WEAK SOLUTIONS OF A GAS-LIQUID DRIFT-FLUX MODEL WITH GENERAL SLIP LAW FOR WELLBORE OPERATIONS

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Abstract. In this work we study a compressible gas-liquid models highly relevant for wellbore operations like drilling. The model is a drift-flux model and is composed of two continuity equations together with a mixture momentum equation. The model allows unequal gas and liquid velocities, dictated by a so-called slip law, which is important for modeling of flow scenarios involving for example counter-current flow. The model is considered in Lagrangian coordinates. The difference in fluid velocities gives rise to new terms in the mixture momentum equation that are challenging to deal with. First, a local (in time) existence result is obtained under suitable assumptions on initial data for a general slip relation. Second, a global in time existence result is proved for small initial data subject to a more specialized slip relation.

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1. Introduction

The drift-flux model is one of the commonly used models nowadays for the prediction of various two-phase flows. It was first developed by Zuber and Findlay [37]. It is used in chemical engineering to predict flow in bubble column reactors, in petroleum applications to model various wellbore operations related to drilling as well as to study production of oil and gas. More recently, it is also used for the study of geothermal energy related problems and injection of CO_2, to mention some of the applications [27]. The drift-flux model remains one of the best available ways to quickly estimate the void fraction in a two-phase system. A one-dimensional transient drift-flux model can be written in the following form:

$$\begin{aligned} &\partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] = 0, \\ &\partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] = 0, \\ &\partial_t[\alpha_g\rho_g u_g + \alpha_l\rho_l u_l] + \partial_x[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + P] = -q + \partial_x[\varepsilon\partial_x u_M], \quad u_M = \alpha_g u_g + \alpha_l u_l, \end{aligned} \tag{1.1}$$

where $\varepsilon \geq 0$. The model is supposed under isothermal conditions. The unknowns are $\rho_l(P), \rho_g(P)$ the liquid and gas densities, α_l, α_g volume fractions of liquid and gas satisfying

$$\alpha_g + \alpha_l = 1, \tag{1.2}$$

and u_l, u_g velocities of liquid and gas, P common pressure for liquid and gas, and q representing external forces like gravity and friction. In the following we assume that the liquid is incompressible whereas the gas phase is described by the polytropic gas law

$$P = C\rho_g^\gamma, \qquad \gamma > 1, \tag{1.3}$$

where, without loss of generality, we choose $C = 1$. Since the momentum is given only for the mixture, we need an additional closure law, a so-called hydrodynamical closure law, which connects

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the two phase velocities. This law should be able to take into account the different *flow regimes*. A commonly used slip relation is in the form [37, 16, 6, 20, 1]

$$u_g = \hat{c}_0 u_M + \hat{c}_1. \tag{1.4}$$

Here $\hat{c}_0$ and $\hat{c}_1$ are flow dependent coefficients. $\hat{c}_0$ is referred to as the *distribution parameter* and $\hat{c}_1$ to as the *drift velocity*. Various discrete schemes have been developed for computing numerical solutions of the compressible two-phase model (1.1)–(1.4), see [22, 11, 26, 12, 5, 6, 2, 13, 23] and references therein. It is well known that it is difficult to solve this model efficiently due to strong nonlinear coupling mechanisms and challenges associated with transition to single-phase regions. Therefore it is of interest to deepen the insight into the finer mechanism of this model, also from a mathematical point of view. In particular, it is desirable to obtain a better understanding of the effect from the slip law (1.4).

The main objective of this paper is two-fold:

- Discuss some mathematical properties of the model (1.1) when it is studied in combination with the general slip law (1.4) where the coefficients $\hat{c}_0 \geq 1$ and $\hat{c}_1 \geq 0$ are assumed to be constant. More precisely, we establish a local in time result guaranteeing existence of weak solutions for this general case.
- Present a global in time existence result of weak solutions when we consider the slip law (1.4) with $\hat{c}_1 = 0$ but $\hat{c}_0 > 1$. Note from (1.4) that $\hat{c}_0 = 1$ and $\hat{c}_1 = 0$ imply that $u_g = u_l$, i.e., no relative motion between the two phases.

We obtain our results by considering the model in Lagrangian variables in a free-boundary setting. The precise description of the model problem is as follows (we refer to Section 2 for a detailed derivation of the model): First, we introduce the variables (c, ρ, u) given as

$$c = \frac{m - k^*}{\rho}, \qquad \rho = n + m - k^*, \qquad u = u_g, \tag{1.5}$$

where

$$m = \alpha_l \rho_l, \qquad n = \alpha_g \rho_g, \tag{1.6}$$

and $k^* = \rho_l(1 - 1/\hat{c}_0)$ represents a minimal mass of liquid that must be present in order to make the slip law well-defined. The model we study in this work takes the following form:

$$\begin{gathered} \partial_t c = 0, \\ \partial_t \rho + \rho^2 \partial_x u = 0, \\ \partial_t u + \partial_x [P(c,\rho) - u^2 g(c\rho) - u h(c\rho) + j(c\rho)] = \partial_x [E(c\rho)\partial_x u], \quad \text{in } 0 < x < 1. \end{gathered} \tag{1.7}$$

with boundary conditions

$$u(0,t) = 0, \qquad \rho(1,t) = c(1,t) = 0, \tag{1.8}$$

or

$$\rho(0,t) = \rho(1,t) = 0, \qquad c(0,t) = c(1,t) = 0, \tag{1.9}$$

and with initial conditions

$$c(x,0) = c_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x), \qquad x \in [0,1], \tag{1.10}$$

where $c(x,t) = c_0(x) = \frac{m_0(x) - k^*}{\rho_0(x)}$. Moreover, the different functions appearing in the mixture momentum equation are given as follows

$$\begin{aligned} &P(c,\rho) = \Big(\frac{[1-c]\rho}{a^* - c\rho}\Big)^{\gamma}, \qquad a^* = \rho_l/\hat{c}_0 \qquad (\text{i.e. } \rho_l = a^* + k^*), \\ &g(c\rho) = k^* \frac{c\rho}{k^* + c\rho}, \quad h(c\rho) = 2\rho_l \Big(\frac{\hat{c}_1}{\hat{c}_0}\Big) \frac{c\rho}{k^* + c\rho}, \quad j(c\rho) = \rho_l^2 \Big(\frac{\hat{c}_1}{\hat{c}_0}\Big)^2 \frac{1}{k^* + c\rho}, \\ &E(c\rho) = \varepsilon(c, c\rho)\rho = \frac{[c\rho]^{\beta+1}}{(a^* - [c\rho])^{\beta+1}}. \end{aligned} \tag{1.11}$$

Some observations:

(i) The terms associated with the functions $g(\cdot)$, $h(\cdot)$, and $j(\cdot)$ appear when some relative motion between the gas and liquid phase is allowed, i.e., when $\hat{c}_0 > 1$ and $\hat{c}_1 > 0$. For this case we derive the local existence result given by Theorem 3.1.

(ii) If $\hat{c}_0 > 1$ and $\hat{c}_1 = 0$, then $h = j = 0$ but $g > 0$. For this case we derive the global existence result given by Theorem 3.2 subject to a smallness assumption on the initial data. Note that the parameter $\delta > 0$ appearing in the characterization of initial data, as described in (3.68), depends on the specified global time $T > 0$. Hence, this result cannot be used to study the long-time behavior of the system in question.

(iii) If $\hat{c}_0 = 1$ and $\hat{c}_1 = 0$, then also $g = 0$ in (1.11). This corresponds to the no-slip case where both phases move with the same velocity. This case has been discussed in a number of works [7, 8, 9, 32, 33, 34, 35].

The results are obtained for the model (2.58) and (2.59) which is directly related to the above model through the transformation (2.57). Hence, the results of Theorem 3.1 and Theorem 3.2 expressed in terms of the variables (c, Q, u) can be transferred to the model (1.7)–(1.11) described in terms of (c, ρ, u). See also Remark 3.1 and Remark 3.2. The model with slip parameters $\hat{c}_0 > 1$ and $\hat{c}_1 = 0$ has been studied in [4] and [31] and local in time existence results have been obtained. However, both the local in time existence result for the general slip where $\hat{c}_0 > 1$ and $\hat{c}_1 > 0$ and the global in time result for the slip with $\hat{c}_0 > 1$ and $\hat{c}_1 = 0$ are new. The main techniques we rely on are the energy method and the continuation method, combined with some rather delicate estimates for the lower limit of masses.

- The central part of the local existence result is Proposition 4.1 which ensures that for a sufficient small time period $[0, T]$,

$$\frac{A}{3}\phi(x)^{\frac{3\alpha}{4}} \le Q \le 2B\phi(x)^{\frac{3\alpha}{4}}, \qquad \int_0^1 Eu_x^2 + \int_0^t\int_0^1 u_s^2 \le 2M,$$

  where A, B, M are constants related to initial data and M is large enough. Here, $Q(c,\rho) = \frac{\rho}{a^*-c\rho}$ and $\phi(x) = 1 - x$ and α is a positive parameter characterizing the mass decay rate at the right boundary where masses vanish. Corresponding to these estimates we have that $|u| \le C + CM^{1/2}$, see (4.7) of Lemma 4.1.

- Similarly, the heart of the matter of the global existence result is Proposition 5.1 which guarantees the following estimates

$$\int_0^1 \left(\frac{u^2}{2} + \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)}\right) \le 2\delta, \qquad \frac{\tilde{B}\delta^{\frac{1}{\gamma-1}}}{2}\phi^{\frac{3\alpha}{4}} \le Q \le 2\tilde{A}\phi^{\frac{3\alpha}{4}},$$

  for a sufficient small $\delta(T)$ for a global time $T > 0$ and where $\tilde{A}, \tilde{B}$ are constants related to initial data. Most interestingly, there is a fine tuned balance between the smallness on the energy estimate and the smallness of the lower limit of Q which results in an estimate of fluid velocity of the form $|u| \le C\delta^{-\frac{\beta+1}{2(\gamma-1)}}\left(\int_0^1 Eu_x^2\right)^{\frac{1}{2}}$, see (5.6) of Lemma 5.1. The fact that the δ-parameter is allowed to appear in the lower bound of Q is exploited in the proof of Lemma 5.4. However, the price to pay for this is that the other lemmas become more difficult because fluid velocity involves a δ^{-1} type of term that must be controlled. The key that is repeatedly used to prove these results is the smallness on the energy, as expressed by Lemma 5.1. As commented before, the fact that $\delta(T)$ depends on global time T (see Remark 5.2) prevents from deducing anything about the long-time behavior of the model. This is a consequence of the new term that accounts for non-equal fluid velocity. For the no-slip case the long-time behavior of the gas-liquid model has been investigated in [19, 10].

These estimates pave the way for deriving the required regularity on u and Q in space and time, see Corollary 4.4 and Corollary 4.5 for the local result (Corollary 5.4 and Corollary 5.5 for the global result), which are sufficient to prove convergence to weak solutions. Techniques that are used are motivated by previous studies of single-phase Navier-Stokes, see for example [24, 18, 21, 28, 29, 3].

The structure of the work is as follows: In section 2 we derive the model (1.7)–(1.11). In Section 3 the local and global existence results are presented together with their respective assumptions on initial data and parameters. In Section 4 a priori estimates for smooth solutions for the local existence result are derived. Then, by using the line method where the continuous system is approximated by a semi-discrete, corresponding estimates are obtained for the semi-discrete approximations. This allows for showing the convergence to limit functions that are shown to be weak solutions. Section 5 is devoted to the study of the model with the slip where $\hat{c}_1 = 0$. Various global in time estimates are obtained under a smallness assumption on the initial data.

2. Derivation of the model

We set $n = \alpha_g \rho_g$ and $m = \alpha_l \rho_l$ in (1.1) and consider the model

$$\begin{aligned} \partial_t n + \partial_x [n u_g] &= 0, \\ \partial_t m + \partial_x [m u_l] &= 0, \\ \partial_t [n u_g + m u_l] + \partial_x [n u_g^2 + m u_l^2 + P(n,m)] &= \partial_x [\varepsilon(n,m) \partial_x u_M], \end{aligned} \tag{2.12}$$

where the mixture fluid velocity u_M is defined as follows:

$$u_M = \alpha_g u_g + \alpha_l u_l, \tag{2.13}$$

and where the pressure law $P(n,m)$ and viscosity term $\varepsilon(n,m)$ are given by

$$P(n,m) = \Big(\frac{n}{\rho_l - m}\Big)^{\gamma}, \qquad \varepsilon(n,m) = \frac{(m-k^*)^{\beta+1}}{(n+m-k^*)(\rho_l - m)^{\beta+1}}, \quad \gamma > 1, \quad \beta > 0, \tag{2.14}$$

together with the constitutive relations

$$\alpha_l + \alpha_g = 1, \quad u_g - \hat{c}_0 u_M - \hat{c}_1 = 0, \quad \rho_l = \rho_{l,0}, \quad \rho_g = \rho_g(P), \tag{2.15}$$

where $\hat{c}_0$ and $\hat{c}_1$ are assumed to be constants. As will be explained in the following the slip law $u_g - \hat{c}_0 u_M - \hat{c}_1 = 0$ requires that the liquid mass is above a critical lower limit k^*, i.e., $m \geq k^*$. This information is taken into account in the viscosity coefficient $\varepsilon(n,m)$. Similarly, the upper limit for the liquid mass $m \leq \rho_l$ is also accounted for in the viscosity term.

We now want to rewrite the model (2.12). Our approach is inspired by the work [14]. Given the slip relation

$$u_g = \hat{c}_0 u_M + \hat{c}_1, \tag{2.16}$$

we introduce α_g^*, α_l^* given by

$$\alpha_g^* = \frac{1}{\hat{c}_0}, \qquad \alpha_l^* = 1 - \alpha_g^*. \tag{2.17}$$

In the following we will assume that

$$\hat{c}_0 > 1, \tag{2.18}$$

implying that $\alpha_g^* < 1$. This is consistent with previous applications of the slip velocity (2.16) in the context of gas-liquid and liquid-oil flow modeling where $\hat{c}_0$ typically is ranging between 1.0 and 1.5. Moreover, in view of (2.16) it follows that

$$u_g = \frac{\hat{c}_0 \alpha_l u_l + \hat{c}_1}{1 - \hat{c}_0 \alpha_g} = \frac{\alpha_l u_l + \hat{c}_1 \alpha_g^*}{\alpha_g^* - \alpha_g} = \frac{\alpha_l u_l + \hat{c}_1 (1 - \alpha_l^*)}{\alpha_l - \alpha_l^*}. \tag{2.19}$$

It is implicitly assumed that $\alpha_g < \alpha_g^*$ (or equivalently, that $\alpha_l > \alpha_l^*$) for this slip law to be valid. From (2.19), we get

$$\alpha_l u_l = u_g(\alpha_l - \alpha_l^*) - (1 - \alpha_l^*)\hat{c}_1. \tag{2.20}$$

Clearly,

$$m u_l = \rho_l \alpha_l u_l = \rho_l u_g (\alpha_l - \alpha_l^*) - \rho_l (1 - \alpha_l^*)\hat{c}_1 = \rho_l u_g(\alpha_l - \alpha_l^*) - d = u_g(m - k^*) - d, \tag{2.21}$$

where the constants d and k^* are defined by

$$d = \rho_l (1 - \alpha_l^*)\hat{c}_1, \qquad k^* = \rho_l \alpha_l^*, \tag{2.22}$$

and recall that the liquid is incompressible, i.e., ρ_l=constant. Now, we introduce the notation

$$\rho = \rho_M - \alpha_l^* \rho_l = \rho_l(\alpha_l - \alpha_l^*) + \alpha_g \rho_g = n + m - k^*, \tag{2.23}$$

where

$$\rho_M = \alpha_g \rho_g + \alpha_l \rho_l.$$

Next, we introduce the variable c defined by

$$c = \frac{m - k^*}{\rho} = \frac{m - \alpha_l^* \rho_l}{\rho} = \frac{\rho_l(\alpha_l - \alpha_l^*)}{\rho}. \tag{2.24}$$

We then apply (2.20) and (2.24) and derive the following relations:

$$m = c\rho + k^*, \tag{2.25}$$

$$c\rho u_g = \rho_l(\alpha_l - \alpha_l^*)u_g = \rho_l[\alpha_l u_l + (1-\alpha_l^*)\hat{c}_1] = m u_l + \rho_l(1-\alpha_l^*)\hat{c}_1 = m u_l + d, \tag{2.26}$$

according to (2.22). In other words, $m u_l = c\rho u_g - d$. Moreover, we have from (2.23), (2.24), and (2.21)

$$1 - c = 1 - \frac{m - k^*}{\rho} = \frac{n}{\rho}, \tag{2.27}$$

$$\rho u_g = (\rho_M - k^*)u_g = n u_g + (m - k^*)u_g = n u_g + m u_l + d. \tag{2.28}$$

The model (2.12), by adding the two mass conservation equations, can be written in the form

$$\begin{aligned} \partial_t m + \partial_x[m u_l] &= 0, \\ \partial_t[n + m] + \partial_x[n u_g + m u_l] &= 0, \\ \partial_t[n u_g + m u_l] + \partial_x[n u_g^2 + m u_l^2] + \partial_x[P(n,m)] &= \partial_x[\varepsilon(n,m)\partial_x u_M]. \end{aligned} \tag{2.29}$$

Employing, respectively, first (2.25) and (2.26), then (2.23) and (2.28), the first and second equation of (2.29) can be rewritten such that we arrive at the following form for the system in question:

$$\begin{aligned} \partial_t[c\rho] + \partial_x[c\rho u_g] &= 0, \\ \partial_t \rho + \partial_x[\rho u_g] &= 0, \\ \partial_t[\rho u_g] + \partial_x[\rho u_g^2] + \partial_x[n u_g^2 + m u_l^2 - \rho u_g^2] + \partial_x[P(n,m)] &= \partial_x[\varepsilon(n,m)\partial_x u_M]. \end{aligned} \tag{2.30}$$

Here we also have used (2.28) again to rewrite the momentum equation. Noting that

$$n u_g^2 - \rho u_g^2 = (k^* - m)u_g^2 = \rho_l(\alpha_l^* - \alpha_l)u_g^2,$$

the mixture momentum equation of (2.30) can be written in the form

$$\partial_t[\rho u_g] + \partial_x[\rho u_g^2] + \partial_x[\rho_l(\alpha_l^* - \alpha_l)u_g^2 + m u_l^2 + P(n,m)] = \partial_x[\varepsilon(n,m)\partial_x u_M]. \tag{2.31}$$

Now, we want to rewrite the last term on the left hand side in terms of the variables (c, ρ, u_g). Firstly, we observe that

$$n = (1-c)\rho, \qquad m = c\rho + k^*. \tag{2.32}$$

Hence, the pressure law $P(n,m)$ takes the form

$$P(n,m) = \Big(\frac{n}{\rho_l - m}\Big)^\gamma = \Big(\frac{[1-c]\rho}{[\rho_l - k^*] - c\rho}\Big)^\gamma = \Big(\frac{[1-c]\rho}{a^* - c\rho}\Big)^\gamma := P(c,\rho), \tag{2.33}$$

where $a^* = \rho_l - k^* = \rho_l \alpha_g^*$. Secondly, we note that

$$\rho_l(\alpha_l^* - \alpha_l)u_g^2 + m u_l^2 = \alpha_l \rho_l[u_l^2 - u_g^2] + k^* u_g^2. \tag{2.34}$$

Next, we observe in view of (2.20) that

$$\alpha_l(u_l - u_g) = -u_g \alpha_l^* - (1-\alpha_l^*)\hat{c}_1, \tag{2.35}$$

$$\alpha_l(u_l + u_g) = u_g(2\alpha_l - \alpha_l^*) - (1-\alpha_l^*)\hat{c}_1. \tag{2.36}$$

Multiplying these two relations we get

$$\alpha_l^2(u_l^2 - u_g^2) = -u_g^2 \alpha_l^*[2\alpha_l - \alpha_l^*] - 2\hat{c}_1 u_g[1-\alpha_l^*][\alpha_l - \alpha_l^*] + \hat{c}_1^2[1-\alpha_l^*]^2. \tag{2.37}$$

Then we have

$$\alpha_l \rho_l(u_l^2 - u_g^2) = -\rho_l u_g^2 \alpha_l^*[2 - \frac{\alpha_l^*}{\alpha_l}] - 2\rho_l \hat{c}_1 u_g[1-\alpha_l^*][1 - \frac{\alpha_l^*}{\alpha_l}] + \rho_l \hat{c}_1^2[1-\alpha_l^*]^2 \frac{1}{\alpha_l}. \tag{2.38}$$

In view of (2.34) and (2.38) we get

$$
\begin{aligned}
G(n,m,u_g) &:= \rho_l(\alpha_l^* - \alpha_l)u_g^2 + m u_l^2 \\
&= \rho_l\alpha_l^* u_g^2\Big[-2 + \frac{\alpha_l^*}{\alpha_l}\Big] + \rho_l\alpha_l^* u_g^2 - 2\rho_l\hat{c}_1 u_g[1-\alpha_l^*][1-\frac{\alpha_l^*}{\alpha_l}] + \rho_l\hat{c}_1^2[1-\alpha_l^*]^2\frac{1}{\alpha_l} \\
&= \rho_l\alpha_l^* u_g^2\Big[-1 + \frac{\alpha_l^*}{\alpha_l}\Big] + 2\rho_l\hat{c}_1 u_g[1-\alpha_l^*][-1+\frac{\alpha_l^*}{\alpha_l}] + \rho_l\hat{c}_1^2[1-\alpha_l^*]^2\frac{1}{\alpha_l} \\
&= k^* u_g^2\Big[\frac{k^*-m}{m}\Big] + 2\rho_l\hat{c}_1 u_g[1-\alpha_l^*]\Big[\frac{k^*-m}{m}\Big] + \rho_l^2\hat{c}_1^2[1-\alpha_l^*]^2\frac{1}{m} \\
&= -k^* u_g^2\Big[\frac{c\rho}{c\rho+k^*}\Big] - 2\rho_l u_g\Big(\frac{\hat{c}_1}{\hat{c}_0}\Big)\Big[\frac{c\rho}{c\rho+k^*}\Big] + \rho_l^2\Big(\frac{\hat{c}_1}{\hat{c}_0}\Big)^2\Big[\frac{1}{c\rho+k^*}\Big] \\
&=: -u_g^2 g(c\rho) - u_g h(c\rho) + j(c\rho),
\end{aligned}
\tag{2.39}
$$

where we have used (2.32) and we have defined the function $g(\cdot)$, $h(\cdot)$, and $j(\cdot)$ as

$$
g(c\rho) = k^*\frac{c\rho}{k^*+c\rho}, \qquad h(c\rho) = 2\rho_l\Big(\frac{\hat{c}_1}{\hat{c}_0}\Big)\frac{c\rho}{k^*+c\rho}, \qquad j(c\rho) = \rho_l^2\Big(\frac{\hat{c}_1}{\hat{c}_0}\Big)^2\frac{1}{k^*+c\rho}. \tag{2.40}
$$

For the viscosity term $\varepsilon(n,m)$ we have

$$
\varepsilon(n,m) = \frac{c(m-k^*)^\beta}{(\rho_l - m)^{\beta+1}} = \frac{c[c\rho]^\beta}{(a^* - [c\rho])^{\beta+1}} := \varepsilon(c, c\rho). \tag{2.41}
$$

Hence, setting $u_g := u$, using (2.39) in the momentum equation (2.31), we obtain a gas-liquid model of the following form:

$$
\begin{aligned}
\partial_t[c\rho] + \partial_x[c\rho u] &= 0, \\
\partial_t[\rho] + \partial_x[\rho u] &= 0, \\
\partial_t[\rho u] + \partial_x[\rho u^2] + \partial_x[P(c,\rho) - u^2 g(c\rho) - u h(c\rho) + j(c\rho)] &= \frac{1}{\hat{c}_0}\partial_x[\varepsilon(c,c\rho)\partial_x u].
\end{aligned}
\tag{2.42}
$$

We may absorb the constant $1/\hat{c}_0$ into the viscosity term ε without loss of any generality.

2.1. **Lagrangian coordinates.** Following the approach of the works [8, 9, 32], which in turn is motivated by studies for the single-phase gas model, we suggest to study the model (2.42), described in terms of the variables (c, ρ, u), in a free boundary setting.

$$
\begin{aligned}
\partial_t[c\rho] + \partial_x[c\rho u] &= 0, \\
\partial_t[\rho] + \partial_x[\rho u] &= 0, \\
\partial_t[\rho u] + \partial_x[\rho u^2] + \partial_x[P(c,\rho) - u^2 g(c\rho) - u h(c\rho) + j(c\rho)] &= \partial_x[\varepsilon(c,c\rho)\partial_x u],
\end{aligned}
\tag{2.43}
$$

with $x \in (a(t), b(t))$ and $t > 0$. Initial data are

$$
\rho(x, t=0) = \rho_0(x), \qquad c(x,t=0) = c_0(x) = \frac{m_0(x) - k^*}{\rho_0(x)}, \qquad u(x,t=0) = u_0(x), \tag{2.44}
$$

for $x \in [a_0, b_0]$ where $a_0 = a(t=0)$ and $b_0 = b(t=0)$. Boundary conditions are set to be as follows:

$$
u(a(t),t) = 0, \qquad \rho(b(t),t) = 0, \qquad c(b(t),t) = 0, \tag{2.45}
$$

or

$$
\rho(a(t),t) = 0, \qquad c(a(t),t) = 0, \qquad \rho(b(t),t) = 0, \qquad c(b(t),t) = 0. \tag{2.46}
$$

Here $a(t)$ and $b(t)$, which separate the gas-liquid mixture and the vacuum like state corresponding to $\rho = 0$ and $c = 0$, satisfy

$$
\frac{da}{dt} = u(a(t),t), \qquad a(0) = a_0, \tag{2.47}
$$

and

$$
\frac{db}{dt} = u(b(t),t), \qquad b(0) = b_0. \tag{2.48}
$$

We can introduce Lagrangian coordinates by using the transformation $(x,t) \to (\xi,\tau)$ given by

$$\xi = \int_{a(t)}^{x} \rho(z,t)\,dz, \qquad \tau = t, \tag{2.49}$$

observing that

$$\int_{a(t)}^{b(t)} \rho(z,t)\,dz = \int_{a_0}^{b_0} \rho(z,t=0)\,dz = \text{constant} = 1.$$

This implies that $[a(t),b(t)]$ is converted into the fixed interval $[0,1]$. Since

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = \rho\frac{\partial}{\partial \xi},$$

we can transform (2.43) into the following form:

$$\begin{aligned} \partial_\tau[c\rho] + [c\rho]\partial_x u &= 0,\\ \partial_\tau \rho + \rho\partial_x u &= 0,\\ \rho\partial_\tau u + \partial_x[P(c,\rho) - u^2 g(c\rho) - uh(c\rho) + j(c\rho)] &= \partial_x[\varepsilon(c,c\rho)\partial_x u], \quad \text{in } 0<\xi<1. \end{aligned} \tag{2.50}$$

In other words,

$$\begin{aligned} \partial_\tau[c\rho] + c\rho^2\partial_\xi u &= 0,\\ \partial_\tau \rho + \rho^2\partial_\xi u &= 0,\\ \partial_\tau u + \partial_\xi[P(c,\rho) - u^2 g(c\rho) - uh(c\rho) + j(c\rho)] &= \partial_\xi[\varepsilon(c,c\rho)\rho\partial_\xi u], \quad \text{in } 0<\xi<1. \end{aligned} \tag{2.51}$$

We now replace (τ,ξ) by (t,x). Moreover, an easy calculation shows that (2.51) corresponds to

$$\begin{aligned} \partial_t c &= 0,\\ \partial_t \rho + \rho^2\partial_x u &= 0,\\ \partial_t u + \partial_x[P(c,\rho) - u^2 g(c\rho) - uh(c\rho) + j(\rho)] &= \partial_x[E(c\rho)\partial_x u], \quad \text{in } 0<x<1, \end{aligned} \tag{2.52}$$

with boundary conditions

$$u(0,t) = 0, \qquad \rho(1,t) = c(1,t) = 0, \tag{2.53}$$

or

$$\rho(0,t) = \rho(1,t) = 0, \qquad c(0,t) = c(1,t) = 0, \tag{2.54}$$

and with initial conditions

$$c(x,0) = c_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x), \qquad x \in [0,1], \tag{2.55}$$

where $c(x,t) = c_0(x) = \frac{m_0(x)-k^*}{\rho_0(x)}$. Moreover, we have that

$$\begin{aligned} P(c,\rho) &= \Big(\frac{[1-c]\rho}{a^*-c\rho}\Big)^\gamma,\\ g(c\rho) &= k^*\frac{c\rho}{k^*+c\rho}, \quad h(c\rho) = 2\rho_l\Big(\frac{\hat{c}_1}{\hat{c}_0}\Big)\frac{c\rho}{k^*+c\rho}, \quad j(c\rho) = \rho_l^2\Big(\frac{\hat{c}_1}{\hat{c}_0}\Big)^2\frac{1}{k^*+c\rho},\\ E(c\rho) &= \varepsilon(c,c\rho)\rho = \frac{[c\rho]^{\beta+1}}{(a^*-[c\rho])^{\beta+1}}. \end{aligned} \tag{2.56}$$

Hence, the model (2.52)–(2.56) is now consistent with the model (1.7)–(1.11).

2.2. **Reformulation.** For the analysis of the model (2.52), it will be convenient to introduce the function $Q(c,\rho)$ given by

$$Q(c,\rho) = \frac{\rho}{a^*-c\rho}, \qquad \text{which corresponds to} \qquad \rho = a^*\frac{Q}{1+cQ}. \tag{2.57}$$

A similar approach was also used in [8, 9], however, for a different model with equal fluid velocities. The following nice relation holds for $Q(c,\rho)$:

$$\begin{aligned} Q(c,\rho)_t &= Q_c c_t + Q_\rho \rho_t = Q_\rho \rho_t \\ &= \Big(\frac{1}{a^*-c\rho} + \frac{c\rho}{(a^*-c\rho)^2}\Big)\rho_t = \frac{a^*}{(a^*-c\rho)^2}\rho_t \\ &= -\frac{a^*\rho^2}{(a^*-c\rho)^2}u_x = -a^* Q(c,\rho)^2 u_x. \end{aligned}$$

Hence, the system (2.52) can be replaced by

$$\begin{cases} \partial_t c = 0, \\ \partial_t Q + a^* Q^2 \partial_x u = 0, \\ \partial_t u + \partial_x[P(c,Q) - u^2 g(cQ) - uh(cQ) + j(cQ)] = \partial_x[E(cQ)\partial_x u], \quad x\in(0,1),\ t>0, \end{cases} \tag{2.58}$$

with

$$\begin{aligned} &P(c,Q) = [(1-c)Q(c,\rho)]^\gamma, \qquad E(c\rho) = \varepsilon(c,c\rho)\rho = [cQ]^{\beta+1} := E(cQ), \\ &g(c\rho) = k^*\frac{c\rho}{c\rho+k^*} = a^*\alpha_l^*\Big(\frac{cQ}{\alpha_l^*+cQ}\Big) := g(cQ), \\ &h(c\rho) = 2\rho_l\Big(\frac{\hat c_1}{\hat c_0}\Big)\frac{c\rho}{c\rho+k^*} = 2\Big(\frac{\hat c_1}{\hat c_0}\Big)a^*\Big(\frac{cQ}{\alpha_l^*+cQ}\Big) := h(cQ), \\ &j(c\rho) = \rho_l^2\Big(\frac{\hat c_1}{\hat c_0}\Big)^2\frac{1}{k^*+c\rho} = \rho_l\Big(\frac{\hat c_1}{\hat c_0}\Big)^2\Big(\frac{1+cQ}{\alpha_l^*+cQ}\Big) := j(cQ), \end{aligned} \tag{2.59}$$

since

$$c\rho = a^*\frac{cQ}{1+cQ}.$$

Boundary conditions for our system (2.58)–(2.59) are (in view of (2.57) and (2.53), (2.54)):

$$u(0,t) = 0,\ (c,Q)(1,t) = 0, \tag{2.60}$$

or

$$(c,Q)(0,t) = 0,\ (c,Q)(1,t) = 0. \tag{2.61}$$

Initial conditions are (in view of (2.57) and (2.55)):

$$c(x,0) = c_0(x), \qquad Q(x,0) = Q_0(x) = \frac{\rho_0}{a^*-c_0\rho_0}, \qquad u(x,0) = u_0(x), \qquad x\in[0,1]. \tag{2.62}$$

3. Main results

Throughout the rest of the paper, we denote $L^p = L^p([0,1])$, $\int_0^1 f = \int_0^1 f\,dx$ when it will not cause any confuse.

3.1. **Local weak solution.** Main assumptions:

$$\begin{cases} \tilde c_1\phi^{\frac{\alpha}{4}} \le c_0 \le \tilde c_2\phi^{\frac{\alpha}{4}},\ \frac{A\tilde c_2}{\tilde c_1}\phi^{\frac{3\alpha}{4}} \le Q_0 \le \frac{B\tilde c_1}{\tilde c_2}\phi^{\frac{3\alpha}{4}},\ \tilde c_1\le\tilde c_2,\ \alpha>0, \\ \phi^{1-\alpha\beta}|(c_0^{\beta+1}Q_0^\beta)_x|^2 \in L^1,\ \phi^{1-\frac{\alpha}{4}}c_{0x}^2 \in L^1, \\ \int_0^1 E(c_0Q_0)u_{0x}^2\,dx \le M, \end{cases} \tag{3.63}$$

for some constant $M>0$ determined by (4.16), and

$$\alpha(\beta+1)<1,\ 0<\beta\le\frac{3}{4},\ \alpha(4\beta+1)\le 2,\ \gamma\ge\frac{4\beta+1}{3},\ \gamma>1, \tag{3.64}$$

where

$$\phi(x) = \begin{cases} x(1-x), & \text{for boundary (2.61)}, \\ 1-x, & \text{for boundary (2.60)}. \end{cases}$$

For boundary condition (2.60), we require the compatibility condition $u_0(0) = 0$.

Theorem 3.1. *Under the assumptions of (3.63) and (3.64), there exists a constant $T_0 > 0$ such that (2.58), (2.62) and (2.61) or (2.60) admits a unique weak solution (c, Q, u) on $[0,1] \times [0, T_0]$ in the sense that*
(A) we have the following regularity:

$$c, Q \in L^\infty([0,T_0]; L^\infty) \cap C^1([0,T_0]; L^2), \ E(cQ)u_x \in L^\infty([0,T_0]; L^2),$$
$$u \in L^\infty([0,T_0]; L^\infty) \cap C^{\frac{1}{2}}([0,T_0]; L^2).$$

Moreover, the following estimates hold:

$$\frac{A}{3}\phi^{\frac{3\alpha}{4}} \le Q \le 2B\phi^{\frac{3\alpha}{4}}, \ \int_0^1 E u_x^2 \, dx + \int_0^t \int_0^1 u_s^2 \, dx \, ds \le 2M,$$

and

$$\|u\|_{L^\infty} + \int_0^1 \left(\phi^{1-\alpha\beta} |(c^{\beta+1}Q^\beta)_x|^2 + \phi^{(\beta-2)\alpha} Q_t^2 \right) dx \le C,$$

for $(x,t) \in [0,1] \times [0,T_0]$, where C depends on $A, B, M, \tilde{c}_1, \tilde{c}_2, \alpha, \beta, \gamma$ and the initial data.

(B) The following equations hold:

$$\begin{cases} \partial_t c = 0, \ \partial_t Q + a^* Q^2 \partial_x u = 0, \ \text{for a.e. } (x,t) \in (0,1) \times (0,T_0], \\ (c,Q)(x,0) = (c_0(x), Q_0(x)), \ \text{for a.e. } x \in [0,1], \\ \displaystyle\int_0^{T_0} \int_0^1 \left[u\varphi_t + \left(P(c,Q) - u^2 g(cQ) - uh(cQ) + j(cQ) - \frac{\rho_l (\frac{\hat{c}_1}{\hat{c}_0})^2}{\alpha_l^*} - E(cQ)u_x \right) \varphi_x \right] dx \, dt \\ \displaystyle + \int_0^1 u_0 \varphi(x,0) \, dx = 0, \end{cases}$$

for any test function $\varphi \in C_0^\infty([0,1] \times [0,T_0))$ (for boundary condition (2.60), $\varphi \in C_0^\infty\big((0,1] \times [0,T_0)\big)$).

Remark 3.1. *Denote $\rho = a^* \frac{Q}{1+cQ}$, then from Theorem 3.1, we get a weak solution (c, ρ, u) on $[0,1] \times [0,T_0]$ to (2.52), (2.55) and (2.53) or (2.54).*

3.2. **Global weak solution.** If $\hat{c}_1 = 0$ in the general slip law (2.16), the system (2.58) becomes

$$\begin{cases} c_t = 0, \\ Q_t + a^* Q^2 u_x = 0, \\ u_t + \left[P(c,Q) - u^2 g(cQ) \right]_x = [E(cQ)u_x]_x, \ x \in (0,1), \ t > 0. \end{cases} \tag{3.65}$$

System (3.65) is supplemented with initial data

$$(c, Q, u)(x, 0) = (c_0, Q_0, u_0), \tag{3.66}$$

and boundary condition

$$u(0,t) = 0, \ c(1,t) = Q(1,t) = 0. \tag{3.67}$$

Main assumptions:

$$
\begin{cases}
B_1\phi^{\frac{\alpha}{4}} \le c_0 \le A_1\phi^{\frac{\alpha}{4}},\ \alpha>0,\\
\tilde{B}\delta^{\frac{1}{\gamma-1}}\phi^{\frac{3\alpha}{4}} \le Q_0 \le \tilde{A}\phi^{\frac{3\alpha}{4}},\\
\int_0^1 \left(\frac{u_0^2}{2}+\frac{(1-c_0)^\gamma Q_0^{\gamma-1}}{a^*(\gamma-1)}\right) \le \delta,\\
(c_0Q_0)^{\frac{\beta+1}{2}}u_{0x}\in L^2,\ \phi^{1-\alpha\beta}|(c_0^{\beta+1}Q_0^{\beta})_x|^2\in L^1,\ \phi^{1-\frac{\alpha}{4}}c_{0x}^2\in L^1,
\end{cases}
\tag{3.68}
$$

and

$$
\begin{cases}
\gamma>\max\{\beta+2,\ 1+4\beta,\ \frac{2(3\beta+1)}{3}+1\},\ \gamma\ge\frac{2+8\beta}{3}-\frac{2}{3\alpha},\\
0<\beta\le\frac{3}{4},\ \alpha(\beta+1)<1,\ \alpha(4\beta+1)\le 2,\ \alpha(4\beta-1)\le 1,
\end{cases}
\tag{3.69}
$$

where $A_1,\ \tilde{A}>1,\ B_1,\ \tilde{B}<1$ and $\phi(x)=1-x$. The coefficient of the lower bound of Q_0 related to $\delta^{\frac{1}{\gamma-1}}$ naturally comes from $(3.68)_3$.

Theorem 3.2. *Under the assumptions of (3.68) and (3.69), for any given $T>0$, there exists a positive constant $C(T)$ such that (3.65), (3.66) and (3.67) admits a unique weak solution (c,Q,u) on $[0,1]\times[0,T]$ with the same regularities as (A) in Theorem 3.1, satisfying the following estimates*

$$
\int_0^1\left(\frac{u^2}{2}+\frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)}\right)dx\le 2\delta,\ \text{and}\ \frac{\tilde{B}\delta^{\frac{1}{\gamma-1}}}{2}\phi^{\frac{3\alpha}{4}}\le Q\le 2\tilde{A}\phi^{\frac{3\alpha}{4}},
$$

and

$$
\|u\|_{L^\infty}+\int_0^1\left(Eu_x^2+\phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x|^2+\phi^{(\beta-2)\alpha}Q_t^2\right)dx+\int_0^t\int_0^1 u_s^2\,dx\,ds\le C,
$$

for $(x,t)\in[0,1]\times[0,T]$, provided $\delta\le C(T)$. Here C may depend on $A_1,B_1,\tilde{A},\tilde{B},\delta,\alpha,\beta,\gamma$ and the initial data. Moreover, the following equations hold:

$$
\begin{cases}
\partial_t c=0,\ \partial_t Q+a^*Q^2\partial_x u=0,\ \text{for a.e. } (x,t)\in(0,1)\times(0,T],\\
(c,Q)(x,0)=(c_0(x),Q_0(x)),\ \text{for a.e. } x\in[0,1],\\
\int_0^T\int_0^1\left[u\varphi_t+\left(P(c,Q)-u^2g(cQ)-E(cQ)u_x\right)\varphi_x\right]dx\,dt+\int_0^1 u_0\varphi(x,0)\,dx=0,
\end{cases}
$$

for any test function $\varphi\in C_0^\infty\big((0,1]\times[0,T)\big)$.

Remark 3.2. *Denote $\rho=a^*\frac{Q}{1+cQ}$, then from Theorem 3.2, we get a weak solution (c,ρ,u) on $[0,1]\times[0,T]$ to (2.52), (2.53) and (2.55) with $\hat{c}_1=0$ (i.e., $h=j\equiv 0$).*

4. Local existence of weak solutions

4.1. *A priori* **estimates.** Assume that the solutions are smooth enough in $[0,1]\times[0,T]$. Then we get a crucial proposition:

Proposition 4.1. *Under the conditions of Theorem 3.1, assume that the solutions are smooth enough, and that*

$$
\frac{A}{3}\phi^{\frac{3\alpha}{4}}\le Q\le 2B\phi^{\frac{3\alpha}{4}}\ \text{and}\ \int_0^1 Eu_x^2+\int_0^t\int_0^1 u_s^2\le 2M,\ (x,t)\in[0,1]\times[0,\tilde{T}]\subseteq[0,1]\times[0,T], \tag{4.1}
$$

then

$$
\frac{A}{2}\phi^{\frac{3\alpha}{4}}\le Q\le\frac{3B}{2}\phi^{\frac{3\alpha}{4}}\ \text{and}\ \int_0^1 Eu_x^2+\int_0^t\int_0^1 u_s^2\le\frac{3M}{2}, \tag{4.2}
$$

for $(x,t)\in[0,1]\times[0,\tilde{T}]$, provided that T is small enough which is determined by (4.10), (4.16) and (4.28).

Suppose Proposition 4.1 is true, we get a useful corollary as follows:

Corollary 4.1. *Under the conditions of Theorem 3.1, assume that the solutions are smooth enough in* $[0,1]\times[0,T]$*, we get*

$$\frac{A}{3}\phi^{\frac{3\alpha}{4}} \le Q \le 2B\phi^{\frac{3\alpha}{4}} \text{ and } \int_0^1 Eu_x^2 + \int_0^t\int_0^1 u_s^2 \le 2M,\ (x,t)\in[0,1]\times[0,T], \tag{4.3}$$

provided that T *is small enough which is determined by (4.10), (4.16) and (4.28).*

Proof. Since

$$A \le \frac{Q_0}{\phi^{\frac{3\alpha}{4}}} \le B, \text{ and } \int_0^1 E(c_0Q_0)u_{0x}^2 \le M,$$

by the continuity of the solution, then there exists a constant $\tilde{T}_1 \in (0,T)$ such that

$$\inf_{x\in[0,1]} \frac{Q}{\phi^{\frac{3\alpha}{4}}} > \frac{A}{3},\ \sup_{x\in[0,1]} \frac{Q}{\phi^{\frac{3\alpha}{4}}} < 2B, \text{ and } \int_0^1 Eu_x^2 + \int_0^t\int_0^1 u_s^2 < 2M, \tag{4.4}$$

for any $t\in[0,\tilde{T}_1]$. We denote the maximal time for (4.4) by $\tilde{T}_2 \in [\tilde{T}_1, T]$. If $\tilde{T}_2 = T$, we have nothing to do. For otherwise, we get from (4.4) and Proposition 4.1

$$\inf_{x\in[0,1]} \frac{Q}{\phi^{\frac{3\alpha}{4}}} \ge \frac{A}{2},\ \sup_{x\in[0,1]} \frac{Q}{\phi^{\frac{3\alpha}{4}}} \le \frac{3B}{2}, \text{ and } \int_0^1 Eu_x^2 + \int_0^t\int_0^1 u_s^2 \le \frac{3M}{2},$$

for any $t\in[0,\tilde{T}_2)\subset[0,T]$. This together with the continuity of the solution w.r.t. time in $[0,T]$ implies that (4.4) is also valid at $\tilde{T}_2$, which is contradicted with the definition of $\tilde{T}_2$. Therefore, we get $\tilde{T}_2 = T$, which shows that (4.3) holds. □

Let's go back to the proof of Proposition 4.1.

Proof of Proposition 4.1:

The proof of this proposition is divided into the following lemmas.

Lemma 4.1. *Under the assumptions of Proposition 4.1, it holds that*

$$\int_0^1 u^4 + \int_0^t\int_0^1 Eu^2u_x^2 \le C_1(1+M^3)T + \int_0^1 u_0^4, \tag{4.5}$$

for $t\in[0,\tilde{T}]$*, where* $C_1 = C_1(A,B)$*.*

Proof. Multiplying $(2.58)_3$ by $4u^3$, integrating by parts over $[0,1]$, and using Cauchy inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 u^4 + 12\int_0^1 Eu^2u_x^2\\
=&12\int_0^1 [P(c,Q) - u^2g(cQ) - uh(cQ) + j(cQ)]u^2u_x - \frac{12\rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2}{\alpha_l^*}\int_0^1 u^2u_x\\
\le&\int_0^1 Eu^2u_x^2 + C_1\int_0^1 u^2Q^{2\gamma-\beta-1}c^{-\beta-1} + C_1\int_0^1 u^4(1+u^2)(cQ)^{1-\beta} + C_1\int_0^1 u^2(cQ)^{-\beta-1}.
\end{aligned}$$

This together with (4.1), $c = c_0$ and Young inequality implies

$$\frac{d}{dt}\int_0^1 u^4 + 11\int_0^1 Eu^2u_x^2 \le C_1\|u\|_{L^\infty}^2 + C_1(\|u\|_{L^\infty}^6 + 1), \tag{4.6}$$

where we have used

$$\frac{\alpha(6\gamma-4\beta-4)}{4} > -1 \text{ and } \alpha(\beta+1) < 1,$$

i.e., $\gamma > \frac{2\beta+2}{3} - \frac{2}{3\alpha}$ and $\alpha(\beta+1) < 1$ (Note that $\alpha > 0$ and $\alpha(\beta+1) < 1$ concludes $\frac{2\beta+2}{3} - \frac{2}{3\alpha} < 0$. Thus from (3.64), we have $\gamma > 1 > 0 > \frac{2\beta+2}{3} - \frac{2}{3\alpha}$). Here $C_1 = C_1(A,B)$.

Claim:

$$\|u\|_{L^\infty} \le C_1 + C_1 M^{\frac{1}{2}}. \tag{4.7}$$

In fact, for boundary (2.61), we obtain from $(2.58)_3$

$$\int_0^1 u = \int_0^1 u_0.$$

This deduces

$$|u(x,t)| \le |\int_0^1 u_0| + |u - \int_0^1 u| \le C_1 + \int_0^1 |u_x| \le C_1 + C_1 M^{\frac{1}{2}},$$

where we have used Hölder inequality, $\alpha(\beta+1) < 1$ and (4.1).

For boundary (2.60), we have

$$|u(x,t)| = |\int_0^x u_y| \le \int_0^1 |u_x| \le C_1 + C_1 M^{\frac{1}{2}}.$$

Substituting (4.7) into (4.6), we get

$$\frac{d}{dt}\int_0^1 u^4 + 11\int_0^1 E u^2 u_x^2 \le C_1(1 + M^3). \tag{4.8}$$

Integrating (4.8) over $[0,t]$, we get (4.5). □

Lemma 4.2. *Under the assumptions of Proposition 4.1, it holds that*

$$Q \le \frac{3B}{2}\phi^{\frac{3\alpha}{4}}, \tag{4.9}$$

for $(x,t) \in [0,1] \times [0,\tilde{T}]$, *provided* T *is sufficiently small such that*

$$B^\beta + \sqrt{MT}C_1 + C_1 T(1+M) \le \left(\frac{3B}{2}\right)^\beta. \tag{4.10}$$

Proof. It follows from $(2.58)_2$, $(2.58)_3$ and $(2.58)_1$ that

$$\left(u + (\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x\right)_t + [P(c,Q) - u^2 g(cQ) - uh(cQ) + j(cQ)]_x = 0. \tag{4.11}$$

Integrating (4.11) over $[x,1] \times [0,t]$, we have

$$\begin{aligned} &c^{\beta+1}Q^\beta + a^*\beta\int_0^t (1-c)^\gamma Q^\gamma \\ =&c_0^{\beta+1}Q_0^\beta + a^*\beta\int_0^t\int_x^1 u_s + a^*\beta\int_0^t \big(u^2 g(cQ) + uh(cQ)\big) - a^*\beta\rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2\int_0^t \frac{(\alpha_l^*-1)cQ}{\alpha_l^*(\alpha_l^* + cQ)}. \end{aligned} \tag{4.12}$$

Multiplying (4.12) by $c_0^{-(\beta+1)}$, and using $(2.58)_1$, Hölder inequality, (4.1) and (4.7), we have

$$\begin{aligned} Q^\beta \le& Q_0^\beta + \frac{a^*\beta\sqrt{2MT}\phi^{\frac{1}{2}}}{c_0^{\beta+1}} + \frac{C_1 T(1+M)\phi^\alpha}{c_0^{\beta+1}} + \frac{C_1 T\phi^\alpha}{c_0^{\beta+1}} \\ \le& B^\beta\phi^{\frac{3\alpha\beta}{4}} + \sqrt{MT}C_1\phi^{\frac{1}{2}-\frac{\alpha(\beta+1)}{4}} + C_1T(1+M)\phi^{\alpha-\frac{\alpha(\beta+1)}{4}} + C_1 T\phi^{\alpha-\frac{\alpha(\beta+1)}{4}} \\ \le& \Big(B^\beta + \sqrt{MT}C_1 + C_1T(1+M)\Big)\phi^{\frac{3\alpha\beta}{4}}, \end{aligned} \tag{4.13}$$

where we have used

$$\frac{1}{2} - \frac{\alpha(\beta+1)}{4} \ge \frac{3\alpha\beta}{4} \text{ and } \frac{3\alpha - \alpha\beta}{4} \ge \frac{3\alpha\beta}{4},$$

i.e., $4\alpha\beta + \alpha \le 2$ and $\beta \le \frac{3}{4}$, and also have used

$$1 - x \le C_1\phi. \tag{4.14}$$

In fact, for boundary condition (2.60), by the definition of ϕ, (4.14) is obvious, so it is for boundary condition (2.61) and $x \in [\frac{1}{2}, 1]$. If $x \in [0, \frac{1}{2}]$, for boundary condition (2.61), we can integrate (4.11) again over $[0, x] \times [0, t]$, and get (4.12) with $\int_x^1 u_s$ replaced by $\int_0^x u_s$. Then, using some similar arguments as (4.13) will produce x instead of $1-x$. Then $x \le C_1\phi$ for $x \in [0, \frac{1}{2}]$. In the following, we do not mention it again when we use (4.14) for boundary condition (2.61), for brevity.

Since $\beta > 0$ from (3.64), we may choose $T > 0$ small enough such that

$$B^\beta + \sqrt{MT}C_1 + C_1T(1+M) \le \left(\frac{3B}{2}\right)^\beta.$$

This combining (4.13) implies

$$Q \le \frac{3B\phi^{\frac{3\alpha}{4}}}{2}.$$

□

Lemma 4.3. *Under the assumptions of Proposition 4.1, it holds that*

$$\int_0^1 Eu_x^2 + \int_0^t\int_0^1 u_s^2 \le \frac{3M}{2}, \tag{4.15}$$

for $t \in [0, \tilde{T}]$, provided T is sufficiently small such that (4.10) and

$$C_1 + C_1(1+M^3)T \le \frac{M}{20} \tag{4.16}$$

are satisfied.

Proof. Multiplying $(2.58)_3$ by u_t, and integrating by parts over $[0, 1]$, we have

$$\begin{aligned}
&\int_0^1 u_t^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 Eu_x^2\\
=&\frac{d}{dt}\int_0^1\left[P - u^2g(cQ) - uh(cQ) + j(cQ) - \frac{\rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2}{\alpha_l^*}\right]u_x + \frac{1}{2}\int_0^1[(cQ)^{\beta+1}]_t u_x^2\\
&-\int_0^1 P_t u_x + \int_0^1[u^2g(cQ)]_t u_x + \int_0^1[uh(cQ)]_t u_x - \int_0^1[j(cQ)]_t u_x\\
=&\frac{d}{dt}\int_0^1\left[P - u^2g(cQ) - uh(cQ) + j(cQ) - \frac{\rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2}{\alpha_l^*}\right]u_x + \sum_{i=1}^5 I_i.
\end{aligned} \tag{4.17}$$

For I_1, we have

$$\begin{aligned}
I_1 =&\frac{\beta+1}{2}\int_0^1 c^{\beta+1}Q^\beta Q_t u_x^2 = -\frac{a^*(\beta+1)}{2}\int_0^1 c^{\beta+1}Q^{\beta+2}u_x^3\\
=&-\frac{a^*(\beta+1)}{2}\int_0^1 Eu_xQu_x^2,
\end{aligned} \tag{4.18}$$

where we have used $(2.58)_2$.

Note from $(2.58)_3$ and (2.61) (or (2.60)) that

$$Eu_x = P(c,Q) - u^2g(cQ) - uh(cQ) + \rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2\frac{(\alpha_l^*-1)cQ}{\alpha_l^*(\alpha_l^*+cQ)} + \int_1^x u_t. \tag{4.19}$$

Substituting (4.19) into (4.18), and using Hölder inequality, (4.7) and Cauchy inequality, we have

$$
\begin{aligned}
I_1 \le & \frac{1}{6}\int_0^1 u_t^2 + C_1\left(\int_0^1 \phi^{\frac{1}{2}} Q u_x^2\right)^2 + C_1(1+M)\int_0^1 cQ^2u_x^2 \\
\le & \frac{1}{6}\int_0^1 u_t^2 + C_1\left(\int_0^1 \phi^{\frac{1}{2}} c^{-\beta-1}Q^{-\beta}Eu_x^2\right)^2 + C_1(1+M)\int_0^1 c^{-\beta}Q^{1-\beta}Eu_x^2 \\
\le & \frac{1}{6}\int_0^1 u_t^2 + C_1(1+M)M,
\end{aligned}
\tag{4.20}
$$

where we have used (4.1),

$$
\frac{1}{2} - \frac{4\alpha\beta+\alpha}{4} \ge 0 \text{ and } \frac{\alpha(3-4\beta)}{4} \ge 0,
$$

i.e., $4\alpha\beta + \alpha \le 2$ and $\beta \le \frac{3}{4}$.

For I_2, we have

$$
\begin{aligned}
I_2 = & -\gamma\int_0^1 (1-c)^\gamma Q^{\gamma-1}Q_t u_x = \gamma a^*\int_0^1 (1-c)^\gamma Q^{\gamma+1}u_x^2 \\
= & \gamma a^*\int_0^1 (1-c)^\gamma c^{-\beta-1}Q^{\gamma-\beta}Eu_x^2 \le C_1M,
\end{aligned}
\tag{4.21}
$$

where we have used

$$
\frac{3\gamma-4\beta-1}{4} \ge 0,
$$

i.e., $\gamma \ge \frac{4\beta+1}{3}$.

For I_3, using Cauchy inequality, (4.1) and (4.7), we have

$$
\begin{aligned}
I_3 \le & C_1\int_0^1 \left[|uu_t g(cQ)| + u^2|cQ_t|\right]|u_x| \\
\le & \frac{1}{6}\int_0^1 u_t^2 + C_1(1+M)\int_0^1 cQ^2u_x^2 \\
\le & \frac{1}{6}\int_0^1 u_t^2 + C_1(1+M)^2.
\end{aligned}
\tag{4.22}
$$

Similarly, for I_4 and I_5, we have

$$
I_4 + I_5 \le \frac{1}{6}\int_0^1 u_t^2 + C_1(1+M)^2. \tag{4.23}
$$

Substituting (4.20), (4.21), (4.22) and (4.23) into (4.17), and integrating the result over $[0,t]$ for $t \le \tilde{T}$, we have

$$
\begin{aligned}
\int_0^t\int_0^1 u_s^2 + \int_0^1 Eu_x^2 \le & 2\int_0^1 \left[P - u^2g(cQ) - uh(cQ) + j(cQ) - \frac{\rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2}{\alpha_l^*}\right]u_x \\
& + C_1(1+M)^2T + C_0,
\end{aligned}
\tag{4.24}
$$

where

$$
C_0 = M - 2\int_0^1 \left[P_0 - u_0^2g(c_0Q_0) - u_0h(c_0Q_0) + j(c_0Q_0) - \frac{\rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2}{\alpha_l^*}\right]u_{0x}.
$$

By (4.17) and Cauchy inequality, we have

$$\begin{aligned}\int_0^t\int_0^1 u_s^2+\int_0^1 Eu_x^2 \leq&\frac{1}{6}\int_0^1 Eu_x^2+C_1\int_0^1 Q^{2\gamma-\beta-1}c^{-\beta-1}\\ &+C_1\int_0^1(u^4+u^2+1)(cQ)^{1-\beta}+C_1(1+M)^2T+C_0\\ \leq&\frac{1}{6}\int_0^1 Eu_x^2+\frac{6M}{5}+C_1+C_1(1+M^3)T,\end{aligned}\tag{4.25}$$

where we have used $\gamma>\frac{2\beta+2}{3}-\frac{2}{3\alpha}$, $\beta\leq 1$, (4.5) and (4.1).

Thus,

$$\int_0^t\int_0^1 u_s^2+\frac{5}{6}\int_0^1 Eu_x^2 \leq C_1+\frac{6M}{5}+C_1(1+M^3)T.\tag{4.26}$$

Taking M sufficiently large such that

$$C_1+C_1(1+M^3)T\leq\frac{M}{20},$$

for some small T. This completes the proof of Lemma 4.3. □

Remark 4.1. *Note that the L^4 (instead of L^2) estimate of u in (4.5) plays a crucial role in (4.25).*

Lemma 4.4. *Under the assumptions of Proposition 4.1, it holds that*

$$Q\geq\frac{A}{2}\phi^{\frac{3\alpha}{4}},\tag{4.27}$$

for $(x,t)\in[0,1]\times[0,\tilde{T}]$, provided T is sufficiently small such that (4.10), (4.16) and

$$C_1T+C_1(1+\sqrt{M})T+C_1\sqrt{MT}\leq\frac{1}{A}\tag{4.28}$$

are satisfied.

Proof. It follows from $(2.58)_2$ that

$$\frac{d}{dt}\left(\frac{\phi^{\frac{3\alpha}{4}}}{Q}\right)=-Q^{-2}Q_t\phi^{\frac{3\alpha}{4}}=a^*u_x\phi^{\frac{3\alpha}{4}}.\tag{4.29}$$

Integrating (4.29) over $[0,t]$, and using (4.19), (4.1), $\alpha_l^*\leq 1$ and (4.7), we have

$$\begin{aligned}\frac{\phi^{\frac{3\alpha}{4}}}{Q}=&\frac{\phi^{\frac{3\alpha}{4}}}{Q_0}+a^*\int_0^t\phi^{\frac{3\alpha}{4}}u_x=\frac{\phi^{\frac{3\alpha}{4}}}{Q_0}+a^*\int_0^t\phi^{\frac{3\alpha}{4}}(cQ)^{-(\beta+1)}Eu_x\\ =&\frac{\phi^{\frac{3\alpha}{4}}}{Q_0}+a^*\int_0^t\phi^{\frac{3\alpha}{4}}(cQ)^{-\beta-1}\left(P(c,Q)-u^2g(cQ)-uh(cQ)+\rho_l(\frac{\hat{c}_1}{\hat{c}_0})^2\frac{(\alpha_l^*-1)cQ}{\alpha_l^*(\alpha_l^*+cQ)}+\int_1^x u_s\right)\\ \leq&\frac{\phi^{\frac{3\alpha}{4}}}{Q_0}+C_1T\phi^{\frac{3\alpha}{4}}\phi^{\frac{3\gamma\alpha}{4}-\alpha(\beta+1)}+a^*\int_0^t\phi^{\frac{3\alpha}{4}}(cQ)^{-\beta-1}\left(-uh(cQ)+\int_1^x u_s\right)\\ \leq&\frac{1}{A}+C_1T\phi^{\frac{3(\gamma+1)\alpha}{4}-\alpha(\beta+1)}+C_1(1+\sqrt{M})T\phi^{\frac{3\alpha}{4}}\phi^{-\beta\alpha}+C_1\sqrt{MT}\phi^{\frac{1}{2}}\phi^{\frac{3\alpha}{4}}\phi^{-\alpha(\beta+1)}.\end{aligned}$$

Let

$$\frac{3(\gamma+1)\alpha}{4}-\alpha(\beta+1)\geq 0,\ \frac{3\alpha}{4}-\beta\alpha\geq 0,\ \frac{1}{2}+\frac{3\alpha}{4}-\alpha(\beta+1)\geq 0,$$

i.e., $\gamma\geq\frac{4\beta+1}{3}$, $\beta\leq\frac{3}{4}$ and $\alpha(4\beta+1)\leq 2$, since $\alpha>0$. Then, taking T small enough such that

$$C_1T+C_1(1+\sqrt{M})T+C_1\sqrt{MT}\leq\frac{1}{A},$$

we get (4.27). □

From Lemmas 4.2, 4.3 and 4.4, we end the proof of Proposition 4.1.

Next, we derive more estimates needed for the compactness arguments of the next section where construction of a weak solution is shown.

Corollary 4.2. *Under the assumptions of Theorem 3.1, it holds that*

$$\|u(\cdot,t)\|_{L^\infty} \le C_2, \tag{4.30}$$

and

$$\|u_x(\cdot,t)\|_{L^r} \le C_2, \tag{4.31}$$

for $t \in [0,T]$ and some $r \in (1,2)$, where $C_2 = C_2(A,B,M)$.

Proof. (4.30) can be obtained by (4.7). In order to get (4.31), note that $\alpha(\beta+1) < 1$, then there exists a constant $r \in (1,2)$ such that

$$\alpha(\beta+1) < \frac{2-r}{r}.$$

This together with (4.3), (4.15) and Hölder inequality deduces

$$\int_0^1 |u_x|^r = \int_0^1 E^{\frac{r}{2}}|u_x|^r E^{-\frac{r}{2}} \le \left(\int_0^1 Eu_x^2\right)^{\frac{r}{2}} \left(\int_0^1 E^{-\frac{r}{2-r}}\right)^{\frac{2-r}{2}} \le C_2.$$

□

Corollary 4.3. *Under the assumptions of Theorem 3.1, it holds that*

$$\int_0^1 \phi^{(\beta-2)\alpha} Q_t^2 \le C_2, \tag{4.32}$$

for $t \in [0,T]$.

Proof. (4.32) can be obtained by $(2.58)_2$ and (4.15). More precisely,

$$\begin{aligned}\int_0^1 \phi^{(\beta-2)\alpha} Q_t^2 \le& C_2 \int_0^1 \phi^{(\beta-2)\alpha} Q^4 u_x^2 = C_2 \int_0^1 \phi^{(\beta-2)\alpha} c^{-1-\beta} Q^{3-\beta} E u_x^2 \\ \le& C_2 \int_0^1 \phi^{(\beta-2)\alpha} \phi^{-\frac{(1+\beta)\alpha}{4}} \phi^{\frac{3\alpha(3-\beta)}{4}} E u_x^2 \\ =& C_2 \int_0^1 E u_x^2 \le C_2.\end{aligned}$$

□

Corollary 4.4. *Under the assumptions of Theorem 3.1, it holds that*

$$\begin{cases} \int_0^1 |Q(x,t) - Q(x,s)|^2 dx \le C_2|t-s|^2, \\ \int_0^1 |u(x,t) - u(x,s)|^2 dx \le C_2|t-s|, \end{cases} \tag{4.33}$$

for $t, s \in [0,T]$.

Proof. (4.33) can be obtained by (4.32), (4.15) and Hölder inequality. More precisely, without loss of generality, we assume $s \le t$. Then

$$\int_0^1 |Q(x,t) - Q(x,s)|^2 dx = \int_0^1 \left|\int_s^t Q_\xi(x,\xi) d\xi\right|^2 dx \le (t-s) \int_0^1 \int_s^t [Q_\xi(x,\xi)]^2 d\xi dx \le C_2|t-s|^2,$$

and

$$\int_0^1 |u(x,t) - u(x,s)|^2 dx = \int_0^1 \left|\int_s^t u_\xi(x,\xi) d\xi\right|^2 dx \le (t-s) \int_0^1 \int_s^t [u_\xi(x,\xi)]^2 d\xi dx \le C_2|t-s|.$$

□

Lemma 4.5. *Under the assumptions of Theorem 3.1, it holds that*

$$\int_0^1 \phi^{1-\alpha\beta} |(c^{\beta+1} Q^\beta)_x|^2 + \int_0^t \int_0^1 \phi^{1-\alpha\beta} (1-c)^\gamma Q^{\gamma+\beta-2} c^{\beta+1} Q_x^2 \le C_2, \tag{4.34}$$

for $t \in [0,T]$.

Proof. Multiplying (4.11) by $(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x\phi^{1-\alpha\beta}$, and integrating by parts over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\phi^{1-\alpha\beta}|(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x|^2\\
=&-\int_0^1\phi^{1-\alpha\beta}u_t(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x-\int_0^1\phi^{1-\alpha\beta}(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x[P(c,Q)]_x\\
&+\int_0^1\phi^{1-\alpha\beta}(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x[u^2g(cQ)]_x+\int_0^1\phi^{1-\alpha\beta}(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x[uh(cQ)]_x\\
&-\int_0^1\phi^{1-\alpha\beta}(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x[j(cQ)]_x=\sum_{i=1}^5 II_i.
\end{aligned}\tag{4.35}$$

For II_1, using Cauchy inequality, we have

$$II_1\le C_2\int_0^1\phi^{1-\alpha\beta}|(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x|^2+C_2\int_0^1u_t^2.\tag{4.36}$$

For II_2, we have

$$\begin{aligned}
II_2=&-\gamma\int_0^1\phi^{1-\alpha\beta}(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x[(1-c)Q]^{\gamma-1}[Q_x-(cQ)_x]\\
=&-\frac{\gamma}{a^*\beta}\int_0^1\phi^{1-\alpha\beta}\left(\beta c^{\beta+1}Q^{\beta-1}Q_x+(\beta+1)c^\beta c_xQ^\beta\right)[(1-c)Q]^{\gamma-1}[Q_x(1-c)-c_xQ]\\
=&-\frac{\gamma}{a^*}\int_0^1\phi^{1-\alpha\beta}(1-c)^\gamma Q^{\gamma+\beta-2}c^{\beta+1}Q_x^2+\frac{\gamma(\beta+1)}{a^*\beta}\int_0^1\phi^{1-\alpha\beta}c^\beta c_x^2Q^{\beta+\gamma}(1-c)^{\gamma-1}\\
&-\frac{\gamma}{a^*\beta}\int_0^1\phi^{1-\alpha\beta}Q_xc_x(1-c)^{\gamma-1}Q^{\gamma+\beta-1}c^\beta\left[(1-c)(\beta+1)-\beta c\right].
\end{aligned}$$

This, together with Cauchy inequality applied to the last term and the fact $c=c_0$, gives

$$\begin{aligned}
II_2\le&-\frac{\gamma}{2a^*}\int_0^1\phi^{1-\alpha\beta}(1-c)^\gamma Q^{\gamma+\beta-2}c^{\beta+1}Q_x^2+C_2\int_0^1\phi^{1-\alpha\beta}c_{0x}^2(1-c_0)^{\gamma-2}Q^{\gamma+\beta}c_0^{\beta-1}\\
&+\frac{\gamma(\beta+1)}{a^*\beta}\int_0^1\phi^{1-\alpha\beta}c_0^\beta c_{0x}^2Q^{\beta+\gamma}(1-c_0)^{\gamma-1}\\
\le&-\frac{\gamma}{2a^*}\int_0^1\phi^{1-\alpha\beta}(1-c)^\gamma Q^{\gamma+\beta-2}c^{\beta+1}Q_x^2+C_2.
\end{aligned}\tag{4.37}$$

For II_3, using (4.7), (4.3) and Cauchy inequality, we have

$$\begin{aligned}
II_3\le&C_2\int_0^1\phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x|^2+C_2\int_0^1\phi^{1-\alpha\beta}Eu_x^2(cQ)^{1-\beta}+C_2\int_0^1\phi^{1-\alpha\beta}|(cQ)_x|^2\\
\le&C_2\int_0^1\phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x|^2+C_2\int_0^1\phi^{1-\alpha\beta}|(cQ)_x|^2+C_2.
\end{aligned}\tag{4.38}$$

Since

$$Q_x=\frac{(c^{\beta+1}Q^\beta)_x}{\beta c^{\beta+1}Q^{\beta-1}}-\frac{(\beta+1)Qc_x}{\beta c},\tag{4.39}$$

we have

$$\begin{aligned}
\int_0^1\phi^{1-\alpha\beta}|(cQ)_x|^2\le&C_2\int_0^1\phi^{1-\alpha\beta}Q^2c_{0x}^2+C_2\int_0^1\phi^{1-\alpha\beta}c^2Q_x^2\\
\le&C_2\int_0^1\phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x|^2c^{-2\beta}Q^{2-2\beta}+C_2\int_0^1\phi^{1-\alpha\beta}Q^2c_{0x}^2\\
\le&C_2\int_0^1\phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x|^2+C_2,
\end{aligned}\tag{4.40}$$

where we have used $\beta \le \frac{3}{4}$ and the fact $c = c_0$.

Substituting (4.40) into (4.38), we have

$$II_3 \le C_2 \int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^{\beta})_x|^2 + C_2. \tag{4.41}$$

Similar to II_3, for II_4 and II_5, we have

$$II_4 + II_5 \le C_2 \int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^{\beta})_x|^2 + C_2. \tag{4.42}$$

Putting (4.36), (4.37), (4.41) and (4.42) into (4.35), and using (4.15) and Gronwall inequality, we get (4.34). □

Corollary 4.5. *Under the assumptions of Theorem 3.1, it holds that*

$$\int_0^1 |Q_x| \le C_2, \tag{4.43}$$

for $t \in [0, T]$.

Proof. (4.43) could be obtained by (4.39), (4.34) and (4.3). More precisely,

$$\begin{aligned}
\int_0^1 |Q_x| \le& C_2 \int_0^1 \frac{|(c^{\beta+1}Q^{\beta})_x|}{c^{\beta+1}Q^{\beta-1}} + C_2 \int_0^1 \frac{|Qc_x|}{c} \\
\le& C_2 \int_0^1 \phi^{\frac{3\alpha(1-\beta)-\alpha(\beta+1)}{4}} |(c^{\beta+1}Q^{\beta})_x| + C_2 \int_0^1 \phi^{\frac{\alpha}{2}} |c_{0x}| \\
\le& C_2 \int_0^1 \phi^{\frac{\alpha(1-2\beta)}{2}} |(c^{\beta+1}Q^{\beta})_x| + C_2 \int_0^1 \phi^{\frac{5\alpha-4}{8}} \phi^{\frac{4-\alpha}{8}} |c_{0x}| \\
\le& C_2 \left(\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^{\beta})_x|^2\right)^{\frac{1}{2}} \left(\int_0^1 \phi^{\alpha(1-\beta)-1}\right)^{\frac{1}{2}} + C_2 \left(\int_0^1 \phi^{1-\frac{\alpha}{4}} |c_{0x}|^2\right)^{\frac{1}{2}} \left(\int_0^1 \phi^{\frac{5\alpha-4}{4}}\right)^{\frac{1}{2}} \\
\le& C_2,
\end{aligned}$$

where we also have used $c = c_0$, Hölder inequality, $\phi^{1-\frac{\alpha}{4}}|c_{0x}|^2 \in L^1$, $\alpha > 0$ and $\beta < 1$. □

4.2. **Construction of weak solution.** For boundary condition (2.61), one can use some arguments like in [29, 30, 4, 31] and references therein to construction a weak solution to (2.58). Here we only sketch the construction of weak solution to (2.58), (2.62) and (2.60). To do this, we use the line method like in [15, 24] which need to be slightly modified. More precisely, we consider systems of 3N ordinary differential equations when N goes to infinity:

$$\begin{cases}
\frac{d}{dt} c_{2i-1}^k(t) = 0, \\
\frac{d}{dt} Q_{2i-1}^k + a^*(Q_{2i-1}^k)^2 \frac{u_{2i}^k - u_{2i-2}^k}{k} = 0, \\
\frac{d}{dt} u_{2i}^k + \frac{P(c_{2i+1}^k, Q_{2i+1}^k) - P(c_{2i-1}^k, Q_{2i-1}^k)}{k} - \frac{(u_{2i+2}^k)^2 g(c_{2i+1}^k Q_{2i+1}^k) - (u_{2i}^k)^2 g(c_{2i-1}^k Q_{2i-1}^k)}{k} \\
- \frac{u_{2i+2}^k h(c_{2i+1}^k Q_{2i+1}^k) - u_{2i}^k h(c_{2i-1}^k Q_{2i-1}^k)}{k} + \frac{j(c_{2i+1}^k Q_{2i+1}^k) - j(c_{2i-1}^k Q_{2i-1}^k)}{k} \\
= \frac{1}{k^2}\left[E(c_{2i+1}^k Q_{2i+1}^k)(u_{2i+2}^k - u_{2i}^k) - E(c_{2i-1}^k Q_{2i-1}^k)(u_{2i}^k - u_{2i-2}^k)\right], \ t > 0,
\end{cases} \tag{4.44}$$

for $i = 1, 2, ..., N$, where $k = \frac{2}{2N+1}$, and the boundary conditions are

$$u_0^k(t) = 0, \ (c_{2N+1}^k, Q_{2N+1}^k)(t) = 0. \tag{4.45}$$

The initial data is given as

$$\begin{cases}
c_{2i-1}^k(0) = c_0\left((2i-1)\frac{k}{2}\right), \\
Q_{2i-1}^k(0) = Q_0\left((2i-1)\frac{k}{2}\right), \\
u_{2i}^k(0) = u_0(ik), \ i = 1, 2, ..., N.
\end{cases} \tag{4.46}$$

When $i = N$, we regard some terms related to u_{2N+2} in $(4.44)_3$ as

$$(u_{2N+2}^k)^2 g(c_{2N+1}^k Q_{2N+1}^k) = u_{2N+2}^k h(c_{2N+1}^k Q_{2N+1}^k) = E(c_{2N+1}^k Q_{2N+1}^k)(u_{2N+2}^k - u_{2N}^k) = 0.$$

In the following, we will use $(c_{2i-1}, Q_{2i-1}, u_{2i})$ instead of $(c_{2i-1}^k, Q_{2i-1}^k, u_{2i}^k)$ when it will not cause any confusion.

Proposition 4.2. *Under the assumptions of Theorem 3.1, if*

$$\begin{aligned}&\frac{A}{3}\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}} \le Q_{2i-1} \le 2B\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}}, \text{ and}\\ &\sum_{i=1}^N E(c_{2i-1}^k Q_{2i-1}^k)\frac{\left(u_{2i}^k - u_{2i-2}^k\right)^2}{k} + \int_0^t \sum_{i=1}^N |\frac{d}{ds}u_{2i}|^2 k \le 2M,\end{aligned} \tag{4.47}$$

for $(x,t) \in [0,1]\times[0,\tilde{T}^k] \subseteq [0,1]\times[0,T^k]$, then

$$\begin{aligned}&\frac{A}{2}\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}} \le Q_{2i-1} \le \frac{3B}{2}\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}}, \text{ and}\\ &\sum_{i=1}^N E(c_{2i-1}^k Q_{2i-1}^k)\frac{\left(u_{2i}^k - u_{2i-2}^k\right)^2}{k} + \int_0^t \sum_{i=1}^N |\frac{d}{ds}u_{2i}|^2 k \le \frac{3M}{2},\end{aligned} \tag{4.48}$$

for $x \in [0,1]\times[0,\tilde{T}^k]$, provided that T^k is small enough.

Corollary 4.6. *Under the conditions of Theorem 3.1, we get*

$$\begin{aligned}&\frac{A}{3}\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}} \le Q_{2i-1} \le 2B\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}}, \text{ and}\\ &\sum_{i=1}^N E(c_{2i-1}^k Q_{2i-1}^k)\frac{\left(u_{2i}^k - u_{2i-2}^k\right)^2}{k} + \int_0^t \sum_{i=1}^N |\frac{d}{ds}u_{2i}|^2 k \le 2M,\end{aligned} \tag{4.49}$$

for $x \in [0,1]\times[0,T^k]$, provided that T^k is small enough.

The proof of Proposition 4.2 is divided into the following discrete version of Lemmas 4.1, 4.2, 4.3 and 4.4, i.e., Lemmas 4.6, 4.7, 4.8 and 4.9.

Lemma 4.6. *Under the assumptions of Proposition 4.2, it holds that*

$$\begin{aligned}&\sum_{i=1}^N u_{2i}^4 k + \int_0^t \sum_{i=1}^N E(c_{2i-1}Q_{2i-1})(u_{2i}^2 + u_{2i}u_{2i-2} + u_{2i-2}^2)\frac{\left(u_{2i}^k - u_{2i-2}^k\right)^2}{k}\\ \le& C_1(1+M^3)T + \sum_{i=1}^N [u_0(ik)]^4 k,\end{aligned} \tag{4.50}$$

for $t \in [0,\tilde{T}^k]$, where $C_1 = C_1(A,B)$, provided that T^k is small.

Lemma 4.7. *Under the assumptions of Proposition 4.2, it holds that*

$$Q_{2i-1} \le \frac{3B}{2}\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}}, \tag{4.51}$$

for $t \in [0,\tilde{T}^k]$, provided that T^k is small.

Lemma 4.8. *Under the assumptions of Proposition 4.2, it holds that*

$$\sum_{i=1}^N E(c_{2i-1}^k Q_{2i-1}^k)\frac{\left(u_{2i}^k - u_{2i-2}^k\right)^2}{k} + \int_0^t \sum_{i=1}^N |\frac{d}{ds}u_{2i}|^2 k \le \frac{3M}{2}, \tag{4.52}$$

for $t \in [0,\tilde{T}^k]$, provided that T^k is small.

Lemma 4.9. *Under the assumptions of Proposition 4.2, it holds that*

$$Q_{2i-1} \geq \frac{A}{2}\phi\left((2i-1)\frac{k}{2}\right)^{\frac{3\alpha}{4}}, \tag{4.53}$$

for $(x,t) \in [0,1] \times [0,\tilde{T}^k]$, *provided that* T^k *is small.*

Corollary 4.7. *Under the assumptions of Theorem 3.1, it holds that*

$$\sup_{1\leq i\leq N} |u_{2i}| \leq C_2, \tag{4.54}$$

and

$$\sum_{i=1}^{N}\left|\frac{u_{2i}-u_{2i-2}}{k}\right|^r k \leq C_2, \tag{4.55}$$

for $t \in [0,T^k]$ *and some* $r \in (1,2)$, *where* $C_2 = C_2(A,B,M)$, *provided that* T^k *is small.*

Corollary 4.8. *Under the assumptions of Theorem 3.1, it holds that*

$$\sum_{i=1}^{N}\phi\left((2i-1)\frac{k}{2}\right)^{(\beta-2)\alpha}|\partial_t Q_{2i-1}|^2 k \leq C_2, \tag{4.56}$$

for $t \in [0,T^k]$, *provided that* T^k *is small.*

Corollary 4.9. *Under the assumptions of Theorem 3.1, it holds that*

$$\begin{cases} \sum_{i=1}^{N}|Q_{2i-1}(t)-Q_{2i-1}(s)|^2 k \leq C_2|t-s|^2, \\ \sum_{i=1}^{N}|u_{2i}(t)-u_{2i}(s)|^2 k \leq C_2|t-s|, \end{cases} \tag{4.57}$$

for $t \in [0,T^k]$, *provided that* T^k *is small.*

Lemma 4.10. *Under the assumptions of Theorem 3.1, it holds that*

$$\sum_{i=1}^{N}\frac{\left[\phi\left((2i-1)\frac{k}{2}\right)\right]^{1-\alpha\beta}}{k}|c_{2i+1}^{\beta+1}Q_{2i+1}^{\beta}-c_{2i-1}^{\beta+1}Q_{2i-1}^{\beta}|^2 \leq C_2, \tag{4.58}$$

for $t \in [0,T^k]$, *provided that* T^k *is small.*

Corollary 4.10. *Under the assumptions of Theorem 3.1, it holds that*

$$\sum_{i=1}^{N}|Q_{2i+1}-Q_{2i-1}| \leq C_2, \tag{4.59}$$

for $t \in [0,T^k]$, *provided that* T^k *is small.*

From the proof of Proposition 4.1, we know that there exists a $T_0 > 0$ independent of k and determined by (4.10), (4.16) and (4.28), such that $T^k \geq T_0$. Similar to some arguments in [24], we define the sequence of approximate solutions (c_k, Q_k, u_k) for $(x,t) \in [0,1]\times[0,T_0]$ as follows:

$$\begin{cases} c_k(x,t) = c_{2i-1}(t), \\ Q_k(x,t) = Q_{2i-1}(t), \\ u_k(x,t) = \frac{1}{k}\left[(x-(i-1)k)\,u_{2i}(t) + (ik-x)u_{2i-2}(t)\right], \end{cases}$$

for $(i-1)k < x \leq ik$, $i = 1,2,...,N$. A direct calculation implies

$$\partial_x u_k(x,t) = \frac{u_{2i}(t)-u_{2i-2}(t)}{k},$$

for $(i-1)k < x \leq ik$, $i = 1,2,...,N$. Then by using Helly's theorem and some similar arguments as those in [24], we get a weak solution to (2.58), (2.62) and (2.60) on $[0,1]\times[0,T_0]$. With the regularities, we can use the standard methods (see for instance [36] and references therein) to get the uniqueness of the solution. We complete the proof of Theorem 3.1.

5. Global existence of weak solution with small data

Here is a crucial proposition in this section:

Proposition 5.1. *Under the assumptions of Theorem 3.2, for any given $T > 0$ (not necessarily small), there exists a positive constant $C(T)$ such that if*

$$\begin{aligned} &\int_0^1 \left(\frac{u^2}{2} + \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} \right) \le 2\delta, \\ &\frac{\tilde{B}\delta^{\frac{1}{\gamma-1}}}{2}\phi^{\frac{3\alpha}{4}} \le Q \le 2\tilde{A}\phi^{\frac{3\alpha}{4}}, \text{ in } [0,1]\times[0,T_1] \subseteq [0,1]\times[0,T], \end{aligned} \tag{5.1}$$

then

$$\begin{aligned} &\int_0^1 \left(\frac{u^2}{2} + \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} \right) \le \frac{3\delta}{2}, \\ &\frac{2\tilde{B}\delta^{\frac{1}{\gamma-1}}}{3}\phi^{\frac{3\alpha}{4}} \le Q \le \frac{3\tilde{A}}{2}\phi^{\frac{3\alpha}{4}}, \text{ in } [0,1]\times[0,T_1], \end{aligned} \tag{5.2}$$

provided $\delta \le C(T)$ which is determined by (5.8), (5.12) and (5.31).

Similar to the proof of Corollary 4.1, based on Proposition 5.1, we get the following corollary:

Corollary 5.1. *Under the conditions of Theorem 3.2, assume that the solutions are smooth enough in $[0,1]\times[0,T]$, we get*

$$\int_0^1 \left(\frac{u^2}{2} + \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} \right) \le 2\delta, \text{ and } \frac{\tilde{B}\delta^{\frac{1}{\gamma-1}}}{2}\phi^{\frac{3\alpha}{4}} \le Q \le 2\tilde{A}\phi^{\frac{3\alpha}{4}}, \tag{5.3}$$

for $(x,t) \in [0,1]\times[0,T]$, provided $\delta \le C(T)$ which is determined by (5.8), (5.12), (5.20), and (5.31).

Proof of Proposition 5.1:

The proof of this proposition is divided into the following lemmas.

Lemma 5.1. *Under the assumptions of Proposition 5.1, it holds that*

$$\int_0^1 \left(\frac{u^2}{2} + \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} \right) + \int_0^t \int_0^1 E u_x^2 \le \frac{3\delta}{2}, \tag{5.4}$$

for $t \in [0,T_1]$.

Proof. Multiplying $(3.65)_3$ by u, integrating by parts over $[0,1]$, and using $(3.65)_2$, (5.1), the fact $c = c_0$ and Hölder inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 + \int_0^1 E u_x^2 = \int_0^1 [P(c,Q) - u^2 g(cQ)]u_x \\ &\le -\frac{d}{dt}\int_0^1 \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} + C_3\left(\int_0^1 E u_x^2\right)^{\frac{1}{2}} \left(\int_0^1 u^4 c^{1-\beta} Q^{1-\beta}\right)^{\frac{1}{2}} \\ &\le -\frac{d}{dt}\int_0^1 \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} + C_3\left(\int_0^1 E u_x^2\right)^{\frac{1}{2}} \|u\|_{L^\infty} \left(\int_0^1 u^2\right)^{\frac{1}{2}}, \end{aligned} \tag{5.5}$$

where $\beta \le 1$ and $C_3 = C_3(A_1, \tilde{A})$. Note that

$$\begin{aligned}\|u\|_{L^\infty} \le \int_0^1 |u_x| \le & \left(\int_0^1 Eu_x^2\right)^{\frac12}\left(\int_0^1 (cQ)^{-(\beta+1)}\right)^{\frac12}\\ \le & C_4\delta^{-\frac{\beta+1}{2(\gamma-1)}}\left(\int_0^1 Eu_x^2\right)^{\frac12}\left(\int_0^1 \phi^{-\alpha(\beta+1)}\right)^{\frac12}\\ \le & C_4\delta^{-\frac{\beta+1}{2(\gamma-1)}}\left(\int_0^1 Eu_x^2\right)^{\frac12},\end{aligned} \tag{5.6}$$

where we have used (5.1), $\alpha(\beta+1) < 1$, $C_4 = C_4(B_1, \tilde{B})$ and $u(0,t) = 0$.

Putting (5.6) into (5.5), we have

$$\frac{d}{dt}\int_0^1 \left(\frac{u^2}{2} + \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)}\right) + \int_0^1 Eu_x^2 \le C_5\left(4\delta^{1-\frac{\beta+1}{\gamma-1}}\right)^{\frac12}\int_0^1 Eu_x^2, \tag{5.7}$$

where $C_5 = C_5(A_1, \tilde{A}, B_1, \tilde{B})$.

(5.4) can be obtained by (5.7), provided that

$$1 - \frac{\beta+1}{\gamma-1} > 0,$$

i.e., $\gamma > \beta + 2$, and that

$$\left(4\delta^{1-\frac{\beta+1}{\gamma-1}}\right)^{\frac12} C_5 \le \frac13. \tag{5.8}$$

□

Remark 5.1. *From the proof of Lemma 5.1, it seems not working for $\hat{c}_1 > 0$. For example, the term $uh(cQ)$ seems difficult to handle by the above approach.*

Lemma 5.2. *Under the assumptions of Proposition 5.1, it holds that*

$$Q \le \frac{3\tilde{A}}{2}\phi^{\frac{3\alpha}{4}}, \tag{5.9}$$

for $(x,t) \in [0,1] \times [0,T_1]$.

Proof. It follows from $(3.65)_2$ and $(3.65)_3$ that

$$\left(u + (\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x\right)_t + [P(c,Q) - u^2 g(cQ)]_x = 0. \tag{5.10}$$

Integrating (5.10) over $[x,1] \times [0,t]$, we have

$$c^{\beta+1}Q^\beta + a^*\beta\int_0^t (1-c)^\gamma Q^\gamma = c_0^{\beta+1}Q_0^\beta + a^*\beta\int_x^1 (u-u_0) + a^*\beta\int_0^t u^2 g(cQ). \tag{5.11}$$

Multiplying (5.11) by $c_0^{-(\beta+1)}$, and using (5.6), (5.1), (5.4) and the fact $c = c_0$, we have

$$\begin{aligned}Q^\beta \le & Q_0^\beta + \frac{4a^*\beta\sqrt{\delta}\phi^{\frac12}}{c_0^{\beta+1}} + \frac{a^*\beta}{c_0^{\beta+1}}\int_0^t u^2 g(cQ)\\ \le & Q_0^\beta + C_5\sqrt{\delta}\phi^{\frac12 - \frac{\alpha(\beta+1)}{4}} + C_5\int_0^t u^2 c^{-\beta}Q\\ \le & Q_0^\beta + C_5\sqrt{\delta}\phi^{\frac{3\alpha\beta}{4}} + C_5\int_0^t\int_0^1 Eu_x^2\delta^{-\frac{\beta+1}{\gamma-1}}\phi^{\frac{3\alpha-\alpha\beta}{4}}\\ \le & (\tilde{A})^\beta\phi^{\frac{3\alpha\beta}{4}} + C_5\sqrt{\delta}\phi^{\frac{3\alpha\beta}{4}} + C_5\delta^{1-\frac{\beta+1}{\gamma-1}}\phi^{\frac{3\alpha\beta}{4}},\end{aligned}$$

where we have used

$$\frac12 - \frac{\alpha(\beta+1)}{4} \ge \frac{3\alpha\beta}{4} \text{ and } \frac{3\alpha-\alpha\beta}{4} \ge \frac{3\alpha\beta}{4},$$

i.e., $4\alpha\beta+\alpha\le 2$ and $\beta\le\frac{3}{4}$, since $\alpha>0$. Note that $\beta>0$ and $\gamma>\beta+2$, we may choose $\delta>0$ small enough such that

$$(\tilde{A})^{\beta}+C_5\sqrt{\delta}+C_5\delta^{1-\frac{\beta+1}{\gamma-1}}\le\left(\frac{3\tilde{A}}{2}\right)^{\beta}. \tag{5.12}$$

Then, we get

$$Q\le\frac{3\tilde{A}\phi^{\frac{3\alpha}{4}}}{2}.$$

□

Lemma 5.3. *Under the assumptions of Proposition 5.1, it holds that*

$$\int_0^1 Eu_x^2+\int_0^t\int_0^1 u_s^2\le C_5, \tag{5.13}$$

for $t\in[0,T_1]$.

Proof. Multiplying $(3.65)_3$ by u_t, and integrating by parts over $[0,1]$, we have

$$\begin{aligned}\int_0^1 u_t^2+\frac{1}{2}\frac{d}{dt}\int_0^1 Eu_x^2 =&\frac{d}{dt}\int_0^1[P-u^2g(cQ)]u_x+\frac{1}{2}\int_0^1[(cQ)^{\beta+1}]_t u_x^2\\ &-\int_0^1 P_t u_x+\int_0^1[u^2g(cQ)]_t u_x\\ =&\frac{d}{dt}\int_0^1[P-u^2g(cQ)]u_x+\sum_{i=1}^3 III_i.\end{aligned} \tag{5.14}$$

For III_1, similar to (4.18), we have

$$III_1=-\frac{a^*(\beta+1)}{2}\int_0^1 Eu_xQu_x^2. \tag{5.15}$$

Integrating $(3.65)_3$ over $[x,1]$, and using (3.67), we have

$$Eu_x=P(c,Q)-u^2g(cQ)+\int_1^x u_t. \tag{5.16}$$

Substituting (5.16) into (5.15), and using Hölder inequality, Cauchy inequality, (5.6) and (5.1), we have

$$\begin{aligned}III_1\le&\frac{1}{4}\int_0^1 u_t^2+C_5\left(\int_0^1\phi^{\frac{1}{2}}Qu_x^2\right)^2+\frac{a^*(\beta+1)}{2}\int_0^1 Qu_x^2u^2g(cQ)\\ \le&\frac{1}{4}\int_0^1 u_t^2+C_5\left(\int_0^1\phi^{\frac{1}{2}}c^{-\beta-1}Q^{-\beta}Eu_x^2\right)^2\\ &+C_5\delta^{-\frac{\beta+1}{\gamma-1}}\int_0^1 Eu_x^2\int_0^1 c^{-\beta}Q^{1-\beta}Eu_x^2\\ \le&\frac{1}{4}\int_0^1 u_t^2+C_5\delta^{-\frac{2\beta}{\gamma-1}}\left(\int_0^1 Eu_x^2\right)^2+C_5\delta^{-\frac{\beta+1}{\gamma-1}}\left(\int_0^1 Eu_x^2\right)^2\\ \le&\frac{1}{4}\int_0^1 u_t^2+C_5\delta^{-\frac{\beta+1}{\gamma-1}}\left(\int_0^1 Eu_x^2\right)^2,\end{aligned} \tag{5.17}$$

where

$$\frac{1}{2}-\frac{4\alpha\beta+\alpha}{4}\ge 0 \text{ and } \frac{\alpha(3-4\beta)}{4}\ge 0,$$

i.e., $4\alpha\beta+\alpha\le 2$ and $\beta\le\frac{3}{4}$, since $\alpha>0$.

For III_2, similar to (4.21), we have

$$\begin{aligned} III_2 =&\gamma a^* \int_0^1 (1-c)^\gamma c^{-\beta-1} Q^{\gamma-\beta} E u_x^2 \\ \le& C_5 \int_0^1 E u_x^2, \end{aligned} \tag{5.18}$$

where

$$\frac{3\gamma - 4\beta - 1}{4} \ge 0,$$

i.e., $\gamma \ge \frac{4\beta+1}{3}$.

For III_3, using Cauchy inequality, $(3.65)_2$, (5.6) and (5.1), we have

$$\begin{aligned} III_3 \le& C_5 \int_0^1 \left[|u u_t g(cQ)| + u^2 |cQ_t| \right] |u_x| \\ \le& \frac{1}{4}\int_0^1 u_t^2 + C_5 \int_0^1 u^2 (cQ)^2 + C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \left(\int_0^1 E u_x^2 \right)^2 \\ \le& \frac{1}{4}\int_0^1 u_t^2 + C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \int_0^1 E u_x^2 + C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \left(\int_0^1 E u_x^2 \right)^2 . \end{aligned} \tag{5.19}$$

Substituting (5.17), (5.18) and (5.19) into (5.14), and integrating the result over $[0,t]$ for $t \le T_1$, we have

$$\begin{aligned} \int_0^t \int_0^1 u_s^2 + \int_0^1 E u_x^2 \le& 2\int_0^1 [P - u^2 g(cQ)] u_x + C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \int_0^t \left(\int_0^1 E u_x^2 \right)^2 + C_5 \\ \le& \frac{1}{2}\int_0^1 E u_x^2 + C_5 \int_0^1 Q^{2\gamma-\beta-1} c^{-\beta-1} + C_5 \int_0^1 u^4 (cQ)^{1-\beta} \\ &+ C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \int_0^t \left(\int_0^1 E u_x^2 \right)^2 + C_5 \\ \le& \frac{1}{2}\int_0^1 E u_x^2 + C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \int_0^1 E u_x^2 \int_0^1 u^2 + C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \int_0^t \left(\int_0^1 E u_x^2 \right)^2 + C_5, \end{aligned}$$

where we have used (5.4), $\gamma > \beta + 2$, Cauchy inequality, (5.1), (5.6) and $\beta \le 1$. By using (5.1) and the smallness assumption on δ

$$C_5 \delta^{1-\frac{\beta+1}{\gamma-1}} \le \frac{1}{4}, \tag{5.20}$$

the second term on the right hand side can be controlled by the second term on the left hand side. Thus,

$$\int_0^t \int_0^1 u_s^2 + \frac{1}{4}\int_0^1 E u_x^2 \le C_5 \delta^{-\frac{\beta+1}{\gamma-1}} \int_0^t \left(\int_0^1 E u_x^2 \right)^2 + C_5, \tag{5.21}$$

where we have used (5.4) and $\gamma > \beta + 2$.

Note from (5.4) that

$$\delta^{-\frac{\beta+1}{\gamma-1}} \int_0^t \int_0^1 E u_x^2 \le \frac{3}{2} \delta^{1-\frac{\beta+1}{\gamma-1}}.$$

Hence, combining (5.21), $\gamma > \beta + 2$ and Gronwall inequality, it can be concluded that (5.13) holds. □

Lemma 5.4. *Under the assumptions of Proposition 5.1, it holds that*

$$Q \ge \frac{2\tilde{B} \delta^{\frac{1}{\gamma-1}} \phi^{\frac{3\alpha}{4}}}{3}, \tag{5.22}$$

for $(x,t) \in [0,1] \times [0,T_1]$.

Proof. It follows from $(3.65)_2$ that

$$\frac{d}{dt}\left(\frac{\phi^{\frac{3\alpha}{4}}}{Q}\right) = -Q^{-2}\phi^{\frac{3\alpha}{4}}Q_t = a^*\phi^{\frac{3\alpha}{4}}u_x. \tag{5.23}$$

Integrating (5.23) over $[0,t]$, and using (5.16), we have

$$\begin{aligned}
\frac{\phi^{\frac{3\alpha}{4}}}{Q} =&\frac{\phi^{\frac{3\alpha}{4}}}{Q_0} + a^*\int_0^t \phi^{\frac{3\alpha}{4}}u_x = \frac{\phi^{\frac{3\alpha}{4}}}{Q_0} + a^*\int_0^t \phi^{\frac{3\alpha}{4}}Q^{-1-\beta}c^{-\beta-1}Eu_x\\
=&\frac{\phi^{\frac{3\alpha}{4}}}{Q_0} + a^*\int_0^t \phi^{\frac{3\alpha}{4}}Q^{-1-\beta}c^{-\beta-1}\left(P - u^2g + \int_1^x u_s\right)\\
\le&\frac{\phi^{\frac{3\alpha}{4}}}{Q_0} + a^*\int_0^t \phi^{\frac{3\alpha}{4}}(1-c)^\gamma Q^{\gamma-1-\beta}c^{-\beta-1} + a^*\int_0^t \phi^{\frac{3\alpha}{4}}Q^{-1-\beta}c^{-\beta-1}\left(\int_1^x u_s\right)\\
=&\frac{\phi^{\frac{3\alpha}{4}}}{Q_0} + IV_1 + IV_2.
\end{aligned}$$

For IV_1, we have

$$IV_1 \le C_5 t, \tag{5.24}$$

where we have used

$$\gamma \ge \beta + 1 \text{ and } \frac{3\gamma - 4\beta - 1}{4} \ge 0,$$

i.e., $\gamma \ge \beta + 1$ and $\gamma \ge \frac{4\beta+1}{3}$.

For IV_2, we have

$$\begin{aligned}
IV_2 =&a^*\phi^{\frac{3\alpha}{4}}Q^{-1-\beta}c^{-\beta-1}\int_1^x u - a^*\phi^{\frac{3\alpha}{4}}Q_0^{-1-\beta}c_0^{-\beta-1}\int_1^x u_0\\
&+ a^*(1+\beta)\int_0^t \phi^{\frac{3\alpha}{4}}Q^{-2-\beta}c^{-\beta-1}Q_s\int_1^x u\\
\le&C_5\delta^{\frac{1}{2}-\frac{\beta+1}{\gamma-1}}\phi^{\frac{1}{2}-\frac{\alpha(4\beta+1)}{4}} - (a^*)^2(1+\beta)\int_0^t \phi^{\frac{3\alpha}{4}}Q^{-\beta}c^{-\beta-1}u_x\int_1^x u\\
=&C_5\delta^{\frac{1}{2}-\frac{\beta+1}{\gamma-1}}\phi^{\frac{1}{2}-\frac{\alpha(4\beta+1)}{4}} + IV_2^1 + IV_2^2 + IV_2^3,
\end{aligned} \tag{5.25}$$

where

$$IV_2^1 = -(a^*)^2(1+\beta)\int_0^t \phi^{\frac{3\alpha}{4}}Q^{-1-2\beta}c^{-2\beta-2}(Eu_x - P + u^2g)\int_1^x u,$$

$$IV_2^2 = (a^*)^2(1+\beta)\int_0^t \phi^{\frac{3\alpha}{4}}Q^{-1-2\beta}c^{-2\beta-2}u^2g\int_1^x u,$$

and

$$IV_2^3 = -(a^*)^2(1+\beta)\int_0^t \phi^{\frac{3\alpha}{4}}(1-c)^\gamma Q^{\gamma-1-2\beta}c^{-2\beta-2}\int_1^x u.$$

For IV_2^1, using (5.16), Hölder inequality, (5.1) and (5.13), we have

$$\begin{aligned}
IV_2^1 \le&C_5\delta^{-\frac{1+2\beta}{\gamma-1}}\phi^{-\frac{(4\beta+1)\alpha}{2}}\int_0^t\int_x^1 |u_s|\int_x^1 |u|\\
\le&C_5\delta^{\frac{1}{2}-\frac{1+2\beta}{\gamma-1}}\phi^{1-\frac{(4\beta+1)\alpha}{2}}\int_0^t\left(\int_0^1 u_s^2\right)^{\frac{1}{2}}\\
\le&C_5 t^{\frac{1}{2}}\delta^{\frac{1}{2}-\frac{1+2\beta}{\gamma-1}}\phi^{1-\frac{(4\beta+1)\alpha}{2}}.
\end{aligned} \tag{5.26}$$

For IV_2^2, using (5.1), (5.4), (5.6) and Hölder inequality, we have

$$\begin{aligned} IV_2^2 \leq & C_5 \delta^{\frac{1}{2}-\frac{1+2\beta}{\gamma-1}} \delta^{-\frac{\beta+1}{\gamma-1}} \phi^{\frac{1}{2}+\alpha-\frac{(4\beta+1)\alpha}{2}} \int_0^t \int_0^1 E u_x^2 \\ \leq & C_5 \delta^{\frac{3}{2}-\frac{3\beta+2}{\gamma-1}} \phi^{\frac{1+\alpha-4\alpha\beta}{2}}. \end{aligned} \tag{5.27}$$

Similarly, for IV_2^3, we have

$$IV_2^3 \leq C_5 t \delta^{\frac{1}{2}} \phi^{\frac{1}{2}+\frac{(3\gamma-8\beta-2)\alpha}{4}}. \tag{5.28}$$

Putting (5.26), (5.27) and (5.28) into (5.25), we have

$$\begin{aligned} IV_2 \leq & C_5 \delta^{\frac{1}{2}-\frac{1+\beta}{\gamma-1}} \phi^{\frac{1}{2}-\frac{\alpha(4\beta+1)}{4}} + C_5 t^{\frac{1}{2}} \delta^{\frac{1}{2}-\frac{1+2\beta}{\gamma-1}} \phi^{1-\frac{(4\beta+1)\alpha}{2}} \\ & + C_5 \delta^{\frac{3}{2}-\frac{3\beta+2}{\gamma-1}} \phi^{\frac{1+\alpha-4\alpha\beta}{2}} + C_5 t \delta^{\frac{1}{2}} \phi^{\frac{1}{2}+\frac{(3\gamma-8\beta-2)\alpha}{4}}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\phi^{\frac{3\alpha}{4}}}{Q} \leq & \frac{\phi^{\frac{3\alpha}{4}}}{Q_0} + C_5 t + C_5 \delta^{\frac{1}{2}-\frac{1+\beta}{\gamma-1}} \phi^{\frac{1}{2}-\frac{\alpha(4\beta+1)}{4}} + C_5 t^{\frac{1}{2}} \delta^{\frac{1}{2}-\frac{1+2\beta}{\gamma-1}} \phi^{1-\frac{(4\beta+1)\alpha}{2}} \\ & + C_5 \delta^{\frac{3}{2}-\frac{3\beta+2}{\gamma-1}} \phi^{\frac{1+\alpha-4\alpha\beta}{2}} + C_5 t \delta^{\frac{1}{2}} \phi^{\frac{1}{2}+\frac{(3\gamma-8\beta-2)\alpha}{4}}. \end{aligned} \tag{5.29}$$

Multiplying (5.29) by $\frac{2\tilde{B}\delta^{\frac{1}{\gamma-1}}}{3}$, we have

$$\begin{aligned} \frac{2\tilde{B}\delta^{\frac{1}{\gamma-1}}\phi^{\frac{3\alpha}{4}}}{3Q} \leq & \frac{2}{3} + C_5 \delta^{\frac{1}{\gamma-1}} t + C_5 \delta^{\frac{1}{2}-\frac{\beta}{\gamma-1}} \phi^{\frac{1}{2}-\frac{\alpha(4\beta+1)}{4}} + C_5 t^{\frac{1}{2}} \delta^{\frac{1}{2}-\frac{2\beta}{\gamma-1}} \phi^{1-\frac{(4\beta+1)\alpha}{2}} \\ & + C_5 \delta^{\frac{3}{2}-\frac{3\beta+1}{\gamma-1}} \phi^{\frac{1+\alpha-4\alpha\beta}{2}} + C_5 t \delta^{\frac{1}{2}+\frac{1}{\gamma-1}} \phi^{\frac{1}{2}+\frac{(3\gamma-8\beta-2)\alpha}{4}} \\ \leq & \frac{2}{3} + C_5 \delta^{\frac{1}{\gamma-1}} t + C_5 \delta^{\frac{1}{2}-\frac{\beta}{\gamma-1}} + C_5 t^{\frac{1}{2}} \delta^{\frac{1}{2}-\frac{2\beta}{\gamma-1}} + C_5 \delta^{\frac{3}{2}-\frac{3\beta+1}{\gamma-1}} \\ & + C_5 t \delta^{\frac{1}{2}+\frac{1}{\gamma-1}}, \end{aligned} \tag{5.30}$$

where we have used

$$\alpha(4\beta+1) \leq 2, \ \alpha(4\beta-1) \leq 1, \ \text{and } 2 + (3\gamma - 8\beta - 2)\alpha \geq 0.$$

Taking δ sufficiently small such that

$$\frac{2}{3} + C_5 \delta^{\frac{1}{\gamma-1}} T + C_5 \delta^{\frac{1}{2}-\frac{\beta}{\gamma-1}} + C_5 T^{\frac{1}{2}} \delta^{\frac{1}{2}-\frac{2\beta}{\gamma-1}} + C_5 \delta^{\frac{3}{2}-\frac{3\beta+1}{\gamma-1}} + C_5 T \delta^{\frac{1}{2}+\frac{1}{\gamma-1}} \leq 1, \tag{5.31}$$

where we have used

$$\frac{1}{2} > \frac{2\beta}{\gamma-1}, \ \text{and } \frac{3}{2} > \frac{3\beta+1}{\gamma-1},$$

i.e., $\gamma > 1 + 4\beta$ and $\gamma > \frac{2(3\beta+1)}{3} + 1$.

By (5.30) and (5.31), we get (5.22). □

Remark 5.2. *Note that it is the condition* (5.31) *that forces δ to depend on time $T > 0$. It is also interesting to note the term $C_5 t$ on the left hand side of* (5.29)*. This term is made small by multiplying by a term of the form δ^p for some appropriate choice of p. This illustrates one reason why the δ-dependence appears in the lower limit as seen in (5.22).*

From Lemmas 5.1, 5.2 and 5.4, we end the proof of Proposition 5.1.

Corollary 5.2. *Under the assumptions of Theorem 3.2, it holds that*

$$\|u(\cdot,t)\|_{L^\infty} \leq C_6, \tag{5.32}$$

and

$$\|u_x(\cdot,t)\|_{L^r} \leq C_6, \tag{5.33}$$

for $t \in [0,T]$ and some $r \in (1,2)$, where $C_6 = C_6(\delta, A_1, \tilde{A}, B_1, \tilde{B})$.

Proof. (5.32) can be easily obtained by (5.6) and (5.13). Following the proof of (4.31) one can deduce (5.33). □

Corollary 5.3. *Under the assumptions of Theorem 3.2, it holds that*

$$\int_0^1 \phi^{(\beta-2)\alpha} Q_t^2 \le C_6, \tag{5.34}$$

for $t \in [0,T]$.

Proof. Similar to the proof of Corollary 4.3, (5.34) can be obtained by $(3.65)_2$ and (5.13). □

Corollary 5.4. *Under the assumptions of Theorem 3.2, it holds that*

$$\begin{cases} \int_0^1 |Q(x,t)-Q(x,s)|^2 dx \le C_6|t-s|^2, \\ \int_0^1 |u(x,t)-u(x,s)|^2 dx \le C_6|t-s|, \end{cases} \tag{5.35}$$

for $t, s \in [0,T]$.

Proof. Similar to the proof of Corollary 4.4, (5.35) can be obtained by (5.34), (5.13) and Hölder inequality. □

Lemma 5.5. *Under the assumptions of Theorem 3.2, it holds that*

$$\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x|^2 + \int_0^t\int_0^1 \phi^{1-\alpha\beta}(1-c)^\gamma Q^{\gamma+\beta-2}c^{\beta+1}Q_x^2 \le C_6, \tag{5.36}$$

for $t \in [0,T]$.

Proof. Multiplying (5.10) by $\left(u + (\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x\right)\phi^{1-\alpha\beta}$, and integrating by parts over $[0,1]$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \phi^{1-\alpha\beta}|u+(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x|^2 \\ =&-\int_0^1 \phi^{1-\alpha\beta}\left[u+(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x\right][P(c,Q)]_x + \int_0^1 \phi^{1-\alpha\beta}\left[u+(\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x\right]\left[u^2 g(cQ)\right]_x \\ =&V_1+V_2. \end{aligned} \tag{5.37}$$

For V_1, we have

$$\begin{aligned} V_1 =& -\frac{d}{dt}\int_0^1 \phi^{1-\alpha\beta}\frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} + (1-\alpha\beta)\int_0^1 \phi^{-\alpha\beta}Pu \\ &-\frac{\gamma}{a^*\beta}\int_0^1 \phi^{1-\alpha\beta}\Big(\beta c^{\beta+1}Q^{\beta-1}Q_x + (\beta+1)c^\beta c_x Q^\beta\Big)\Big[(1-c)Q\Big]^{\gamma-1}\Big[Q_x(1-c)-c_xQ\Big] \\ \le& -\frac{d}{dt}\int_0^1 \phi^{1-\alpha\beta}\frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} + C_6\int_0^1 P + C_6\int_0^1 u^2 - \frac{\gamma}{a^*}\int_0^1 \phi^{1-\alpha\beta}(1-c)^\gamma Q^{\gamma+\beta-2}c^{\beta+1}Q_x^2 \\ &-\frac{\gamma}{a^*\beta}\int_0^1 \phi^{1-\alpha\beta}Q_x c_x(1-c)^{\gamma-1}Q^{\gamma-1}\left[(1-c)(\beta+1)c^\beta Q^\beta - \beta c^{\beta+1}Q^\beta\right] \\ &+\frac{\gamma(\beta+1)}{a^*\beta}\int_0^1 \phi^{1-\alpha\beta}c^\beta c_x^2 Q^{\beta+\gamma}(1-c)^{\gamma-1}, \end{aligned}$$

where we have used $(3.65)_2$, integration by parts, Cauchy inequality, (5.3), $\gamma \geq \frac{8\beta}{3}$ and (5.32). This together with Cauchy inequality, (5.6) and the fact $c = c_0$ gives

$$
\begin{aligned}
V_1 \leq & -\frac{d}{dt}\int_0^1 \phi^{1-\alpha\beta}\frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)} + C_6\int_0^1 P + C_6\int_0^1 Eu_x^2 \\
& - \frac{\gamma}{2a^*}\int_0^1 \phi^{1-\alpha\beta}(1-c)^\gamma Q^{\gamma+\beta-2}c^{\beta+1}Q_x^2 \\
& + C_6\int_0^1 \phi^{1-\alpha\beta}c_{0x}^2(1-c_0)^{\gamma-2}Q^{\gamma+\beta}c_0^{\beta-1} + \frac{\gamma(\beta+1)}{a^*\beta}\int_0^1 \phi^{1-\alpha\beta}c_{0x}^2\frac{c_0^\beta}{1-c_0}Q^\beta P.
\end{aligned}
\tag{5.38}
$$

For V_2, using Cauchy inequality, (5.6), (5.4) and the fact $c = c_0$, we have

$$
\begin{aligned}
V_2 = & \frac{1}{a^*\beta}\int_0^1 \phi^{1-\alpha\beta}(c^{\beta+1}Q^\beta)_x[u^2 g(cQ)]_x + \int_0^1 \phi^{1-\alpha\beta}[u^2 g(cQ)]_x u \\
\leq & C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x|^2 + C_6\int_0^1 \phi^{1-\alpha\beta}Eu_x^2(cQ)^{1-\beta} \\
& + C_6\int_0^1 \phi^{1-\alpha\beta}u^2|(c^{\beta+1}Q^\beta)_x||(cQ)_x| + C_6\int_0^1 Eu_x^2 \\
\leq & C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x + u|^2 + C_6\int_0^1 Eu_x^2 \\
& + C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|(\beta+1)c^\beta c_x Q^\beta + \beta c^{\beta+1}Q^{\beta-1}Q_x||c_x Q + cQ_x| \\
\leq & C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x + u|^2 + C_6\int_0^1 Eu_x^2 \\
& + C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|c_0^\beta c_{0x}^2 Q^{\beta+1}| + C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}c_0^{\beta+2}Q^{\beta-1}Q_x^2.
\end{aligned}
$$

Since

$$
Q_x = \frac{(c^{\beta+1}Q^\beta)_x}{\beta c^{\beta+1}Q^{\beta-1}} - \frac{(\beta+1)Qc_x}{\beta c},
$$

we have

$$
\int_0^1 \phi^{1-\alpha\beta}c_0^{\beta+2}Q^{\beta-1}Q_x^2 \leq C_6\int_0^1 \phi^{1-\alpha\beta}c_0^{-\beta}Q^{1-\beta}|(c^{\beta+1}Q^\beta)_x|^2 + C_6\int_0^1 \phi^{1-\alpha\beta}c_0^\beta Q^{\beta+1}c_{0x}^2.
$$

Then

$$
V_2 \leq C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x + u|^2 + C_6\int_0^1 Eu_x^2,
\tag{5.39}
$$

where we have used (5.3) and $\beta \leq \frac{3}{4}$. Substituting (5.38) and (5.39) into (5.37), we get

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_0^1 \phi^{1-\alpha\beta}\left(|u + (\frac{c^{\beta+1}Q^\beta}{a^*\beta})_x|^2 + \frac{(1-c)^\gamma Q^{\gamma-1}}{a^*(\gamma-1)}\right) + \frac{\gamma}{2a^*}\int_0^1 \phi^{1-\alpha\beta}(1-c)^\gamma Q^{\gamma+\beta-2}c^{\beta+1}Q_x^2 \\
\leq & C_6\int_0^1 P + C_6\int_0^1 Eu_x^2 + C_6\int_0^1 \phi^{1-\alpha\beta}c_{0x}^2(1-c_0)^{-2}Q^\beta c_0^{\beta-1}P \\
& + \frac{\gamma(\beta+1)}{a^*\beta}\int_0^1 \phi^{1-\alpha\beta}c_{0x}^2\frac{c_0^\beta}{1-c_0}Q^\beta P + C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x + u|^2 \\
\leq & C_6\int_0^1 P + C_6\int_0^1 Eu_x^2 + C_6\int_0^1 \phi^{1-\frac{\alpha}{4}}c_{0x}^2 P + C_6\int_0^1 Eu_x^2\int_0^1 \phi^{1-\alpha\beta}|(c^{\beta+1}Q^\beta)_x + u|^2,
\end{aligned}
\tag{5.40}
$$

where we have used (5.3).

From (5.11) and the proof of Lemma 5.2, we get

$$\int_0^t P \leq C_6. \tag{5.41}$$

By (5.40), (5.4), (5.41), (3.68) and Gronwall inequality, we get (5.36). □

Similar to (4.43), we get the following corollary.

Corollary 5.5. *Under the assumptions of Theorem 3.2, it holds that*

$$\int_0^1 |Q_x| \leq C_6, \tag{5.42}$$

for $t \in [0, T]$.

Following the similar arguments with the last section, we get a unique weak solution to (3.65)-(3.67) in $[0, 1] \times [0, T]$. This completes the proof of Theorem 3.2.

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