^1([0,T])$ for $h(c,Q)$ given by (49). This estimate relies on assumption (32) which relates the β-parameter to the θ-parameter of the viscosity term and parameter α that characterizes the decay rate of initial masses toward zero at the boundaries.

A main concern of this work is to identify more precisely the role of the wall friction term. More precisely, we seek to identify how the β parameter of the friction term is related to the γ parameter of the pressure law and the θ parameter of the viscous term. This balance between terms representing different forces is manifested itself in Lemma 3.8. In particular, the analysis depends on the fact that initial gas and liquid masses decay to zero at the boundaries at the same rate, expressed by the parameter α as in (26), in order to obtain sufficient control of the frictional term. See Remark 3.2 for details.

Overview. The rest of the paper is structured as follows: In Section 2 we first give necessary background information for deriving the model problem (1)–(5). Then, the assumptions on initial data and important parameters like γ, β, θ, and α are given followed by a precise statement of the main result of this paper, existence of weak solutions. Section 3 contains the estimates ranging from basic energy estimate to pointwise upper and lower limits of masses n, ζ, and velocity u, as well as various higher order regularity estimates. Section 4 gives a brief summary how to get converge to weak solutions by means of a semi-discrete approximation of the original model.

2. The existence result

We consider the following transient, "compressible gas-incompressible liquid" two-phase model (described in Eulerian coordinates)

$$\begin{gathered} \partial_t n + \partial_x[nu] = 0 \\ \partial_t m + \partial_x[mu] = 0 \\ \partial_t[(m+n)u] + \partial_x[(m+n)u^2] + \partial_x p(n,m) = -f(m+n)^{\beta+1}u|u| + \partial_x[\varepsilon(n,m)\partial_x u], \end{gathered} \tag{6}$$

on the interval $x \in (a(t), b(t))$. Here n is the gas mass, m the liquid mass, and u is fluid velocity. Pressure $p(n,m)$ and viscosity $\varepsilon(n,m)$ take the following form:

$$p(n,m) = C\rho_l^\gamma \Big(\frac{n}{\rho_l - m}\Big)^\gamma, \tag{7}$$

$$\varepsilon(n,m) = D\frac{(m+n)^\theta}{(\rho_l - m)^{\theta+1}}, \qquad \theta \in (0, 1/2), \tag{8}$$

where C and D are constants. For simplicity we set $C = D = 1$ in the following. We refer to [12] and references therein for more details concerning p and ε. The first term on the right hand side of the momentum equation describes wall friction effects. The constant $f > 0$ depends on fluid rheology as well as well/pipe diameter. More generally, it also depends on the prevailing flow regime. We assume that $\beta > 0$, see Section 2.1 for the precise assumptions on β. In fact, a main purpose of this work is to identify more precisely the interplay between the parameter β and the parameter γ in (7) and θ in (8).

One special feature of the above two-phase model (6)–(8) is the possible singular behavior associated with the pressure law at transition to pure liquid flow, that is, when $m = \rho_l\alpha_l = \rho_l$ or vacuum in the gas phase corresponding to $\rho_g = 0$. Now, introducing the variable $\zeta = m + n$, the system (6) can be written as

$$\begin{gathered} \partial_t n + \partial_x[nu] = 0 \\ \partial_t \zeta + \partial_x[\zeta u] = 0 \\ \partial_t[\zeta u] + \partial_x[\zeta u^2] + \partial_x p(n,\zeta) = -f\zeta^{\beta+1}u|u| + \partial_x[\varepsilon(n,\zeta)\partial_x u], \qquad x \in (a(t), b(t)). \end{gathered} \tag{9}$$

Motivated by previous studies for single-phase gas models we here propose to study the model (9) in a free boundary setting where the boundary points $a(t)$ and $b(t)$ are moving. More precisely, $a(t), b(t)$ are the particle paths separating the two-phase mixture and the vacuum state $n = m = \zeta = 0$ and is characterized as follows:

$$\begin{aligned} \frac{d}{dt}a(t) &= u(a(t),t), \qquad \text{and} \qquad n(a(t),t) = \zeta(a(t),t) = 0 \\ \frac{d}{dt}b(t) &= u(b(t),t), \qquad \text{and} \qquad n(b(t),t) = \zeta(b(t),t) = 0. \end{aligned} \tag{10}$$

Furthermore, the initial data are specified as follows

$$n(x,0) = n_0(x), \qquad \zeta(x,0) = \zeta_0(x), \qquad u(x,0) = u_0(x), \qquad x \in (a_0, b_0), \tag{11}$$

where $a_0 = a(0)$ and $b_0 = b(0)$. The boundary conditions are set as follows:

$$\zeta, n|_{x=a} = 0, \qquad \zeta, n|_{x=b} = 0. \tag{12}$$

In this work we assume that the initial masses $n_0(x), \zeta_0(x)$ connect to vacuum continuously, i.e., $\inf_{[0,1]} n_0(x) = 0 = \inf_{[0,1]} \zeta_0(x)$. Following along the line of previous studies for the single-phase Navier-Stokes equations [19, 17, 18], it is convenient to replace the moving domain $[a(t), b(t)]$ by a fixed domain by introducing suitable Lagrangian coordinates. First, in view of the particle paths $X_t(x)$ given by

$$\frac{dX_t(x)}{dt} = u(X_t(x), t), \qquad X_0(x) = x,$$

the system (9) takes the form

$$\begin{aligned} \frac{dn}{dt} + nu_x &= 0 \\ \frac{d\zeta}{dt} + \zeta u_x &= 0 \\ \zeta \frac{du}{dt} + p(n,\zeta)_x &= -f\zeta^{\beta+1} u|u| + (\varepsilon(n,\zeta)u_x)_x. \end{aligned} \tag{13}$$

Next, we introduce the coordinate transformation

$$\xi = \int_{a(t)}^{x} \zeta(y,t)\, dy, \qquad \tau = t, \tag{14}$$

such that the free boundary $x = a(t)$ and the free boundary $x = b(t)$, in terms of the (ξ, τ) coordinate system, are given by

$$\xi_{a(t)}(\tau) = 0, \qquad \xi_{b(t)}(\tau) = \int_{a(t)}^{b(t)} \zeta(y,t)\, dy = \int_{a_0}^{b_0} \zeta_0(y)\, dy = \text{const}, \tag{15}$$

where $\int_{a_0}^{b} \zeta_0(y)\, dy$ is the total liquid and gas mass initially, which we normalize to 1. Applying (14) to shift from (x,t) to (ξ,τ) in (13), we get

$$\begin{aligned} n_\tau + (n\zeta)u_\xi &= 0 \\ \zeta_\tau + (\zeta^2)u_\xi &= 0 \\ u_\tau + p(n,\zeta)_\xi &= -f\zeta^\beta u|u| + (\varepsilon(n,\zeta)\zeta u_\xi)_\xi, \qquad \xi \in (0,1), \quad \tau \geq 0. \end{aligned}$$

In the following, we find it convenient to replace the coordinates (ξ,τ) by (x,t) such that the model we shall work with in the rest of this paper is given in the form

$$\begin{aligned} \partial_t n + (n\zeta)\partial_x u &= 0 \\ \partial_t \zeta + \zeta^2 \partial_x u &= 0 \\ \partial_t u + \partial_x p(n,\zeta) &= -f\zeta^\beta u|u| + \partial_x(E(n,\zeta)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{16}$$

with

$$p(n,\zeta) = \Big(\frac{n}{\rho_l - [\zeta - n]}\Big)^\gamma \tag{17}$$

and

$$E(n,\zeta) := \varepsilon(n,\zeta)\zeta = \Big(\frac{\zeta}{(\rho_l - [\zeta - n])}\Big)^{\theta+1}, \qquad 0 < \theta < 1/2. \tag{18}$$

Moreover, in light of (12), boundary conditions are given by

$$\zeta(0,t) = n(0,t) = 0, \qquad \zeta(1,t) = n(1,t) = 0, \tag{19}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad \zeta(x,0) = \zeta_0(x), \quad u(x,0) = u_0(x), \qquad x \in (0,1). \tag{20}$$

2.1. **Main result.** We now state the main result for the model (16)–(20). However, we first give a precise statement of various assumptions on the initial data as well as of relations between important parameters like γ, β, θ, and α. These choices largely follow along the line of the single-phase work [10].

Assumptions. In this paper we use a weight function $\phi(x)$, which is assumed to fulfill

$$0 < \phi(x) < 1, \text{ for } \quad 0 < x < 1, \quad \phi(0) = \phi(1) = 0, \tag{21}$$

$$\phi'(x) \in L^\infty(I), \tag{22}$$

$$(x(1-x)) \leq C\phi(x), \tag{23}$$

From (23), it follows that $\phi(x)^a \in L^1([0,1])$ for every $a > -1$. Furthermore, the above model is subject to the following assumptions:

$$0 < \theta < \frac{1}{2}, \tag{24}$$

$$\gamma > 1, \tag{25}$$

For the initial masses n_0, m_0 it is assumed that there are constants $C_1, C_2, D_1, D_2 > 0$ and a parameter $\alpha > 0$, which is characterized more precisely in (30), such that

$$C_1\phi(x)^\alpha \leq n_0(x) \leq C_2\phi(x)^\alpha, \qquad D_1\phi(x)^\alpha \leq m_0(x) \leq D_2\phi(x)^\alpha, \tag{26}$$

where $D_2 < \rho_l$. Consequently, we have that

$$0 < \frac{D_1}{C_2} \leq \frac{m_0}{n_0}(x) \leq \frac{D_2}{C_1}.$$

For $c_0 = \frac{n_0}{n_0+m_0} = \frac{1}{1+\frac{m_0}{n_0}}$, it follows that $\sup_{x\in[0,1]} c_0(x) < 1$ and $\inf_{x\in[0,1]} c_0(x) > 0$. Hence, the following assumption is made concerning c_0:

$$\sup_{x\in[0,1]} c_0(x) < 1, \qquad \inf_{x\in[0,1]} c_0(x) > 0, \qquad (c_0)_x \in L^\infty([0,1]), \tag{27}$$

Concerning initial fluid velocity u_0, we assume that

$$u_0(x) \in L^\infty([0,1]). \tag{28}$$

For $Q_0 = \frac{n_0+m_0}{\rho_l - m_0}$ we assume that

$$(Q_0^{1+\theta} u_{0x}(x))_x \in L^{2n}([0,1]), \qquad n \in \mathbb{N}. \tag{29}$$

Now, let $\alpha > 0$ introduced in (26) satisfy the following relation

$$\frac{19}{20} + \frac{1}{10}\theta \leq \alpha \leq \frac{1}{2\theta}, \tag{30}$$

where $\nu > 0$ is defined by

$$\nu = (\frac{1}{2} - \theta)(1 + \frac{\theta}{10}). \tag{31}$$

The following restriction is assumed for β

$$\beta > \max\Big(\frac{\nu}{\alpha} + \theta, \frac{1}{2} + \frac{\theta}{2}\Big) > 0. \tag{32}$$

Let $k_1 > 0$ satisfy

$$2\nu < k_1 < \min\Big((2\gamma - 3\theta + 1)\alpha, \frac{60(1-2\theta)}{11(1+3\theta)} - 2\nu, \frac{40(1-2\theta)}{11(1+\theta)} - 2\nu\Big), \tag{33}$$

and, moreover

$$k_1 < \begin{cases} 1 + (1-3\theta)\alpha & \text{if } 0 < \theta < \frac{1}{3}, \\ \frac{20(1-2\theta)}{9-7\theta} + \frac{22(1-3\theta)}{9-7\theta}\nu & \text{if } \frac{1}{3} \leq \theta < \frac{1}{2}. \end{cases} \tag{34}$$

The following control for $Q_0 = \frac{n_0+m_0}{\rho_l - m_0}$ is then required (the first one is only a consequence of (26)):

$$0 \leq Q_0(x) \leq C\phi^\alpha(x), \tag{35}$$

$$\phi^\nu(x)(Q_0^\theta(x))_x \in L^2([0,1]), \tag{36}$$

$$\phi^{k_1}(x) Q_0^{2\theta-2}(x) \in L^1([0,1]), \tag{37}$$

and

$$(Q_0^\gamma(x))_x \in L^{2n}([0,1]), \qquad n \in \mathbb{N}. \tag{38}$$

Then we can state the main theorem.

Theorem 2.1 (Main Result)**.** *Given the assumptions* (24)–(36)*, then the initial-boundary problem* (16)–(20) *possesses a global weak solution* (n,ζ,u) *in the sense that for any* $T>0$*,*

(A) *we have the following regularity:*

$$n,\zeta,u\in L^\infty([0,1]\times[0,T])\cap C^1([0,T];L^2([0,1])),$$
$$E(n,\zeta)u_x\in L^\infty([0,1]\times[0,T])\cap C^{\frac{1}{2}}([0,T];L^2([0,1])).$$

In particular, the following pointwise estimates holds for $\mu>0$*:*

$$(\inf_{x\in[0,1]} c_0)\frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}}\le n(x,t)\le\min\Big\{\rho_l C(T)\phi(x)^\alpha,\frac{\rho_l-\mu}{1-\sup_{[0,1]}c}\Big\},$$
$$\frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}}\le \zeta(x,t)\le\min\Big\{\rho_l C(T)\phi(x)^\alpha,\frac{\rho_l-\mu}{1-\sup_{[0,1]}c}\Big\},$$

$\forall(x,t)\in[0,1]\times[0,T]$ *where the positive constant* μ *only depends on time* T *and the regularity of the initial data as stated in the assumptions.*

(B) *Moreover, the following equations hold,*

$$\begin{aligned}\int_0^\infty\int_0^1\Big[n\varphi_t-n\zeta u_x\varphi\Big]\,dx\,dt+\int_0^1 n_0(x)\varphi(x,0)\,dx=0,\\ \int_0^\infty\int_0^1\Big[\zeta\psi_t-\zeta^2u_x\psi\Big]\,dx\,dt+\int_0^1 \zeta_0(x)\psi(x,0)\,dx=0,\\ \int_0^\infty\int_0^1\Big[u\omega_t+\Big(p(n,\zeta)-E(n,\zeta)u_x\Big)\omega_x-f\zeta^\beta u|u|\omega\Big]\,dx\,dt+\int_0^1 u_0(x)\omega(x,0)\,dx=0,\end{aligned}\tag{39}$$

for any test function $\varphi(x,t),\psi(x,t),\omega(x,t)\in C_0^\infty(D)$,
with $D:=\{(x,t)\,|\,0\le x\le 1,\ t\ge 0\}$.

The proof of Theorem 2.1 is based on a series of priori estimates for approximate solutions of (16)–(20) and a corresponding limit procedure.

3. A priori estimates

In order to obtain the necessary estimates it is convenient to introduce a shift of variables as follows:

3.1. Transformed models.

We introduce the variable

$$c=\frac{n}{\zeta},\tag{40}$$

and see from the first two equations of (16) that

$$\partial_t c=\frac{1}{\zeta}n_t-\frac{n}{\zeta^2}\zeta_t=-\frac{n\zeta}{\zeta}u_x+\frac{n\zeta^2}{\zeta^2}u_x=0.$$

Consequently, the model (16)–(20) then can be written in terms of the variables (c,ζ,u) in the form

$$\begin{aligned}\partial_t c&=0\\ \partial_t\zeta+\zeta^2\partial_x u&=0\\ \partial_t u+\partial_x p(c,\zeta)&=-f\zeta^\beta u|u|+\partial_x(E(c,\zeta)\partial_x u),\qquad x\in(0,1),\end{aligned}\tag{41}$$

with

$$p(c,\zeta)=\Big(\frac{c\zeta}{\rho_l-[1-c]\zeta}\Big)^\gamma\tag{42}$$

and

$$E(c,\zeta)=\frac{\zeta^{\theta+1}}{(\rho_l-[1-c]\zeta)^{\theta+1}},\qquad 0<\theta<1/2.\tag{43}$$

Moreover, boundary conditions are given by

$$\begin{aligned} &\zeta(0,t)=0, \qquad &&\zeta(1,t)=0, \\ &c(0,t)=c_0(0), \qquad &&c(1,t)=c_0(1), \qquad t\geq 0, \end{aligned} \tag{44}$$

whereas initial data are

$$c(x,0)=c_0(x), \quad \zeta(x,0)=\zeta_0(x), \quad u(x,0)=u_0(x), \qquad x\in(0,1). \tag{45}$$

As remarked before, the model (41)–(45) possibly contains singular behavior associated with the pressure term p and viscosity term E. It is clear from these functions that ζ must obey an upper limit strong enough to ensure that these functions do not blow up. For that purpose we introduce the quantity $Q(c,\zeta)=\frac{\zeta}{\rho_l-(1-c)\zeta}$ and deduce a reformulated model in terms of the variables (c,Q,u). That is, we introduce the variable

$$Q(c,\zeta)=\frac{\zeta}{\rho_l-(1-c)\zeta}, \qquad \text{(which implies that } \zeta=\rho_l\frac{Q}{1+(1-c)Q}), \tag{46}$$

implicitly assuming $\zeta\geq 0$ and $\zeta<\frac{\rho_l}{1-c}$, and observe that

$$\begin{aligned} Q(c,\zeta)_t=\Big(\frac{\zeta}{\rho_l-(1-c)\zeta}\Big)_t &= \Big(\frac{1}{\rho_l-(1-c)\zeta}+\frac{(1-c)\zeta}{(\rho_l-(1-c)\zeta)^2}\Big)\zeta_t \\ &= \frac{\rho_l}{(\rho_l-(1-c)\zeta)^2}\zeta_t=-\rho_l\frac{\zeta^2}{(\rho_l-(1-c)\zeta)^2}u_x=-\rho_l Q(c,\zeta)^2u_x, \end{aligned}$$

in view of the second equation of (41). Consequently, we rewrite the model (41) in the form

$$\begin{aligned} &\partial_t c=0 \\ &\partial_t Q+\rho_l Q^2u_x=0 \\ &\partial_t u+\partial_x p(cQ)=-h(c,Q)u|u|+\partial_x(E(Q)\partial_x u), \qquad x\in(0,1), \end{aligned} \tag{47}$$

with

$$p(cQ)=(cQ)^\gamma, \tag{48}$$

and

$$h(c,Q)=f\rho_l^\beta\Big(\frac{Q}{1+(1-c)Q}\Big)^\beta, \tag{49}$$

and

$$E(Q)=Q^{\theta+1}, \qquad 0<\theta<1/2. \tag{50}$$

This model is then subject to the boundary conditions

$$\begin{aligned} &Q(0,t)=0, \qquad &&Q(1,t)=0, \\ &c(0,t)=c_0(0), \qquad &&c(1,t)=c_0(1), \qquad t\geq 0, \end{aligned} \tag{51}$$

In addition, we have the corresponding initial data

$$c(x,0)=c_0(x), \quad Q(x,0)=\frac{\zeta_0(x)}{\rho_l-(1-c_0(x))\zeta_0(x)}, \quad u(x,0)=u_0(x), \qquad x\in(0,1). \tag{52}$$

In particular, the first equation of (47) gives that

$$c(x,t)=c_0(x), \qquad t>0. \tag{53}$$

Remark 3.1. *It is interesting to compare the result of Theorem 2.1 to the main result of* [9]*. A main difference is that in* [9] *the viscosity coefficient* $\varepsilon(n,m)$ *is of the form*

$$\varepsilon(n,m)=\frac{n^\theta}{(\rho_l-m)^{\theta+1}},$$

which implies that it appears in the form $E(c,Q)=c^\theta Q^{\theta+1}$ *in the transformed model similar to* (47)*, however, where* $Q=\frac{m}{\rho_l-m}$ *and* $c=\frac{n}{m}$*. As a consequence, different estimates explicitly depend on the decay rate of* $c=c_0$*, which is assumed to be of the form*

$$C_1\phi(x)^{\alpha/2}\leq c_0(x)\leq C_2\phi(x)^{\alpha/2}, \qquad c_0=\frac{n_0}{m_0}.$$

In the current work we need a different decay rate for the initial masses n_0 and m_0 as stated in (26) *in order to obtain necessary control of the friction term.*

3.2. **A priori estimates.** We are now ready to establish some important estimates. We let C and $C(T)$ denote a generic positive constant depending only on the initial data and the given time T, respectively. We also note that a constant C can change from one line to another in a sequence of calculations.

In particular, we note from (49) that for $\beta > 0$

$$h(c,Q) = f\rho_l^\beta\Big(\frac{1}{\frac{1}{Q}+(1-c)}\Big)^\beta \le f\rho_l^\beta\Big(\frac{1}{1-c}\Big)^\beta \le C, \qquad Q \ge 0, \tag{54}$$

in view of assumption (27).

Remark 3.2. *Note that the assumption* $\sup_{x\in[0,1]} c_0 < 1$ *as given by* (27)*, which in turn is a result of assumption* (26) *requiring equal decay rate at the boundaries for gas and liquid, is essential for the estimate* (54)*. This estimate is crucial for the result of Lemma 3.2.*

Lemma 3.1 (Energy estimate)**.** *Under the assumptions of Theorem 2.1 we have the basic energy estimate*

$$\begin{aligned}&\int_0^1\Big(\frac{1}{2}u^2+\frac{c^\gamma}{\gamma-1}Q^{\gamma-1}\Big)dx+\int_0^t\int_0^1 Q^{1+\theta}u_x^2\,dx\,ds+\int_0^t\int_0^1 h(c,Q)u^2|u|dxds\\&=\int_0^1\Big(\frac{1}{2}u_0^2+\frac{c_0^\gamma}{\gamma-1}Q_0^{\gamma-1}\Big)\,dx \le C, \qquad \forall t\in[0,T].\end{aligned} \tag{55}$$

Proof. Start by summing equation (47)(b) multiplied by $\frac{c^\gamma Q^\gamma}{\rho_l Q^2}$ with equation (47)(c) multiplied by u to obtain

$$\frac{c^\gamma Q^\gamma Q_t}{\rho_l Q^2}+c^\gamma Q^\gamma u_x+uu_t+u(c^\gamma Q^\gamma)_x = u(Q^{1+\theta}u_x)_x - h(c,Q)u^2|u|. \tag{56}$$

Then rewrite equation (56) as

$$\frac{d}{dt}\Big(\frac{1}{2}u^2+\int_0^Q\frac{c^\gamma\xi^\gamma}{\rho_l\xi^2}d\xi\Big)+(c^\gamma Q^\gamma u)_x = u(Q^{1+\theta}u_x)_x - h(c,Q)u^2|u|, \tag{57}$$

and integrate it over $[0,1]\times[0,t]$ to yield

$$\begin{aligned}&\int_0^1\Big(\frac{1}{2}u^2+\frac{c^\gamma}{\gamma-1}\frac{Q^{\gamma-1}}{\rho_l}\Big)dx+\int_0^t\int_0^1 Q^{1+\theta}u_x^2dxds\\=&\int_0^1\Big(\frac{1}{2}u_0^2+\frac{c_0^\gamma}{\gamma-1}\frac{Q_0^{\gamma-1}}{\rho_l}\Big)dx+\int_0^t(Q^{1+\theta}uu_x)\Big|_{x=0}^{x=1}ds\\-&\int_0^t(c^\gamma Q^\gamma u)\Big|_{x=0}^{x=1}ds-\int_0^t\int_0^1 h(c,Q)u^2|u|dxds.\end{aligned} \tag{58}$$

Now invoking the boundary conditions (51) and the assumptions on the initial data we arrive at the conclusion (55). □

Now, we derive a pointwise upper bound on Q. We first present an upper bound which does not depend on the weighting function $\phi(x)$. Then, in Corollary 3.1 we present a more refined upper bound by making use of the higher order regularity of u as given by Lemma 3.3.

Lemma 3.2. *Under the assumptions of Theorem 2.1 we have the pointwise upper bound*

$$Q(x,t) \le C(T), \qquad \forall (x,t)\in[0,1]\times[0,T]. \tag{59}$$

Proof. Multiplying equation (47)(b) by $\theta Q^{\theta-1}$, we observe that

$$(Q^\theta)_t = -\rho_l\theta Q^{1+\theta}u_x. \tag{60}$$

We then integrate equation (60) over $[0,t]$ and, moreover, equation (47)(c) over $[0,x]$ (or alternatively over$[x,1]$), which gives

$$Q^{\theta}(x,t) = Q_0^{\theta}(x) - \rho_l\theta\int_0^t (Q^{1+\theta}u_x)(x,s)ds \tag{61}$$

and

$$Q^{1+\theta}u_x = (cQ)^{\gamma} + \int_0^x u_t dy + \int_0^x h(c,Q)u|u|dy = (cQ)^{\gamma} - \int_x^1 u_t dy - \int_x^1 h(c,Q)u|u|dy. \tag{62}$$

Putting $x=1$ in this last equation, using the boundary conditions, and integrating in time over $[0,t]$ reveals that

$$\int_0^x (u_0 - u(y,t))dy - \int_0^t\int_0^x h(c,Q)u|u|dy = -\int_x^1 (u_0 - u(y,t))dy + \int_0^t\int_x^1 h(c,Q)u|u|dy, \tag{63}$$

a fact which will be used in the following. We further substitute equation (62) into equation (61), and exploit the boundary conditions such that

$$\begin{aligned} Q^{\theta}(x,t) + \rho_l\theta\int_0^t (cQ)^{\gamma}(x,s)ds = Q_0^{\theta}(x) \quad &+ \quad \rho_l\theta\Big(\int_0^x u_0(y)dy - \int_0^x u(y,t)dy\Big) \\ &- \quad \rho_l\theta\int_0^t\int_0^x h(c,Q)u|u|dyds. \end{aligned} \tag{64}$$

We can then estimate $Q^{\theta}(x,t)$ as follows

$$Q^{\theta}(x,t) \quad\le\quad Q_0^{\theta} + C\int_0^x |u_0(y)|dy + C\int_0^x |u(y,t)|dy + C\int_0^t\int_0^x h(c,Q)u^2 dyds \tag{65}$$

for $0<x<1$, $0<t\le T$. Moreover using assumption (28), (54), Lemma 3.1 and the Hölder inequality, we find that

$$\begin{aligned} Q^{\theta}(x,t) \quad &\le\quad Q_0^{\theta} + Cx + C\Big(\int_0^1 u^2(y,t)dy\Big)^{\frac12}x^{\frac12} + C\Big(\int_0^t\int_0^1 u^2 dyds\Big) \\ &\le\quad Q_0^{\theta} + Cx^{\frac12} + CT, \end{aligned} \tag{66}$$

where we have used that $x \le x^{1/2}$ for $x\in[0,1]$. However, using equation (63) in equation (64) we can similarly deduce that

$$Q^{\theta}(x,t) \quad\le\quad Q_0^{\theta} + C(1-x)^{\frac12} + CT. \tag{67}$$

Finally, combining (66) and (67) and exploiting the fact that $\min(x,1-x)\le 2x(1-x)$ (for $0<x<1$) lead us to the following estimate

$$Q^{\theta}(x,t) \quad\le\quad Q_0^{\theta} + C(x(1-x))^{\frac12} + CT \le C\phi(x)^{\alpha\theta} + C(x(1-x))^{\frac12} + C(T), \tag{68}$$

where we use assumption (35) on the initial data Q_0. Clearly, we can conclude that the estimate (59) holds. □

Lemma 3.3. *Under the assumptions of Theorem 2.1 we have the following higher order estimate for any integer m*

$$\int_0^1 u^{2m}dx + m(2m-1)\int_0^t\int_0^1 u^{2m-2}Q^{1+\theta}u_x^2 dxds + 2m\int_0^t\int_0^1 h(c,Q)u^{2m}|u|dxds \le C(T). \tag{69}$$

We omit the proof of Lemma 3.3 for brevity. It can be proved using similar arguments as in [12]. A key step is that we make use of the pointwise upper bound of Q given by (59).

However, equipped with the higher order control on u as given by Lemma 3.3, we can derive a more refined upper bound for Q that depends on $\phi(x)$.

Corollary 3.1. *Under the assumptions of Theorem 2.1 we have the pointwise upper bound*

$$Q(x,t) \le C(T)\phi^{\alpha}(x), \qquad \forall (x,t)\in[0,1]\times[0,T]. \tag{70}$$

Proof. We only have to revisit the last term of (65), which is the friction related term. Clearly, we can estimate as follows

$$\int_0^t \int_0^x h(c,Q)u^2 dyds \le Cx^{\frac{1}{2}} \int_0^t \Big(\int_0^1 u^4\,dx\Big)^{\frac{1}{2}} ds \le CTx^{\frac{1}{2}}.$$

Following the same arguments as used in Lemma 3.2, we conclude that (68) is refined to

$$Q^\theta(x,t) \le C\phi(x)^{\alpha\theta} + C(T)(x(1-x))^{\frac{1}{2}}. \tag{71}$$

But, since $0<\alpha\le \frac{1}{2\theta}$, according to (30), the conclusion (70) follows. □

The next lemma largely follow arguments used for single-phase analysis and the friction term does not cause additional problems since it appears as a non-negative term that can be ignored, see (76).

Lemma 3.4. *Under the assumptions of Theorem 2.1 and for $2\nu=(1-2\theta)(1+\frac{\theta}{10})$ we have the following upper bound*

$$\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2\,dx \le C(T), \tag{72}$$

Proof. Using equation (60) in combination with the momentum equation (47)(c) we obtain

$$(Q^\theta)_{xt} = -\theta\rho_l(u_t + ((cQ)^\gamma)_x) - \theta\rho_l h(c,Q)u|u|. \tag{73}$$

We then multiply this equation $\phi^{2\nu}(Q^\theta)_x$ and integrate it over $[0,1]\times[0,t]$ to yield

$$\begin{aligned}
&\frac{\theta^2}{2}\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2dx + \rho_l\theta^2\gamma\int_0^t\int_0^1 \phi^{2\nu}c^\gamma Q^{\gamma+\theta-2}Q_x^2dxds = \frac{\theta^2}{2}\int_0^1 \phi^{2\nu}Q_0^{2\theta-2}Q_{0x}^2dx\\
-\;& \rho_l\theta\int_0^t\int_0^1 \phi^{2\nu}u_t(Q^\theta)_xdxds - \rho_l\theta\int_0^t\int_0^1 \phi^{2\nu}h(c,Q)u|u|(Q^\theta)_xdxds\\
-\;& \rho_l\theta^2\gamma\int_0^t\int_0^1 \phi^{2\nu}c^{\gamma-1}Q^{\gamma+\theta-1}Q_xc_xdxds
\end{aligned} \tag{74}$$

Using partial integration we can rewrite as follows

$$\begin{aligned}
&\frac{\theta^2}{2}\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2dx + \rho_l\theta^2\gamma\int_0^t\int_0^1 \phi^{2\nu}c^\gamma Q^{\gamma+\theta-2}Q_x^2dxds = \frac{\theta^2}{2}\int_0^1 \phi^{2\nu}Q_0^{2\theta-2}Q_{0x}^2dx\\
-\;& \rho_l\theta\int_0^1 \phi^{2\nu}u(Q^\theta)_xdx + \rho_l\theta\int_0^1 \phi^{2\nu}u_0(Q_0^\theta)_xdx + \rho_l\theta\int_0^t\int_0^1 \phi^{2\nu}u(Q^\theta)_{xt}dxds\\
-\;& \rho_l\theta\int_0^t\int_0^1 \phi^{2\nu}h(c,Q)u^2(Q^\theta)_xdxds - \rho_l\theta^2\gamma\int_0^t\int_0^1 \phi^{2\nu}c^{\gamma-1}Q^{\gamma+\theta-1}Q_xc_xdxds\\
:=\;& I_0+I_1+I_2+I_3+I_4+I_5.
\end{aligned} \tag{75}$$

Furthermore, estimating the quantities I_0, I_1, I_2, I_3, I_4 and I_5 (see Appendix A) we arrive at the inequality

$$\begin{aligned}
&\frac{\theta^2}{4}\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2dx + \rho_l\theta^2\gamma\int_0^t\int_0^1 \phi^{2\nu}c^\gamma Q^{\gamma+\theta-2}Q_x^2dxds\\
+\;& (\rho_l\theta)^2\int_0^t\int_0^1 \phi^{2\nu}h(c,Q)u^2|u|dxds \le C(T) + C\int_0^t\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2dx.
\end{aligned} \tag{76}$$

Clearly, this implies that

$$\frac{\theta^2}{4}\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2dx \le C(T) + C\int_0^t\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2dx, \tag{77}$$

and an application of Gronwall's lemma proves (72). □

The next lemma is obtained by following along the line of [10]. The main difference is the appearance of a new non-negative term on the left hand side of the inequality (78) due to the friction term.

Lemma 3.5. *Under the assumptions of Theorem 2.1 where k_1 is characterized by (33), for any integer $m > 0$ and for $\alpha_1 = (1-\frac{1}{2^m})(\theta-1) < 0$, we have the following upper bound*

$$\int_0^1 \phi^{k_1} Q^{\alpha_1} u^2 dx + \int_0^t\int_0^1 \phi^{k_1} Q^{1+\theta+\alpha_1} u_x^2\, dxds + \int_0^t\int_0^1 \phi^{k_1} h(c,Q) Q^{\alpha_1} u^2 |u| dxds \le C(T). \tag{78}$$

Proof. First let

$$\alpha_m = \frac{\theta-1}{2}, \tag{79}$$

and, moreover, define α_{m-1} as,

$$\alpha_{m-1} = \frac{\alpha_m}{2} + \frac{\theta-1}{2} = \frac{3}{2}\alpha_m. \tag{80}$$

It follows from the equations (47)(b) and (c) that

$$\begin{aligned}(\phi^{k_1} Q^{\alpha_m} u^{2^m})_t &= -\alpha_m \rho_l \phi^{k_1} Q^{1+\alpha_m} u^{2^m} u_x + 2^m \phi^{k_1} Q^{\alpha_m} u^{2^m-1} (Q^{1+\theta} u_x)_x \\ &\quad - 2^m \phi^{k_1} Q^{\alpha_m} u^{2^m-1} (c^\gamma Q^\gamma)_x - 2^m \phi^{k_1} h(c,Q) Q^{\alpha_m} u^{2^m} |u|.\end{aligned} \tag{81}$$

We integrate equation (81) over $[0,1]\times[0,t]$, which after application of partial integration and the boundary conditions yields

$$\begin{aligned}&\int_0^1 \phi^{k_1} Q^{\alpha_m} u^{2^m} dx + 2^m(2^m-1)\int_0^t\int_0^1 \phi^{k_1} Q^{1+\theta+\alpha_m} u^{2^m-2} u_x^2 dxds \\ &+ 2^m \int_0^t\int_0^1 \phi^{k_1} h(c,Q) Q^{\alpha_m} u^{2^m} |u| dxds = \int_0^1 \phi^{k_1} Q_0^{\alpha_m} u_0^{2^m} dx - \alpha_m \rho_l \int_0^t\int_0^1 \phi^{k_1} Q^{1+\alpha_m} u^{2^m} u_x dxds \\ &- 2^m \alpha_m \int_0^t\int_0^1 \phi^{k_1} Q^{\theta+\alpha_m} u^{2^m-1} Q_x u_x dxds - 2^m k_1 \int_0^t\int_0^1 \phi^{k_1-1} Q^{1+\theta+\alpha_m} u^{2^m-1} u_x \phi'(x) dxds \\ &- 2^m\gamma \int_0^t\int_0^1 \phi^{k_1} c^\gamma Q^{\gamma+\alpha_m-1} u^{2^m-1} Q_x dxds - 2^m\gamma \int_0^t\int_0^1 \phi^{k_1} c^{\gamma-1} Q^{\gamma+\alpha_m} u^{2^m-1} c_x dxds \\ &:= \sum_{i=1}^6 I_i^m \le C(T),\end{aligned} \tag{82}$$

where the estimation of I_i^m (for $i = 1,2,3,4,5,6$) is given in Appendix B, see (152)–(157). Obviously, equation (82) is also valid for α_{m-1} and $m-1$ (instead of α_m and m and with the exception of the inequality part, which must be proved), and thus we obtain

$$\begin{aligned}&\int_0^1 \phi^{k_1} Q^{\alpha_{m-1}} u^{2^{m-1}} dx + 2^{m-1}(2^{m-1}-1)\int_0^t\int_0^1 \phi^{k_1} Q^{1+\theta+\alpha_{m-1}} u^{2^{m-1}-2} u_x^2 dxds \\ &\quad + 2^{m-1}\int_0^t\int_0^1 \phi^{k_1} h(c,Q) Q^{\alpha_{m-1}} u^{2^{m-1}} |u| dxds \\ &= \int_0^1 \phi^{k_1} Q_0^{\alpha_{m-1}} u_0^{2^{m-1}} dx - \alpha_{m-1}\rho_l \int_0^t\int_0^1 \phi^{k_1} Q^{1+\alpha_{m-1}} u^{2^{m-1}} u_x dxds \\ &\quad - 2^{m-1}\alpha_{m-1}\int_0^t\int_0^1 \phi^{k_1} Q^{\theta+\alpha_{m-1}} u^{2^{m-1}-1} Q_x u_x dxds \\ &\quad - 2^{m-1} k_1 \int_0^t\int_0^1 \phi^{k_1-1} Q^{1+\theta+\alpha_{m-1}} u^{2^{m-1}-1} u_x \phi'(x) dxds \\ &\quad - 2^{m-1}\gamma \int_0^t\int_0^1 \phi^{k_1} c^\gamma Q^{\gamma+\alpha_{m-1}-1} u^{2^{m-1}-1} Q_x dxds \\ &\quad - 2^{m-1}\gamma \int_0^t\int_0^1 \phi^{k_1} c^{\gamma-1} Q^{\gamma+\alpha_{m-1}} u^{2^{m-1}-1} c_x dxds := \sum_{i=1}^6 I_i^{m-1} \le C(T),\end{aligned} \tag{83}$$

where the estimation of I_i^{m-1} (for $i = 1,2,3,4,5,6$) follows from the estimates in Appendix B, see (158)–(162). These estimates, in turn, depend on the estimate (82). The recurrence relation (80) then implies that $\alpha_k = (2-\frac{1}{2^{m-k}})(\frac{\theta-1}{2})$ for $k=1,\dots,m$. In particular, $\alpha_1 = (1-\frac{1}{2^m})(\theta-1)$. We can thus conclude by induction that

$$\int_0^1 \phi^{k_1} Q^{\alpha_1} u^2 dx + \int_0^t \int_0^1 \phi^{k_1} Q^{1+\theta+\alpha_1} u_x^2 dxds + \int_0^t \int_0^1 \phi^{k_1} h(c,Q) Q^{\alpha_1} u^2 |u| dxds \le C(T), \quad (84)$$

and the proof is completed. □

Lemma 3.6. *Under the assumptions of Theorem 2.1 and for any integer $m > 0$ and for $\beta_1 = (2-\frac{1}{2^m})(\theta-1) < 0$, we have*

$$\int_0^1 \phi^{k_1} Q^{\beta_1} dx \le C(T). \quad (85)$$

Proof. From equation (47)(b) it follows that

$$(\phi^{k_1} Q^{\beta_1})_t = -\beta_1 \rho_l \phi^{k_1} Q^{1+\beta_1} u_x. \quad (86)$$

Integrate (86) over $[0,1]\times[0,t]$ to obtain

$$\int_0^1 \phi^{k_1} Q^{\beta_1} dx = \int_0^1 \phi^{k_1} Q_0^{\beta_1} dx - \beta_1 \rho_l \int_0^t \int_0^1 \phi^{k_1} Q^{1+\beta_1} u_x dxds. \quad (87)$$

Furthermore, we obtain an estimate for $\int_0^1 \phi^{k_1} Q^{\beta_1} dx$ from (87) by using the Cauchy inequality as follows:

$$\begin{aligned}\int_0^1 \phi^{k_1} Q^{\beta_1} dx &\le \int_0^1 \phi^{k_1} Q_0^{\beta_1} dx + C \int_0^t \int_0^1 \phi^{\frac{k_1}{2}} Q^{\frac{1+\theta+\alpha_1}{2}} u_x \phi^{\frac{k_1}{2}} Q^{1+\beta_1} Q^{\frac{-(1+\theta+\alpha_1)}{2}} dxds \\ &\le \int_0^1 \phi^{k_1} Q_0^{\beta_1} dx + C \int_0^t \int_0^1 \phi^{k_1} Q^{1+\theta+\alpha_1} u_x^2 dxds + C \int_0^t \int_0^1 \phi^{k_1} Q^{1+2\beta_1-\theta-\alpha_1} dxds.\end{aligned} \quad (88)$$

Now notice, in view of assumptions (35) and (37), that

$$\int_0^1 \phi^{k_1} Q_0^{\beta_1} dx \le C.$$

Moreover,

$$\int_0^t \int_0^1 \phi^{k_1} Q^{1+\theta+\alpha_1} u_x^2 dxds \le C(T),$$

due to Lemma 3.5. Thus by using these two latter facts and the fact that $1+2\beta_1-\theta-\alpha_1 = \beta_1$, (88) can be written as

$$\int_0^1 \phi^{k_1} Q^{\beta_1} dx \quad \le \quad C(T) + C \int_0^t \int_0^1 \phi^{k_1} Q^{\beta_1} dxds. \quad (89)$$

After an application of Gronwall's lemma we arrive at the conclusion (85). □

Lemma 3.7. *Under the assumptions of Theorem 2.1, and for $k_2 = \nu + \frac{k_1}{2}$ where $k_1 > 2\nu$, we have the following pointwise lower bound on Q*

$$Q(x,t) \ge C(T) \phi^{\frac{11k_2}{10(1-2\theta)}}(x), \qquad \forall (x,t) \in [0,1]\times[0,T]. \quad (90)$$

Proof. It follows from the Sobolev inequality $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$ that

$$\begin{aligned}[\phi^{k_2} Q^{\beta_2}](x,t) &\le C \int_0^1 |\phi^{k_2} Q^{\beta_2}| dx + C \int_0^1 |(\phi^{k_2} Q^{\beta_2})_x| dx \\ &\le C \int_0^1 |\phi^{k_2} Q^{\beta_2}| dx + C \int_0^1 |\phi^{k_2-1} Q^{\beta_2}| dx + C \int_0^1 \phi^{k_2} |(Q^{\beta_2})_x| dx.\end{aligned} \quad (91)$$

Choosing β_2 such that $\beta_2 = \theta + (1-\frac{1}{2^{m+1}})(\theta-1)$, and noting that $\frac{\beta_1}{2} = (1-\frac{1}{2^{m+1}})(\theta-1)$ then it's clear that

$$\beta_2 = \theta + \frac{\beta_1}{2}, \qquad \beta_2 - \beta_1 = \theta - \frac{\beta_1}{2} > 0, \qquad 2\beta_2 - \beta_1 = 2\theta > 0, \quad (92)$$

and it is also clear that $\beta_2 < 0$ for m large enough since $0 < \theta < \frac{1}{2}$. Some further simple manipulations including application of the Cauchy inequality, Young's inequality with $p = \frac{\beta_1}{\beta_2}$ and $q = \frac{\beta_1}{\beta_1 - \beta_2}$, and Corollary 3.1 then gives

$$\begin{aligned}
\phi^{k_2} Q^{\beta_2}(x,t) &\le C\int_0^1 \phi^{\nu} Q^{\beta_2 - \frac{\beta_1}{2}} \phi^{\frac{k_1}{2}} Q^{\frac{\beta_1}{2}}\,dx + C\int_0^1 \phi^{k_2} Q^{\beta_2 - 1}|Q_x|\,dx + C\int_0^1 \phi^{k_2-1} Q^{\beta_2}\,dx \\
&\le C\int_0^1 \phi^{2\nu} Q^{2\beta_2-\beta_1}\,dx + C\int_0^1 \phi^{k_1} Q^{\beta_1}\,dx + C\int_0^1 \phi^{\nu+\frac{k_1}{2}} Q^{\beta_2 - 1 + \theta - \theta}|Q_x|\,dx \\
&\quad + C\int_0^1 \phi^{k_2 - 1 - k_1\frac{\beta_2}{\beta_1}} \phi^{k_1\frac{\beta_2}{\beta_1}} Q^{\beta_2}\,dx \qquad (93)\\
&\le C\int_0^1 \phi^{2\nu+2\alpha\theta}\,dx + C\int_0^1 \phi^{k_1} Q^{\beta_1}\,dx + C\int_0^1 \phi^{k_1} Q^{2\beta_2-2\theta}\,dx \\
&\quad + C\int_0^1 \phi^{2\nu} Q^{2\theta-2} Q_x^2\,dx + C\int_0^1 \phi^{k_1} Q^{\beta_1}\,dx + C\int_0^1 \phi^{(k_2-1-k_1\frac{\beta_2}{\beta_1})(\frac{\beta_1}{\beta_1-\beta_2})}\,dx.
\end{aligned}$$

Moreover, application of (92), Lemmas 3.4 and 3.6, in addition to the fact that $2\nu + 2\alpha\theta > -1$ and $(k_2 - 1 - k_1\frac{\beta_2}{\beta_1})\frac{\beta_1}{\beta_1-\beta_2} > -1$ (the latter for sufficiently large m), allow us to conclude that

$$\phi^{k_2} Q^{\beta_2}(x,t) \quad \le \quad C(T). \tag{94}$$

Finally, since $\beta_2 < 0$ and $2\theta - 1 < \beta_2 < \frac{10}{11}(2\theta - 1)$ for sufficiently large m, it follows from (94) that $Q(x,t) \ge C(T)\phi^{-\frac{k_2}{\beta_2}} \ge C(T)\phi^{\frac{11k_2}{10(1-2\theta)}}$. □

Equipped with the upper and lower limits on $Q(c,\zeta)$, this pointwise control can be transferred to the masses n and ζ. We also can derive BV-estimates for these mass variables by relying on Lemma 3.4 and assumption (27). These results are summed up in the following two corollaries.

Corollary 3.2. *We have the upper and lower bounds*

$$C(T)\phi^{\frac{11k_2}{10(1-2\theta)}} \le Q(x,t) \le C(T)\phi(x)^{\alpha}, \tag{95}$$

$$\frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}} \le \zeta(x,t) \le \min\Big\{\rho_l C(T)\phi(x)^{\alpha}, \frac{\rho_l - \mu}{1 - \sup_{[0,1]} c}\Big\}, \tag{96}$$

$$(\inf_{x\in[0,1]} c_0)\frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}} \le n(x,t) \le \min\Big\{\rho_l C(T)\phi(x)^{\alpha}, \frac{\rho_l - \mu}{1 - \sup_{[0,1]} c}\Big\}, \tag{97}$$

where $\mu > 0$ is a small constant.

Proof. The first estimate (95) follows from Corollary 3.1 and Lemma 3.7. For the second estimate (96) we observe that

$$\zeta = \rho_l \frac{Q}{1 + (1-c)Q}, \tag{98}$$

for $\rho_l - (1-c)\zeta > 0$. Consequently, in view of (95) and (46), it follows that

$$\zeta \le \rho_l Q \le \min\Big\{\rho_l C(T)\phi(x)^{\alpha}, \frac{\rho_l - \mu}{1 - \sup_{[0,1]} c}\Big\},$$

for an appropriate choice of $\mu > 0$. Moreover, (95) and (98) also imply that

$$\zeta \ge \frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}}.$$

In view of the fact that $n(x,t) = \zeta(x,t)c_0(x)$ and assumption (27), the last estimate (97) follows. □

Corollary 3.3. *We have the estimates*

$$\int_0^1 |\partial_x \zeta|\,dx \le C(T), \qquad \int_0^1 |\partial_x n|\,dx \le C(T), \tag{99}$$

for a suitable constant $C(T)$.

Proof. It follows that

$$Q(c,\zeta)_x = Q_c c_x + Q_\zeta \zeta_x = -Q^2 c_x + \rho_l \zeta^{-2} Q^2 \zeta_x. \tag{100}$$

For $x \in (0,1)$ where $Q > 0$ we can rewrite in the form

$$\begin{aligned}
\zeta_x &= \rho_l^{-1}\Big(\frac{\zeta}{Q}\Big)^2 Q_x + \rho_l^{-1}\zeta^2 c_x \\
&= \rho_l\Big(\frac{1}{1+(1-c)Q}\Big)^2 Q_x + \rho_l^{-1}\zeta^2 c_x \\
&= \phi(x)^{-\nu} Q^{1-\theta}\rho_l\Big(\frac{1}{1+(1-c)Q}\Big)^2 \phi(x)^{\nu} Q^{\theta-1} Q_x + \rho_l^{-1}\zeta^2 c_x.
\end{aligned}$$

Consequently, using Cauchy inequality, Corollary 3.1, Lemma 3.4, and assumption (27), we get

$$\begin{aligned}
\int_0^1 |\zeta_x|\,dx &\le C\int_0^1 \phi(x)^{-2\nu} Q^{2(1-\theta)}dx + C\int_0^1 \phi(x)^{2\nu} Q^{2-2\theta} Q_x^2 dx + C\int_0^1 |c_x| dx \\
&\le C(T) + C\int_0^1 \phi(x)^{-2\nu+2(1-\theta)\alpha}dx \le C(T),
\end{aligned} \tag{101}$$

since $1 + 2(1-\theta)\alpha > 2\nu$. Clearly, we also have the estimate

$$\int_0^1 |n_x|\,dx \le \int_0^1 |\zeta||c_x|\,dx + \int_0^1 c|\zeta_x|\,dx \le C(T),$$

in view of assumption (27), Corollary 3.2, and estimate (101) of $\int |\zeta_x|\,dx$. □

Lemma 3.8. *For a given integer* $n > 0$*, and under the assumptions of Theorem 2.1, we can prove that*

$$\int_0^1 u_t^{2n} dx + n(2n-1)\int_0^t\int_0^1 Q^{\theta+1} u_{xt}^2 u_t^{2n-2} dxds \le C(T). \tag{102}$$

Proof. We differentiate the third equation of (47) with respect to time t, multiply the resulting equation by $2nu_t^{2n-1}$ and integrate over $[0,1]\times[0,t]$, and obtain

$$\begin{aligned}
&\int_0^1 u_t^{2n}(x,t)\,dx + 2n\int_0^t\int_0^1 p(cQ)_{xt} u_t^{2n-1} dxds \\
&= \int_0^1 u_t^{2n}(x,0)dx - 2n\int_0^t\int_0^1 (h(c,Q)u|u|)_t u_t^{2n-1}dxds + 2n\int_0^t\int_0^1 (Q^{\theta+1}u_x)_{xt} u_t^{2n-1} dxds.
\end{aligned} \tag{103}$$

First, it follows that

$$\int_0^1 u_t^{2n}(x,0)\,dx \le C(T), \tag{104}$$

by considering the momentum equation of (47) at time $t = 0$

$$(u_0)_t + p(c_0Q_0)_x = -h(c_0,Q_0)u_0|u_0| + (Q_0^{\theta+1}u_{0,x})_x,$$

together with assumptions (24)–(29), and (38), as well as estimate (54). We also note that

$$\begin{aligned}
&\int_0^t\int_0^1 [p(cQ) - (Q^{\theta+1}u_x)]_{xt} u_t^{2n-1} dxds \\
&\quad= \int_0^t ([p(cQ) - (Q^{\theta+1}u_x)]_t u_t^{2n-1})\Big|_{x=0}^{x=1} ds - \int_0^t\int_0^1 [p(cQ) - (Q^{\theta+1}u_x)]_t (u_t^{2n-1})_x dxds \\
&\quad= -\int_0^t\int_0^1 [p(cQ) - (Q^{\theta+1}u_x)]_t (u_t^{2n-1})_x dxds,
\end{aligned} \tag{105}$$

by application of the boundary conditions (51). Moreover, using the second equation of (47) it follows that

$$\begin{aligned}
&\int_0^t\int_0^1 (Q^{\theta+1}u_x)_t(u_t^{2n-1})_x dxds \qquad (106)\\
&= (2n-1)\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2u_t^{2n-2}dxds - (2n-1)(\theta+1)\rho_l\int_0^t\int_0^1 Q^{\theta+2}u_x^2u_{xt}u_t^{2n-2}dxds\\
&= (2n-1)\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2u_t^{2n-2}dxds + I_1^{(n)},
\end{aligned}$$

and

$$\int_0^t\int_0^1 p(cQ)_t(u_t^{2n-1})_x dxds = -\gamma\rho_l(2n-1)\int_0^t\int_0^1 c^\gamma Q^{\gamma+1}u_xu_{xt}u_t^{2n-2}dxds = I_2^{(n)}, \qquad (107)$$

and

$$\begin{aligned}
&\int_0^t\int_0^1 (h(c,Q)u|u|)_t u_t^{2n-1}\,dx\,ds\\
&\quad = -\rho_l\int_0^t\int_0^1 h_Q(c,Q)Q^2u_xu|u|u_t^{2n-1}dxds + 2\int_0^t\int_0^1 h(c,Q)|u|u_t^{2n}dxds \qquad (108)\\
&\quad = I_3^{(n)} + I_4^{(n)}.
\end{aligned}$$

Moreover, using the "epsilon version" Cauchy inequality (i.e. $ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}$) it is found that

$$I_1^{(n)} \le \frac{2n-1}{4}\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2u_t^{2n-2}\,dx\,ds + C\int_0^t\int_0^1 Q^{\theta+3}u_x^4u_t^{2n-2}\,dx\,ds \qquad (109)$$

$$= \frac{2n-1}{4}\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2u_t^{2n-2}\,dx\,ds + I_{11}^{(n)}, \qquad (110)$$

where we have used $\varepsilon = \frac{1}{4(\theta+1)\rho_l}$. Similarly, we have for $I_2^{(n)}$ by using $\varepsilon = \frac{1}{4\gamma\rho_l}$

$$\begin{aligned}
I_2^{(n)} &\le \frac{2n-1}{4}\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2u_t^{2n-2}\,dx\,ds + C\int_0^t\int_0^1 c^{2\gamma}Q^{2\gamma+1-\theta}u_x^2u_t^{2n-2}\,dx\,ds\\
&= \frac{2n-1}{4}\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2u_t^{2n-2}dxds + I_{22}^{(n)}.
\end{aligned} \qquad (111)$$

Combining (103)–(111), we get

$$\begin{aligned}
&\int_0^1 u_t^{2n}(x,t)\,dx + n(2n-1)\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2u_t^{2n-2}\,dx\,ds\\
&\quad\le C(1 + I_{11}^{(n)} + I_{22}^{(n)} + I_3^{(n)} + I_4^{(n)}).
\end{aligned} \qquad (112)$$

The proof now proceeds by induction. We first show that

$$I_{11}^{(1)} \le C(T) + C(T)\int_0^t V_{11}(s)\int_0^1 u_t^2dxds, \qquad (113)$$

$$I_{22}^{(1)} \le C(T), \qquad (114)$$

$$I_3^{(1)} \le C(T) + C(T)\int_0^t\int_0^1 u_t^2dxds, \qquad (115)$$

$$I_4^{(1)} \le C(T)\int_0^t V_4(s)\int_0^1 u_t^2dxds, \qquad (116)$$

for appropriate choices of V_{11} and V_4 where $V_{11}(s), V_4(s) \in L^1([0,T])$. We refer to Appendix C for details. An application of Gronwall's lemma, in view of (112), then let us conclude that

$$\int_0^1 u_t^2(x,t)\,dx + \int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2dxds \le C(T). \qquad (117)$$

Moreover, assuming that Lemma 3.8 holds for $n-1$, i.e. that

$$\int_0^1 u_t^{2n-2}(x,t)\,dx + (n-1)(2n-3)\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2 u_t^{2n-4}dxds \le C(T), \tag{118}$$

we can show that

$$I_{11}^{(n)} \le C(T) + C(T)\int_0^t\int_0^1 u_t^{2n}dxds, \tag{119}$$

$$I_{22}^{(n)} \le C(T), \tag{120}$$

$$I_3^{(n)} \le C(T) + C(T)\int_0^t\int_0^1 u_t^{2n}dxds, \tag{121}$$

$$I_4^{(n)} \le C(T)\int_0^t V_4(s)\int_0^1 u_t^{2n}dxds, \tag{122}$$

where $V_4(s) \in L^1([0,T])$. Estimation details are again left to Appendix C. Lemma 3.8 thus follows by another application of Gronwall's lemma. □

Lemma 3.9. *Under the assumptions of Theorem 2.1 we have the estimates*

$$\|Q^{\theta+1}u_x\|_{L^\infty(D_T)} \le C(T), \tag{123}$$

$$\int_0^1 |(Q^{\theta+1}u_x)_x|\,dx \le C(T), \tag{124}$$

$$\int_0^1 |Q_x|\,dx \le C(T), \tag{125}$$

for a suitable constant $C(T)$ and where $D_T = [0,1]\times[0,T]$.

Proof. Using the Cauchy inequality, (54) and Lemma 3.1, Corollary 3.1, and Lemma 3.8, it follows from (62) that

$$Q^{1+\theta}u_x \le c^\gamma Q^\gamma + \int_0^1 |u_t|dx + C\int_0^1 u^2dx \le C + C\int_0^1 u_t^2dx \le C(T). \tag{126}$$

This proves (123). Similarly, again using the Cauchy inequality, (54), Lemmas 3.1, 3.4, 3.8, and Corollary 3.1, it follows from (62) that

$$\begin{aligned}
&\int_0^1 |(Q^{1+\theta}u_x)_x|dx \\
&\le \int_0^1 |(c^\gamma Q^\gamma)_x|dx + \int_0^1 |u_t|dx + C\int_0^1 h(c,Q)u^2dx \\
&\le C\int_0^1 |c_x|dx + C\int_0^1 \phi(x)^{-\nu}Q^{\gamma-\theta}\phi(x)^{\nu}Q^{\theta-1}|Q_x|dx + \int_0^1 |u_t|dx + C\int_0^1 h(c,Q)u^2dx \\
&\le C(T) + C\int_0^1 \phi(x)^{-2\nu}Q^{2(\gamma-\theta)}dx + C\int_0^1 \phi(x)^{2\nu}Q^{2(\theta-1)}Q_x^2dx + C\int_0^1 u_t^2dx \\
&\le C(T),
\end{aligned}$$

where we use that $2(\gamma-\theta)\alpha - 2\nu > -1$. Finally, (125) follows since

$$\begin{aligned}
\int_0^1 |Q_x|dx &\le \int_0^1 |\phi^\nu Q^{\theta-1}Q_xQ^{1-\theta}\phi^{-\nu}|dx \\
&\le 2\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2dx + 2\int_0^1 \phi^{-2\nu}Q^{2-2\theta}dx \le C(T),
\end{aligned} \tag{127}$$

by the Cauchy inequality, Corollary 3.1, Lemma 3.4, and the fact that $2(1-\theta)\alpha - 2\nu > -1$. □

Lemma 3.10. *Under the assumptions of Theorem 2.1, we have the following estimates for the velocity u*

$$\int_0^1 |u_x(x,t)|dx \le C(T), \tag{128}$$

$$|u(x,t)| \le C(T). \tag{129}$$

Proof. Using assumption (27), (54), (62), Lemmas 3.3 and 3.8 as well as the Hölder inequality with $p = 2n$ and $q = \frac{2n}{2n-1}$, we can obtain the estimate

$$\begin{aligned}
\int_0^1 |u_x(x,t)|dx &\le \int_0^1 c^\gamma Q^{\gamma-1-\theta}dx + \int_0^1 Q^{-1-\theta}\int_0^x |u_t|dydx + \int_0^1 Q^{-1-\theta}\int_0^x h(c,Q)u^2dydx \\
\le\ & C\int_0^1 Q^{\gamma-1-\theta}dx + C\int_0^1 Q^{-1-\theta}\Big(x(1-x)\Big)^{\frac{2n-1}{2n}}\Big(\int_0^1 (u_t)^{2n}dy\Big)^{\frac{1}{2n}}dx \\
&+C\int_0^1 Q^{-1-\theta}\Big(x(1-x)\Big)^{\frac{2n-1}{2n}}\Big(\int_0^1 u^{4n}dy\Big)^{\frac{1}{2n}}dx \\
\le\ & C\int_0^1 Q^{\gamma-1-\theta}dx + C\int_0^1 Q^{-1-\theta}\phi^{\frac{2n-1}{2n}}dx.
\end{aligned} \tag{130}$$

Furthermore, using Corollary 3.1 and Lemma 3.7 as well as the fact that when $0 < \theta < \frac{1}{2}$, $k_2 = \nu + \frac{k_1}{2}$, and $2\nu < k_1 < \frac{40(1-2\theta)}{11(1+\theta)} - 2\nu < \frac{20(1-2\theta)}{11\theta} - 2\nu$, then $\frac{-11k_2\theta}{10(1-2\theta)} > -1$, and for n sufficiently large, $\frac{2n-1}{2n} - \frac{11k_2(1+\theta)}{10(1-2\theta)} > -1$, we can conclude that

$$\int_0^1 Q^{\gamma-1-\theta}dx \le \max_{x\in[0,1]}(Q^{\gamma-1})\int_0^1 \phi^{\frac{-11k_2\theta}{10(1-2\theta)}}dx \le C(T), \tag{131}$$

and

$$\int_0^1 Q^{-1-\theta}\phi^{\frac{2n-1}{2n}}dx \le C\int_0^1 \phi^{\frac{2n-1}{2n}-\frac{11k_2(1+\theta)}{10(1-2\theta)}}dx \le C(T). \tag{132}$$

This proves (128). Finally, Sobolov's embedding theorem $|u| \le C\int_0^1 |u|dx + C\int_0^1 |u_x|dx$, the Cauchy inequality, the energy estimate and (128) directly gives the desired result (129). ☐

Lemma 3.11. *Under the assumptions of Theorem 2.1, we have for $0 < s < t \le T$ that*

$$\int_0^1 |Q(x,t) - Q(x,s)|^2dx \le C(T)|t-s|^2, \tag{133}$$

$$\int_0^1 |\zeta(x,t) - \zeta(x,s)|^2dx \le C(T)|t-s|^2, \tag{134}$$

$$\int_0^1 |n(x,t) - n(x,s)|^2dx \le C(T)|t-s|^2, \tag{135}$$

$$\int_0^1 |u(x,t) - u(x,s)|^2dx \le C(T)|t-s|^2, \tag{136}$$

$$\int_0^1 |(Q^{\theta+1}u_x)(x,t) - (Q^{\theta+1}u_x)(x,s)|^2dx \le C(T)|t-s|. \tag{137}$$

Proof. Using (47) b), the Hölder inequality, Corollary 3.1, Lemma 3.9, we see that

$$\begin{aligned}
\int_0^1 |Q(x,t) - Q(x,s)|^2dx &= \int_0^1 \Big|\int_s^t Q_\eta(x,\eta)d\eta\Big|^2dx = \rho_l\int_0^1\Big|\int_s^t (Q^2u_x)(x,\eta)d\eta\Big|^2dx \\
\le\ & C|t-s|\int_s^t\int_0^1 (Q^2u_x)^2(x,\eta)dxd\eta \le C|t-s|\int_s^t \max_{x\in[0,1]}(Q^{2-2\theta})\int_0^1 (Q^{1+\theta}u_x)^2(x,\eta)dxd\eta \\
\le\ & C(T)|t-s|^2.
\end{aligned} \tag{138}$$

Thus equation (133) is established. The estimate (134) follows by observing that

$$(\rho_l - (1-c)\zeta)^2Q_t = \rho_l\zeta_t.$$

Hence, the calculations in (138) can be used directly to establish the L^2-continuity in time of ζ. Next, using the relation $n(x,t) = c_0(x)\zeta(x,t)$, the estimate (134) implies (135). In a similar manner equation (136) follows, since

$$\begin{aligned} &\int_0^1 |u(x,t) - u(x,s)|^2 dx = \int_0^1 \Big| \int_s^t u_\eta(x,\eta) d\eta \Big|^2 dx = \\ \leq \quad & |t-s| \int_s^t \int_0^1 (u_t)^2(x,\eta) dx d\eta \leq C(T)|t-s|^2. \end{aligned} \tag{139}$$

due to Lemma 3.8.

Finally, we can prove (137) by the following argument. Again using Hölder's inequality we get that

$$\begin{aligned} &\int_0^1 |(Q^{1+\theta}u_x)(x,t) - (Q^{1+\theta}u_x)(x,s)|^2 dx = \int_0^1 \Big| \int_s^t (Q^{1+\theta}u_x)_\eta(x,\eta) d\eta \Big|^2 dx \\ \leq \quad & |t-s| \int_s^t \int_0^1 \Big((Q^{1+\theta}u_x)_\eta \Big)^2 (x,\eta) dx d\eta \\ \leq \quad & |t-s| \int_s^t \int_0^1 \Big((Q^{1+\theta}u_{x\eta} - \rho_l(1+\theta)Q^{2+\theta}u_x^2 \Big)^2 (x,\eta) dx d\eta \\ \leq \quad & C|t-s| \int_s^t \int_0^1 \Big(Q^{2+2\theta}u_{x\eta}^2 + Q^{4+2\theta}u_x^4 \Big)(x,\eta) dx d\eta \\ \leq \quad & C|t-s| \Big(\int_s^t \max_{x\in[0,1]} (Q^{1+\theta}) \int_0^1 Q^{1+\theta}u_{x\eta}^2 dx d\eta + \int_s^t \max_{x\in[0,1]} (Q^{3+\theta}u_x^2) \int_0^1 Q^{1+\theta}u_x^2 dx d\eta \Big) \\ \leq \quad & C|t-s|, \end{aligned} \tag{140}$$

by Lemmas 3.1 and 3.8 and the fact that

$$Q^{1+\theta} \leq C(T), \tag{141}$$

and

$$\begin{aligned} & Q^{3+\theta}u_x^2 = Q^{1-\theta}[Q^{\theta+1}u_x]^2 \\ = \quad & Q^{1-\theta}\Big((cQ)^\gamma + \int_0^x u_t dy + \int_0^x h(c,Q)u|u| dy \Big)^2 \\ \leq \quad & CQ^{1-\theta}\Big((cQ)^{2\gamma} + (\int_0^x u_t dy)^2 + (\int_0^x h(c,Q)u|u| dy)^2 \Big) \\ \leq \quad & CQ^{1-\theta}\Big((cQ)^{2\gamma} + x(1-x)\int_0^1 u_t^2 dy + x(1-x)\int_0^1 h(c,Q)^2 u^4 dy \Big) \\ \leq \quad & C\phi^{(2\gamma+1-\theta)\alpha} + C\phi^{(1-\theta)\alpha+1} \leq C(T), \end{aligned} \tag{142}$$

which again follows from Corollary 3.1, Lemma 3.3 and 3.8, as well as (54) and (62), respectively. □

4. Construction of weak solutions

In order to construct weak solutions to the initial-boundary problem (IBVP) (16)–(20), we apply the line method [20] where a system of ODEs is derived that can approximate the original model. For the details we refer to [9], which in turn is based on single-phase works like [10]. Semi-discrete version of the various lemmas can be obtained, and in combination with Helly's theorem, the result of Theorem 2.1 follows, see [13, 14, 19, 17, 18, 25, 20, 26, 27, 22] and references therein for details.

References

[1] C.S. Avelar, P.R. Ribeiro, and K. Sepehrnoori, Deepwater gas kick simulation, *J. Pet. Sci. Eng.* **67**, 13–22, 2009.

[2] C.E. Brennen, Fundamentals of Multiphase Flow, *Cambridge University Press, New York, 2005.*

[3] J.M Delhaye, M. Giot and M.L. Riethmuller, Thermohydraulics of Two-Phase Systems for Industrial Design and Nuclear Engineering, *Von Karman Institute, McGraw-Hill, New York, 1981.*
[4] S. Evje, Global weak solutions for a compressible gas-liquid model with well-formation interaction, *submitted*, 2010.
[5] S. Evje, Weak solutions for a gas-liquid model relevant for describing gas-kick in oil wells, accepted in *SIAM J. Math. Anal.*, 2011.
[6] S. Evje and K.-K. Fjelde, Hybrid flux-splitting schemes for a two-phase flow model, *J. Comput. Phys.* **175** (2), 674–701, 2002.
[7] S. Evje and K.-K. Fjelde, On a rough AUSM scheme for a one dimensional two-phase model, *Computers & Fluids* **32** (10), 1497–1530, 2003.
[8] S. Evje and K.H. Karlsen, Global weak solutions for a viscous liquid-gas model with singular pressure law, *Comm. Pure Appl. Anal* **8** (6), pp 1867-1894, 2009.
[9] S. Evje, T. Flåtten and H.A. Friis, Global weak solutions for a viscous liquid-gas model with transition to single-phase gas flow and vacuum, *Nonlinear Analysis TMA* **70**, pp 3864-3886, 2009.
[10] D. Fang and T. Zhang, Compressible Navier-Stokes equations with vacuum state in one dimension, *Commun. Pure Appl. Anal.* **3** (4), 675–694, 2004.
[11] D. Fang and T. Zhang, A note on compressible Navier-Stokes equations with vacuum state in one dimension, *Nonlinear Analysis TMA* **58** (5-6), 719–731, 2004.
[12] H.A. Friis and S. Evje, Global weak solutions for a gas-liquid model with external forces and general pressure law, *SIAM J. Appl. Math.* **71** (2), pp. 409–442, 2011.
[13] D. Hoff, Construction of solutions for compressible, isentropic Navier-Stokes equations in one space dimension with nonsmooth initial data, *Proc. Roy. Soc. Edinburgh Sect. A* **103**, (1986), 301-315.
[14] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, *Trans. AMS* **303** (11), 169-181, 1987.
[15] M. Ishii, Thermo-Fluid Dynamic Theory of Two-Phase Flow, *Eyrolles, Paris, 1975*,
[16] S. Jiang, Z. Xin, P. Zhang, Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity, *Meth. Appl. Anal.* **12** (3), 239–252, 2005.
[17] T.-P. Liu, Z. Xin, T. Yang, Vacuum states for compressible flow, *Discrete Continuous Dyn. Sys.* **4**, 1–32, 1998.
[18] T. Luo, Z. Xin, T. Yang, Interface behavior of compressible Navier-Stokes equations with vacuum, *SIAM J. Math. Anal.* **31**, 1175–1191, 2000.
[19] M. Okada, Free-boundary value problems for the equation of one-dimensional motion of viscous gas, *Japan J. Indust. Appl. Math.* **6**, 161–177, 1989.
[20] M. Okada, S. Matusu-Necasova, T. Makino, Free-boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity, *Ann. Univ. Ferrara Sez VII(N.S.)* **48**, 1–20, 2002.
[21] S. Saisorn and S. Wongwises, An inspection of viscosity model for homogeneous two-phase flow pressure drop prediction in a horizontal circular micro-channel, *Int. Comm. Heat Mass Trans.* **35**, 833–838, 2008.
[22] L. Yao, W. Wang, Compressible Navier-Stokes equations with density-dependent viscosity, vacuum and gravitational force in the case of general pressure, *Acta Mathematica Scientia* **28B**(4), 801–817, 2008.
[23] L. Yao and C.J. Zhu, Free boundary value problem for a viscous two-phase model with mass-dependent viscosity, *J. Diff. Eqs.* **247** (10), pp 2705–2739, 2009.
[24] L. Yao and C.J. Zhu, Existence and uniqueness of global weak solution to a two-phase flow model with vacuum, *Math. Ann.* **349** (4), pp 903–928, 2010.
[25] T. Yang, Z.-A. Yao, C.-J. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum, *Comm. Partial Diff. Eq.* **26**, 965–981, 2001.
[26] T. Yang, H. Zhao, A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity, *J. Diff. Eq.* **184**, 163–184, 2002.
[27] T. Yang, C. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum, *Commun. Math. Phys.* **230**, 329–363, 2002.

Appendix A: Some estimates connected to lemma 3.4

In this Appendix we estimate the quantities I_i (for $i = 0, 1, 2, 3, 4, 5$), which are used in the proof of Lemma 3.4.

Estimate for I_0. Using assumption (36) it follows that

$$I_0 = \frac{\theta^2}{2} \int_0^1 \phi^{2\nu} Q_0^{2\theta-2} Q_{0x}^2 dx \leq C. \tag{143}$$

Estimate for I_1. Using the Cauchy's inequality $ab \leq \frac{1}{4\varepsilon} a^2 + \varepsilon b^2$, and the energy estimate (55), we obtain that

$$I_1 = -\rho_l \theta \int_0^1 \phi^{2\nu} u (Q^\theta)_x dx \leq C(\varepsilon, T) + \varepsilon \int_0^1 \phi^{2\nu} Q^{2\theta-2} Q_x^2 dx. \tag{144}$$

Clearly, we can choose $\varepsilon = \theta^2/4$ such that the second term on the right hand side of (144) can be absorbed in the corresponding term on the left hand side of (75).

Estimate for I_2. Using the Cauchy inequality, assumption (28) as well as the arguments used above for I_0, we arrive at the conclusion

$$I_2 = \rho_l\theta \int_0^1 \phi^{2\nu} u_0 (Q_0^\theta)_x dx \le C + C \int_0^1 \phi^{2\nu} Q_0^{2\theta-2} Q_{0x}^2 dx \le C. \tag{145}$$

Estimate for I_3. Using equation (73) we get

$$\begin{aligned} & I_3 = \rho_l\theta \int_0^t \int_0^1 \phi^{2\nu} u (Q^\theta)_{xt} dxds = \\ - \quad & (\rho_l\theta)^2 \int_0^t \int_0^1 \phi^{2\nu} u u_t dxds - (\rho_l\theta)^2 \gamma \int_0^t \int_0^1 \phi^{2\nu} u c^{\gamma-1} c_x Q^\gamma dxds \\ - \quad & (\rho_l\theta)^2 \gamma \int_0^t \int_0^1 \phi^{2\nu} u c^\gamma Q^{\gamma-1} Q_x dxds - (\rho_l\theta)^2 \int_0^t \int_0^1 \phi^{2\nu} h(c,Q) u^2 |u| dxds \\ := \quad & I_{31} + I_{32} + I_{33} - (\rho_l\theta)^2 \int_0^t \int_0^1 \phi^{2\nu} h(c,Q) u^2 |u| dxds. \end{aligned} \tag{146}$$

We can further estimate I_{31}, I_{32} and I_{33} as follows.

$$\begin{aligned} & I_{31} = -(\rho_l\theta)^2 \int_0^t \int_0^1 \phi^{2\nu} u u_t dxds = -(\rho_l\theta)^2 \int_0^t \int_0^1 \phi^{2\nu} \frac{1}{2} \frac{d}{dt}(u^2) dxds \\ = \quad & -(\rho_l\theta)^2 \frac{1}{2} \int_0^1 \phi^{2\nu} (u^2) dxds + (\rho_l\theta)^2 \frac{1}{2} \int_0^1 \phi^{2\nu} (u_0^2) dxds \le C, \end{aligned} \tag{147}$$

due to the energy estimate (55) and assumption (28). Moreover, we see that

$$I_{32} = -(\rho_l\theta)^2 \gamma \int_0^t \int_0^1 \phi^{2\nu} u c^{\gamma-1} c_x Q^\gamma dxds \le C(T), \tag{148}$$

using the Cauchy inequality where we split u from the remaining part, followed by application of Lemma 3.1, Lemma 3.2 and assumption(27). Finally, we find that

$$\begin{aligned} & I_{33} = -(\rho_l\theta)^2 \gamma \int_0^t \int_0^1 \phi^{2\nu} u c^\gamma Q^{\gamma-1} Q_x dxds \\ \le \quad & C \int_0^t \int_0^1 \phi^{2\nu} u^2 c^{2\gamma} Q^{2\gamma-2\theta} dxds + C \int_0^t \int_0^1 \phi^{2\nu} Q^{2\theta-2} Q_x^2 dxds \\ \le \quad & C(T) + C \int_0^t \int_0^1 \phi^{2\nu} Q^{2\theta-2} Q_x^2 dxds \end{aligned} \tag{149}$$

where we have used the Cauchy inequality, Lemma 3.1, Lemma 3.2, and assumption(27).

Estimate for I_4. Using the Cauchy inequality, estimate (54), and the estimate of Lemma 3.3 we get

$$I_4 = -\rho_l\theta \int_0^t \int_0^1 \phi^{2\nu} h(c,Q) u^2 (Q^\theta)_x dxds \tag{150}$$

$$\begin{aligned} \le \quad & C \int_0^t \int_0^1 \phi^{2\nu} h(c,Q)^2 u^4 dxds + C \int_0^t \int_0^1 \phi^{2\nu} (Q^\theta)_x^2 dxds \\ \le \quad & C(T) + C \int_0^t \int_0^1 \phi^{2\nu} Q^{2\theta-2} Q_x^2 dxds. \end{aligned}$$

Estimate for I_5. Using the Cauchy inequality, assumption (27), and Lemma 3.2 we finally get that

$$\begin{aligned} I_5 &= -\rho_l \theta^2 \gamma \int_0^t \int_0^1 \phi^{2\nu} c^{\gamma-1} Q^{\gamma+\theta-1} Q_x c_x dx ds \\ &\leq C(T) + C \int_0^t \int_0^1 \phi^{2\nu} Q^{2\theta-2} Q_x^2 dx ds. \end{aligned} \tag{151}$$

Appendix B: Some estimates connected to lemma 3.5

In this Appendix we estimate the quantities I_i^m and I_i^{m-1} (for $i = 1, 2, 3, 4, 5, 6$), which are used in the proof of Lemma (3.5). The arguments goes along the line of e.g. [10, 22], which in turn build upon central works like [25, 26, 27]. The inclusion of the frictional term does not pose any additional problems in this lemma since it appears as a non-negative term on the right hand side of (82). However, for completeness we include the proof. Note that the equations (79) and (80) are extensively used throughout these proofs. We start by estimating I_i^m (for $i = 1, 2, 3, 4, 5, 6$).

Estimate for I_1^m. Exploiting assumptions (28), (37), and the fact that $k_1 > 0$, we easily conclude that

$$I_1^m = \int_0^1 \phi^{k_1} Q_0^{\alpha_m} u_0^{2^m} dx \leq C. \tag{152}$$

Estimate for I_2^m. Using the Cauchy inequality, Lemma 3.1, and Lemma 3.3, we have

$$\begin{aligned} I_2^m &= -\alpha_m \rho_l \int_0^t \int_0^1 \phi^{k_1} Q^{1+\alpha_m} u^{2^m} u_x dx ds \\ &\leq C \int_0^t \int_0^1 \phi^{2k_1} u^{2^{m+1}} dx ds + C \int_0^t \int_0^1 Q^{1+\theta} u_x^2 dx ds \leq C(T). \end{aligned} \tag{153}$$

Estimate for I_3^m. Using the Cauchy inequality, Lemma 3.3, and Lemma 3.4, and noticing that $k_1 > 2\nu$ (i.e. $2k_1 > 2\nu$), it is clear that

$$\begin{aligned} I_3^m &= -2^m \alpha_m \int_0^t \int_0^1 \phi^{k_1} Q^{\theta+\alpha_m} u^{2^m-1} Q_x u_x dx ds \\ &\leq C \int_0^t \int_0^1 Q^{\theta+1} u^{2^{m+1}-2} u_x^2 dx ds + C \int_0^t \int_0^1 \phi^{2k_1} Q^{2\theta-2} Q_x^2 dx ds \leq C(T). \end{aligned} \tag{154}$$

Estimate for I_4^m. Using the Cauchy inequality, Lemma 3.3, Corollary 3.1, and noticing that $|\phi'(x)|$ is limited, we can perform the following estimates

$$\begin{aligned} I_4^m &= -2^m k_1 \int_0^t \int_0^1 \phi^{k_1-1} Q^{1+\theta+\alpha_m} u^{2^m-1} u_x \phi'(x) dx ds \\ &\leq C \int_0^t \int_0^1 \phi^{k_1-1} Q^{1+\theta+\alpha_m} |u^{2^m-1}||u_x| dx ds \\ &\leq C \int_0^t \int_0^1 Q^{1+\theta} u^{2^{m+1}-2} u_x^2 dx ds + C \int_0^t \int_0^1 \phi^{2k_1-2} Q^{1+\theta+2\alpha_m} dx ds \\ &\leq C(T) + C \int_0^t \int_0^1 \phi^{2k_1-2} Q^{2\theta} dx ds \leq C(T) + C \int_0^t \int_0^1 \phi^{2\theta\alpha+2k_1-2} dx ds \leq C(T), \end{aligned} \tag{155}$$

where the last inequality can be deduced since $2\theta\alpha + 2k_1 - 2 > -1$, when $k_1 > 2\nu$ and $\alpha \geq \frac{19}{20} + \frac{1}{10}\theta$, in view of assumptions (30) and (31).

Estimate for I_5^m. Using the Cauchy inequality, assumptions (27), Lemma 3.3, Corollary 3.1, and Lemma 3.4 we obtain the estimates

$$\begin{aligned} I_5^m &= -2^m\gamma\int_0^t\int_0^1 \phi^{k_1}c^{\gamma}Q^{\gamma+\alpha_m-1}u^{2^m-1}Q_x\,dxds \\ &\le C\int_0^t\int_0^1 u^{2^{m+1}-2}dxds + C\int_0^t\int_0^1 \phi^{2k_1}Q^{2\gamma+2\alpha_m-2}Q_x^2\,dxds \\ &\le C(T) + C\int_0^t\int_0^1 \phi^{k_3}Q^{2\theta-2}Q_x^2\,dxds \le C(T). \end{aligned} \tag{156}$$

Note that the last inequality follows since $k_3 > -1$, where $k_3 = 2k_1 + \alpha k_2$ and k_2 is defined such that $2\gamma + 2\alpha_m - 2 = k_2 + (2\theta - 2)$ (i.e. $k_2 = 2\gamma - \theta - 1 \ge 0$). Clearly, $k_3 \ge 2k_1 > 2\nu$, implying that $\phi(x)^{k_3} \le \phi(x)^{2\nu}$.

Estimate for I_6^m. Again using the Cauchy inequality, Lemma 3.3, Corollary 3.1, as well as assumptions (27), we obtain the estimates

$$\begin{aligned} I_6^m &= -2^m\gamma\int_0^t\int_0^1 \phi^{k_1}c^{\gamma-1}Q^{\gamma+\alpha_m}u^{2^m-1}c_x\,dxds \\ &\le C\int_0^t\int_0^1 u^{2^{m+1}-2}dxds + C\int_0^t\int_0^1 \phi^{2k_1}Q^{2\gamma+2\alpha_m}dxds \\ &\le C(T) + C\int_0^t\int_0^1 \phi^{2k_1+\alpha(2\gamma+2\alpha_m)}dxds \le C(T), \end{aligned} \tag{157}$$

since $2k_1 + \alpha(2\gamma + 2\alpha_m) > -1$.

Next we estimate I_i^{m-1} (for $i = 1, 2, 3, 4, 5, 6$). In particular, we shall make use of the estimate (82).

Estimate for I_1^{m-1}. Similarly as for I_1^m, using (79) and (80), we estimate that

$$I_1^{m-1} = \int_0^1 \phi^{k_1}Q_0^{\alpha_{m-1}}u_0^{2^{m-1}}dx \le C(T). \tag{158}$$

Estimate for I_2^{m-1}. Using the Cauchy inequality, estimate (82), the fact that $2+2\alpha_{m-1}-\alpha_m = 1+\theta$, and Lemma 3.1 we obtain

$$\begin{aligned} I_2^{m-1} &= -\alpha_{m-1}\rho_l\int_0^t\int_0^1 \phi^{k_1}Q^{1+\alpha_{m-1}}u^{2^{m-1}}u_x\,dxds \\ &= -\alpha_{m-1}\rho_l\int_0^t\int_0^1 \phi^{k_1}Q^{\frac{\alpha_m}{2}}u^{2^{m-1}}Q^{1+\alpha_{m-1}-\frac{\alpha_m}{2}}u_x\,dxds \\ &\le C\int_0^t\int_0^1 \phi^{k_1}Q^{\alpha_m}u^{2^m}dxds + C\int_0^t\int_0^1 Q^{2+2\alpha_{m-1}-\alpha_m}u_x^2\,dxds \\ &\le C(T) + C\int_0^t\int_0^1 Q^{1+\theta}u_x^2\,dxds \le C(T). \end{aligned}$$

Estimate for I_3^{m-1}. Using the Cauchy inequality, estimate (82), Lemma 3.4, the relation $2\alpha_{m-1} - \alpha_m = \theta - 1$, and $2\nu < k_1$, we have

$$\begin{aligned} I_3^{m-1} &= -2^{m-1}\alpha_{m-1}\int_0^t\int_0^1 \phi^{k_1}Q^{\theta+\alpha_{m-1}}u^{2^{m-1}-1}Q_xu_x\,dxds \\ &\le C\int_0^t\int_0^1 \phi^{\frac{k_1}{2}}Q^{\frac{1}{2}+\frac{\theta}{2}+\frac{\alpha_m}{2}}\phi^{\frac{k_1}{2}}Q^{\frac{\theta}{2}+\alpha_{m-1}-\frac{1}{2}-\frac{\alpha_m}{2}}u^{2^{m-1}-1}Q_xu_x\,dxds \\ &\le C\int_0^t\int_0^1 \phi^{k_1}Q^{1+\theta+\alpha_m}u^{2^m-2}u_x^2\,dxds + C\int_0^t\int_0^1 \phi^{k_1}Q^{\theta+2\alpha_{m-1}-1-\alpha_m}Q_x^2\,dxds \\ &\le C(T) + C\int_0^t\int_0^1 \phi^{k_1}Q^{2\theta-2}Q_x^2\,dxds \le C(T). \end{aligned} \tag{159}$$

Estimate for I_4^{m-1}. Using the Cauchy inequality, estimate (82), and Corollary 3.1, we have

$$\begin{aligned} I_4^{m-1} &= -2^{m-1}k_1\int_0^t\int_0^1 \phi^{k_1-1}Q^{1+\theta+\alpha_{m-1}}u^{2^{m-1}-1}u_x\phi'(x)dxds \\ &\le C\int_0^t\int_0^1 \phi^{k_1-1}Q^{1+\theta+\alpha_{m-1}}|u^{2^{m-1}-1}u_x|dxds \\ &\le C\int_0^t\int_0^1 \phi^{\frac{k_1}{2}-1}Q^{\frac{1}{2}+\frac{\theta}{2}+\alpha_m}\phi^{\frac{k_1}{2}}Q^{\frac{1}{2}+\frac{\theta}{2}+\frac{\alpha_m}{2}}|u^{2^{m-1}-1}u_x|dxds \\ &\le C\int_0^t\int_0^1 \phi^{k_1}Q^{1+\theta+\alpha_m}u^{2^m-2}u_x^2dxds + C\int_0^t\int_0^1 \phi^{k_1-2}Q^{1+\theta+2\alpha_m}dxds \\ &\le C(T)+C\int_0^t\int_0^1 \phi^{k_1-2+2\theta\alpha}dxds \le C(T), \end{aligned} \tag{160}$$

since $k_1-2+2\theta\alpha > -1-2\theta-\frac{1}{5}\theta^2+\frac{1}{10}\theta+\frac{19}{10}\theta+\frac{1}{5}\theta^2=-1$, in light of assumptions (30)–(33).

Estimate for I_5^{m-1}. Using the assumptions on c given by (27), the Cauchy inequality, Lemma 3.3, Corollary 3.1, and Lemma 3.4 we obtain

$$\begin{aligned} I_5^{m-1} &= -2^{m-1}\gamma\int_0^t\int_0^1 \phi^{k_1}c^\gamma Q^{\gamma+\alpha_{m-1}-1}u^{2^{m-1}-1}Q_xdxds \\ &\le C\int_0^t\int_0^1 u^{2^m-2}dxds + C\int_0^t\int_0^1 \phi^{2k_1}Q^{2\gamma+2\alpha_{m-1}-2}Q_x^2dxds \\ &\le C(T)+C\int_0^t\int_0^1 \phi^{2k_1+\alpha\tilde{k}_2}Q^{2\theta-2}Q_x^2dxds \le C(T), \end{aligned} \tag{161}$$

where $\tilde{k}_2=2\gamma+3\alpha_m-2\theta=2\gamma+\frac{3}{2}(\theta-1)-2\theta=2\gamma-\frac{1}{2}(\theta+3)>0$ and $k_1>2\nu$.

Estimate for I_6^{m-1}. Again, using the assumptions on c and c_x given by (27), the Cauchy inequality, Corollary 3.1, and Lemma 3.3, we obtain

$$\begin{aligned} I_6^{m-1} &= -2^{m-1}\gamma\int_0^t\int_0^1 \phi^{k_1}c^{\gamma-1}Q^{\gamma+\alpha_{m-1}}u^{2^{m-1}-1}c_xdxds \\ &\le C\int_0^t\int_0^1 u^{2^m-2}dxds + C\int_0^t\int_0^1 \phi^{2k_1}Q^{2\gamma+2\alpha_{m-1}}dxds \\ &\le C(T)+C\int_0^t\int_0^1 \phi^{2k_1+2\alpha\gamma+3\alpha\alpha_m}dxds \le C(T), \end{aligned} \tag{162}$$

since $2k_1+2\alpha\gamma+3\alpha\alpha_m=2k_1+[2\gamma+\frac{3}{2}(\theta-1)]\alpha>0$.

Appendix C: Some estimates connected to lemma 3.8

In this Appendix we estimate the quantities $I_{11}^{(1)}, I_{22}^{(1)}, I_3^{(1)}$ and $I_4^{(1)}$ and, moreover, $I_{11}^{(n)}, I_{22}^{(n)}, I_3^{(n)}$ and $I_4^{(n)}$, which are all used in the proof of Lemma 3.8. We estimate as follows:

$$I_{11}^{(1)} = C\int_0^t\int_0^1 Q^{\theta+3}u_x^4dxds \le C\int_0^t \max_{x\in[0,1]}(\phi^{-k_1}Q^{2-\alpha_1}u_x^2)V(s)ds \tag{163}$$

where $V(s) = \int_0^1 \phi^{k_1} Q^{\theta+1+\alpha_1} u_x^2 dx$. Exploiting (62), estimate (54), Lemma 3.3, Corollary 3.1 and the Hölder inequality, it follows that

$$\begin{aligned}
\phi^{-k_1} Q^{2-\alpha_1} u_x^2 &= \phi^{-k1} Q^{-\alpha_1-2\theta} [Q^{\theta+1} u_x]^2 \\
&= \phi^{-k_1} Q^{-\alpha_1-2\theta} \Big((cQ)^\gamma + \int_0^x u_t dy + \int_0^x h(c,Q) u|u| dy \Big)^2 \\
&\le C \phi^{-k_1} Q^{-\alpha_1-2\theta} \Big((cQ)^{2\gamma} + \Big(\int_0^x u_t dy \Big)^2 + \Big(\int_0^x h(c,Q) u|u| dy \Big)^2 \Big) \\
&\le C \phi^{-k_1} Q^{-\alpha_1-2\theta} \Big((cQ)^{2\gamma} + x(1-x) \int_0^1 u_t^2 dy + x(1-x) \int_0^1 h(c,Q)^2 u^4 dy \Big) \\
&\le C \phi^{-k_1+(2\gamma-\alpha_1-2\theta)\alpha} + C \phi^{1-k_1} Q^{-(\alpha_1+2\theta)} \int_0^1 u_t^2 dy + C \phi^{1-k_1} Q^{-(\alpha_1+2\theta)} \\
&\le C(T) \int_0^1 u_t^2 dx + C(T),
\end{aligned}$$

where the last inequality comes from the following facts:

(1) For $0 < \theta < \frac{1}{2}$ and $2\nu < k_1 < (2\gamma - 3\theta + 1)\alpha$ it is clear that $-k_1 + (2\gamma - \alpha_1 - 2\theta)\alpha \ge 0$.

(2) For $0 < \theta < \frac{1}{3}$ and $2\nu < k_1 < 1 + (1 - 3\theta)\alpha$, we have (for sufficiently large m) that $1 - k_1 - (\alpha_1 + 2\theta)\alpha \ge 0$. Here we also first have used the fact that $-(\alpha_1 + 2\theta) > 0$ such that $Q^{-(\alpha_1+2\theta)} \le C\phi(x)^{-(\alpha_1+2\theta)\alpha}$, according to Corollary 3.1.

(3) For $\frac{1}{3} \le \theta < \frac{1}{2}$ and $2\nu < k_1 < \frac{20(1-2\theta)}{9-7\theta} + \frac{22(1-3\theta)}{9-7\theta}\nu$ it is clear that $-(\alpha_1 + 2\theta) < 0$. Consequently, according to Lemma 3.7, $Q^{-(\alpha_1+2\theta)} \le C\phi(x)^{\frac{-11k_2(\alpha_1+2\theta)}{10(1-2\theta)}}$. However, $1 - k_1 - \frac{11k_2(\alpha_1+2\theta)}{10(1-2\theta)} \ge 0$.

Consequently, we have

$$I_{11}^{(1)} \le C(T) + C(T) \int_0^t V(s) \int_0^1 u_t^2 \, dx \, ds, \tag{164}$$

where $V(s) \in L^1([0,T])$, in view of Lemma 3.5. Moreover, we have

$$\begin{aligned}
I_{22}^{(1)} &= C \int_0^t \int_0^1 c^{2\gamma} Q^{2\gamma+1-\theta} u_x^2 dx ds \\
&\le C \int_0^t \max_{x\in[0,1]} (\phi^{-k_1} Q^{2\gamma-\alpha_1-2\theta}) \int_0^1 \phi^{k_1} Q^{1+\theta+\alpha_1} u_x^2 dx ds \le C(T),
\end{aligned} \tag{165}$$

in view of assumption (27), Lemma 3.5, Corollary 3.1, and the fact that when $0 < \theta < \frac{1}{2}$ and $2\nu < k_1 < (2\gamma - 3\theta + 1)\alpha$, then $2\gamma - (\alpha_1 + 2\theta) > 0$ and $(2\gamma - \alpha_1 - 2\theta)\alpha - k_1 \ge 0$, for sufficiently large m.

We must further estimate $I_3^{(1)}$ and $I_4^{(1)}$. Using the Cauchy inequality, we have

$$\begin{aligned}
I_3^{(1)} &= -\rho_l \int_0^t \int_0^1 h_Q(c,Q) Q^2 u_x u|u| u_t dx ds \\
&\le \rho_l \max_{x\in[0,1]} (|h_Q(c,Q) Q^{\frac{3}{2}-\frac{\theta}{2}}|) \int_0^t \int_0^1 |Q^{\frac{1}{2}+\frac{\theta}{2}} u^2 u_x u_t| dx ds \\
&\le \rho_l \max_{x\in[0,1]} (|h_Q(c,Q) Q^{\frac{3}{2}-\frac{\theta}{2}}|) \Big(\int_0^t \int_0^1 Q^{1+\theta} u^4 u_x^2 dx ds + \int_0^t \int_0^1 u_t^2 dx ds \Big) \\
&\le C(T) + C(T) \int_0^t \int_0^1 u_t^2 dx ds,
\end{aligned} \tag{166}$$

in light of Lemma 3.3 and the fact that $\beta + \frac{1}{2} - \frac{\theta}{2} > 0$ (for $\beta > 0$) such that

$$h_Q(c,Q) Q^{\frac{3}{2}-\frac{\theta}{2}} \le C(T), \tag{167}$$

by Lemma 3.2, where we have also used that

$$h_Q(c,Q) = f\beta\rho_l^\beta\Big(\frac{Q}{1+(1-c)Q}\Big)^{\beta-1}\Big(\frac{1}{1+(1-c)Q}\Big)^2. \tag{168}$$

For later use we also note that

$$h_c(c,Q) = f\rho_l^\beta\beta\Big(\frac{Q}{1+(1-c)Q}\Big)^{\beta+1}. \tag{169}$$

Furthermore, we also get that

$$I_4^{(1)} \;=\; 2\int_0^t\int_0^1 h(c,Q)|u|u_t^2\,dxds \le 2\int_0^t \max_{x\in[0,1]}(h(c,Q)|u|)\int_0^1 u_t^2\,dxds. \tag{170}$$

Now, the Sobolev embedding theorem gives

$$\begin{aligned} |h(c,Q)u| &\le C\int_0^1 |h(c,Q)u|dx + C\int_0^1 |(h(c,Q)u)_x|dx \\ &\le C\int_0^1 h(c,Q)^2dx + \int_0^1 u^2dx + C\int_0^1 |(h(c,Q)u)_x|dx \\ &\le C(T) + \int_0^1 |(h(c,Q)u)_x|dx = C(T) + W(s). \end{aligned} \tag{171}$$

where we have used Cauchy's inequality, Lemma 3.1, and estimate (54). Next, we estimate

$$\begin{aligned} W(s) &= \int_0^1 |(h(c,Q)u)_x|dx \\ &\le \int_0^1 |h_c(c,Q)c_xu|dx + \int_0^1 |h_Q(c,Q)Q_xu|dx + \int_0^1 |h(c,Q)u_x|dx \\ &= W_A(s) + W_B(s) + W_C(s) \end{aligned} \tag{172}$$

as follows:

$$W_A(s) \;=\; \int_0^1 |h_c(c,Q)c_xu|dx \le C\int_0^1 |u|dx \le C(T), \tag{173}$$

and

$$\begin{aligned} W_B(s) &= \int_0^1 |h_Q(c,Q)Q_xu|dx \\ &\le \int_0^1 \phi^{2\nu}Q^{2(\theta-1)}Q_x^2dx + \int_0^1 \phi^{-2\nu}h_Q(c,Q)^2Q^{2(1-\theta)}u^2dx \\ &\le C(T) + \max_{x\in[0,1]}(\phi^{-2\nu}h_Q(c,Q)^2Q^{2(1-\theta)})\int_0^1 u^2dx \\ &\le C(T) + C(T)\max_{x\in[0,1]}(\phi^{2\beta\alpha-2\theta\alpha-2\nu}) \le C(T), \end{aligned} \tag{174}$$

since $\beta > \frac{\nu}{\alpha} + \theta > \theta$, and

$$\begin{aligned} W_C(s) &= \int_0^1 |h(c,Q)u_x|dx \le \int_0^1 Q^{\theta+1}u_x^2dx + \int_0^1 |h(c,Q)^2Q^{-(\theta+1)}|dx \\ &\le \int_0^1 Q^{\theta+1}u_x^2dx + \max_{x\in[0,1]}(h(c,Q)^2Q^{-(\theta+1)}) \\ &\le \int_0^1 Q^{\theta+1}u_x^2dx + \max_{x\in[0,1]}(\phi^{2\beta\alpha-\alpha-\theta\alpha}) \in L^1([0,T]), \end{aligned} \tag{175}$$

if $2\beta > 1+\theta$. Here we also have applied Hölder's and Cauchy inequalities, Lemma 3.1, Corollary 3.1, and Lemma 3.4, (168) and (49). Consequently, we can conclude that

$$I_4^{(1)} \le \int_0^t [C(T) + W(s)]\int_0^1 u_t^2\,dxds, \tag{176}$$

where $W(s) \in L^1([0,T])$. Hence, we have shown (113)–(116).

It is now time to estimate the quantities $I_{11}^{(n)}, I_{22}^{(n)}, I_3^{(n)}$ and $I_4^{(n)}$. First, by the induction assumption equation (118), we get that

$$I_{11}^{(n)} = C\int_0^t\int_0^1 Q^{\theta+3}u_x^4 u_t^{2n-2}dxds \le C(T)\int_0^t \max_{x\in[0,1]}(Q^{\theta+3}u_x^4)ds. \tag{177}$$

It follows that

$$\begin{aligned}
Q^{\theta+3}u_x^4 &= Q^{-1-3\theta}[Q^{\theta+1}u_x]^4\\
&= Q^{-1-3\theta}\Big((cQ)^\gamma + \int_0^x u_t dy + \int_0^x h(c,Q)u|u|dy\Big)^4\\
&\le CQ^{-1-3\theta}\Big((cQ)^{4\gamma} + (\int_0^x u_t dy)^4 + (\int_0^x h(c,Q)u|u|dy)^4\Big)\\
&\le CQ^{-1-3\theta}\Big((cQ)^{4\gamma} + (x(1-x))^{\frac{4n-2}{n}}\Big(\int_0^1 u_t^{2n}dy\Big)^{\frac{2}{n}} + (x(1-x))^{\frac{4n-2}{n}}\Big(\int_0^1 h(c,Q)^{2n}u^{4n}dy\Big)^{\frac{2}{n}}\Big)\\
&\le C(T)\phi^{(4\gamma-1-3\theta)\alpha} + C(T)\phi^{\frac{4n-2}{n}}Q^{-1-3\theta}\Big(\int_0^1 u_t^{2n}dy\Big)^{\frac{2}{n}} + C(T)\phi^{\frac{4n-2}{n}}Q^{-1-3\theta},
\end{aligned}$$

where we have used (62), (54), Lemma 3.3, Corollary 3.1 and the Hölder inequality with $p = \frac{4n}{4n-2}$ and $q = 2n$. Now exploiting that for $0 < \theta < \frac{1}{2}$ and $2\nu < k_1 < \frac{60(1-2\theta)}{11(1+3\theta)} - 2\nu$, we have for any $n > 1$ that

$$\frac{4n-2}{n} - \frac{11k_2(1+3\theta)}{10(1-2\theta)} \ge 0, \qquad k_2 = \nu + \frac{k_1}{2}, \tag{178}$$

which implies that

$$\max_{x\in[0,1]}(\phi^{\frac{4n-2}{n}}Q^{-1-3\theta}) \le C(T)\phi^{\frac{4n-2}{n}-\frac{11k_2(1+3\theta)}{10(1-2\theta)}} \le C(T), \tag{179}$$

by Lemma 3.7, and thus also that

$$\max_{x\in[0,1]}(Q^{\theta+3}u_x^4) \le C\Big(\int_0^1 u_t^{2n}dx\Big)^{\frac{2}{n}} + C(T), \tag{180}$$

such that

$$I_{11}^{(n)} \le C(T)\Big[1 + \int_0^t\Big(\int_0^1 u_t^{2n}dx\Big)^{\frac{2}{n}}ds\Big]. \tag{181}$$

Finally, Young's inequality with $p = \frac{n}{2}$ and $q = \frac{n}{n-2}$ gives $(\int_0^1 u_t^{2n}dx)^{\frac{2}{n}} \le \frac{2}{n}\int_0^1 u_t^{2n}dx + \frac{n-2}{2}$, which leads us to conclude that

$$I_{11}^{(n)} \le C(T) + C(T)\int_0^t\int_0^1 u_t^{2n}dxds. \tag{182}$$

Moreover, we have by the induction assumption equation (118) that

$$I_{22}^{(n)} = C\int_0^t\int_0^1 c^{2\gamma}Q^{2\gamma+1-\theta}u_x^2u_t^{2n-2}dxds \le C\int_0^t \max_{x\in[0,1]}(Q^{2\gamma-1-3\theta}[Q^{1+\theta}u_x]^2)ds \tag{183}$$

Exploiting (62), (54), (117), Lemma 3.3, Corollary 3.1, and the Hölder inequality, we can now estimate as follows

$$\begin{aligned}
&Q^{2\gamma-1-3\theta}[Q^{\theta+1}u_x]^2 = Q^{2\gamma-1-3\theta}\Big((cQ)^\gamma + \int_0^x u_t dy + \int_0^x h(c,Q)u|u|dy\Big)^2\\
&\le CQ^{2\gamma-1-3\theta}\Big((cQ)^{2\gamma} + (\int_0^x u_t dy)^2 + (\int_0^x h(c,Q)u|u|dy)^2\Big)\\
&\le CQ^{2\gamma-1-3\theta}\Big((cQ)^{2\gamma} + x(1-x)\int_0^1 u_t^2dy + x(1-x)\int_0^1 h(c,Q)^2u^4dy\Big)\\
&\le C\phi^{(4\gamma-1-3\theta)\alpha} + C\phi^{1+2\gamma\alpha}\phi^{-\frac{11k_2(1+3\theta)}{10(1-2\theta)}} \le C(T),
\end{aligned}$$

since we have that $(4\gamma - 1 - 3\theta)\alpha > 0$ and

$$1 + 2\gamma\alpha - \frac{11k_2(1+3\theta)}{10(1-2\theta)} > 1 + 2\gamma\alpha - \frac{30}{11} = 2\gamma\alpha - \frac{19}{11} > 0.$$

This corresponds to $\gamma\alpha > \frac{19}{22}$, that is, $\alpha > \frac{19}{22}$ which clearly holds in view of assumption (30). Consequently,

$$I_{22}^{(n)} \le C(T). \tag{184}$$

Furthermore, we get that

$$\begin{aligned} I_3^{(n)} &= -\rho_l \int_0^t \int_0^1 h_Q(c,Q)Q^2 u_x u|u|u_t^{2n-1}\,dxds \\ &\le \max_{x\in[0,1]}(|h_Q(c,Q)Q^2u_x|)\int_0^t\int_0^1 u^2|u_t^{2n-1}|\,dxds \\ &\le C\max_{x\in[0,1]}(|h_Q(c,Q)Q^2u_x|)\Big(\int_0^t\int_0^1 u^{4n}\,dxds + \int_0^t\int_0^1 u_t^{2n}\,dxds\Big) \\ &\le C(T) + C(T)\int_0^t\int_0^1 u_t^{2n}\,dxds, \end{aligned} \tag{185}$$

where we have applied Young's inequality with $p = 2n$ and $q = \frac{2n}{2n-1}$, Lemma 3.3 as well as the fact that

$$\begin{aligned} & |h_Q(c,Q)Q^2u_x| \\ \le\; & |h_Q(c,Q)Q^{1-\theta}|\Big|\int_0^x u_t\,dy + (cQ)^\gamma + \int_0^x h(c,Q)u|u|\,dy\Big| \\ \le\; & Ch_Q(c,Q)Q^{1-\theta}\Big(\phi(x)^{1/2}\Big(\int_0^1 u_t^2\,dy\Big)^{1/2} + (cQ)^\gamma + \phi(x)^{1/2}\Big(\int_0^1 u^2\,dy\Big)^{1/2}\Big) \\ \le\; & C(T)[\phi^{\alpha(\beta-\theta)+1/2} + \phi^{\alpha(\beta-\theta)+\gamma}] \le C(T), \end{aligned} \tag{186}$$

due to (54), (62), (117), (168), Lemma 3.1, Corollary 3.1, and Hölder inequality. Also note that we must use that $\beta \ge \theta$, which is already ensured by assumption (32).

Finally, we obtain

$$\begin{aligned} I_4^{(n)} &= 2\int_0^t\int_0^1 h(c,Q)|u|u_t^{2n}\,dxds \le \int_0^t \max_{x\in[0,1]}(h(c,Q)|u|)\int_0^1 u_t^{2n}\,dxds \\ &\le \int_0^t [C(T) + W(s)]\int_0^1 u_t^{2n}\,dx\,ds, \end{aligned} \tag{187}$$

with $W(s) \in L^1([0,T])$ by precisely the same arguments as for $I_4^{(1)}$, see estimate (176).