A COMPRESSIBLE TWO-PHASE MODEL WITH PRESSURE-DEPENDENT WELL-RESERVOIR INTERACTION

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Abstract. This paper deals with a two-phase compressible gas-liquid model relevant for modeling of gas-kick flow scenarios in oil wells. To make the model more realistic we include a natural *pressure-dependent* well-formation interaction term allowing for modeling of dynamic gas influx/efflux. More precisely, the interaction between well and surrounding formation is controlled by a term of the form $A = q_w(P_w - P)$ which appears in the gas continuity equation where q_w is a rate constant, P_w is a critical pressure whereas P is pressure in the well. Consequently, an additional coupling mechanism is added to the mass and momentum equations. We obtain a global existence result for the new model. One consequence of the existence result is that as long as the well initially is filled with a mixture of gas and liquid, the system will regulate itself (in finite time) in such a way that there does not exist any point along the well where all the gas vanishes, e.g., by escaping into the formation. Similarly, the result guarantees that neither will any pure gas region appear in finite time, despite that gas is free to enter the well from the formation as long as the well pressure P is lower than the critical pressure P_w.

Key words. two-phase flow, well-reservoir flow, weak solutions, Lagrangian coordinates, free boundary problem

AMS subject classifications. 76T10, 76N10, 65M12, 35L60

1. Introduction.

In this work we study a compressible gas-liquid two-phase model where we have included a pressure-controlled gas influx/efflux term relevant for the study of gas-kick flow scenarios in oil wells. In Lagrangian variables the model takes the following form

$$
\begin{aligned}
\partial_t n + (n[\rho - n])\partial_x u &= q_w n[P_w - P(n,\rho)] \\
\partial_t \rho + (\rho[\rho - n])\partial_x u &= q_w n[P_w - P(n,\rho)] \\
g(n,\rho)\partial_t u + \partial_x P(n,\rho) &= \partial_x(E(n,\rho)\partial_x u), \qquad x \in (0,1).
\end{aligned} \tag{1}
$$

Here n is the gas mass, ρ is the total mass (sum of gas and liquid mass), u is fluid velocity which is the same for both the gas and liquid phase, q_w is a constant that characterizes the well-formation interaction, P_w is a constant reference pressure (critical pressure) that determines whether gas will enter the well from the surrounding formation ($P_w > P$) or gas from the well will flow into the formation ($P_w < P$). Moreover, the function $g(n,\rho)$ appearing in the mixture momentum equation is given by

$$
g(n,\rho) = \frac{\rho}{\rho - n}, \tag{2}
$$

and is produced when we go from Eulerian to Lagrangian variables, we refer to Section 2 for details. Pressure $P(n,\rho)$ takes the form

$$
P(n,\rho) = \Big(\frac{n}{\rho_l - [\rho - n]}\Big)^{\gamma}, \qquad \gamma > 1, \tag{3}
$$

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where ρ_l is liquid density assumed to be constant. The mixture viscosity coefficient $E(n,\rho)$ is given by

$$E(n,\rho) = \Big(\frac{\rho}{\rho_l - [\rho - n]}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{4}$$

Moreover, boundary conditions are given by

$$P(n,\rho) = E(n,\rho)u_x, \qquad \text{at } x = 0,1, \quad t \geq 0, \tag{5}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{6}$$

The model is derived from a general drift-flux formulation. Details are provided in the next section.

In a recent paper [5] we studied a similar model but where the well-formation interaction was characterized by a rate function $A(x,t)$ assumed to possess certain properties like $L^\infty([0,1])$ boundedness and $H^1([0,1])$ regularity. More precisely, the model took the following form:

$$\begin{aligned} \partial_t n + (n\rho)\partial_x u &= nA \\ \partial_t \rho + \rho^2 \partial_x u &= nA \\ \partial_t u + \partial_x P(n,\rho) &= -u\frac{n}{\rho}A + \partial_x(E(n,\rho)\partial_x u), \qquad x \in (0,1). \end{aligned} \tag{7}$$

The main difference between the model (7) and (1) is the pressure dependent well-formation term

$$A(x,t) = q_w[P_w - P(n,\rho)]. \tag{8}$$

In many application it is much more realistic to assume a pressure sensitive well-formation interaction term as given by (8). For example, when drilling a well, control of pressure in the open hole section is crucial for the operation. The pressure should remain below the *fracture pressure* and above the *pore pressure* of the formation. If the pressure in a section drops below the pore pressure, formation gas may leak into the well. This is called a kick and has to be handled with care in order to avoid a blow-out situation [1]. In this context P_w corresponds to the given fracture pressure or pore pressure. However, the term (8) also introduces a tighter coupling between the continuity equations and the momentum equation adding new challenges as far as existence, uniqueness, and stability issues are concerned.

We obtain an existence result (Theorem 3.1) for the model (1)–(6), equipped with the interaction term (8), for a class of weak solutions under suitable regularity conditions on the initial data n_0, ρ_0, and u_0. The key point leading to this result is the possibility to obtain sufficient pointwise control on the gas mass n and total mass ρ, upper as well as lower limits. More precisely, by assuming initially that the gas mass n and liquid mass m (i.e., liquid mass $m = \rho - n$) do not disappear or blow up on $[0,1]$, that is,

$$C^{-1} \leq n(x,0) \leq C, \qquad 0 < \mu \leq m(x,0) = \rho(x,0) - n(x,0) \leq \rho_l - \mu < \rho_l,$$

for a suitable constant $C > 0$ and $\mu > 0$, then the same will be true for the masses n and $m = \rho - n$ for all $t \in [0,T]$ for any specified time $T > 0$. This nice feature allows us to obtain various estimates which ultimately ensure convergence to a weak solution. A main tool in this analysis is the introduction of a suitable variable transformation allowing for application of ideas and techniques inspired by those used in [16, 14, 19, 17, 12] in previous studies of the single-phase Navier-Stokes equations. More precisely, we introduce the quantities c and $Q(\rho, c)$ defined by

$$c = \frac{n}{\rho}, \qquad Q = \frac{\rho}{\rho_l - [1-c]\rho}. \tag{9}$$

Consequently, the model (1) described in terms of (n, ρ, u) is converted into a system described in terms of (c, Q, u). In this sense the approach of this work follows along the same line as [9, 10, 20]. Special challenges we deal with in this work are:

- The energy estimate gives an upper bound of terms of the form $q_w \int_0^t \int_0^1 u^2 h(c)[cQ]^\gamma \,dx\,ds$ and $q_w \int_0^t \int_0^1 h(c)[cQ]^{2\gamma-1} \,dx\,ds$ with $h(c) = \frac{c}{1-c}$. These terms appear due to the well-formation term (8) and the control of these is directly exploited to obtain a pointwise upper bound of Q. In this sense the model (1) relies on new arguments compared to the model without well-formation interaction terms [9, 10, 20, 21]. It is also quite different from the arguments used in [5] where we take advantage of the fact that we know that the term $A(x,t)$ in (7) is pointwise bounded.

- In order to show that c and Q^β is in $W^{1,2}(I)$ for $I = (0,1)$, we rely on arguments where the estimates of c_x and $(Q^\beta)_x$ in $L^2(I)$ are coupled together, see Lemma 4.3 and 4.4. Again this is due to the fact that we do not control the well formation term (8) appearing in (1). It lives its own life dictated by the pressure behavior $P(n, \rho)$, in contrast to the analysis of the model (7) where we assume that we have the necessary control of $A(x,t)$, i.e., $A(\cdot, t) \in W^{1,2}(I)$.

The rest of this paper is organized as follows. In Section 2 we derive the model (1) starting from a general drift-flux model. In Section 3 we state precisely the main theorem and its assumptions. In Section 4 we describe a priori estimates for the auxiliary model obtained from (1) by using the variable transformation (9). In Section 5 we briefly explain how these estimates then imply convergence to a weak solution.

2. Derivation of the model. Many well operations in the context of petroleum engineering involve gas-liquid flow in a wellbore where there is some interaction with the surrounding reservoir. For examples of such models in the context of *single-phase* flow we refer to [7, 8] and references therein. In this paper we consider a compressible, transient *two-phase* gas-liquid model with inclusion of well-reservoir interaction. For instance, gas-kick refers to a situation where gas flows into the well from the formation at some regions along the wellbore. As this gas ascends in the well it will typically experience a lower pressure. This leads to decompression of the gas, which in turn, potentially can provoke blow-out like scenarios, see [1, 5, 6] and references therein for more details.

The dynamics of the two-phase well flow is supposed to be dictated by a compressible gas-liquid model of the drift-flux type. More precisely, this model is given as

$$\begin{aligned} \partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] &= [\alpha_g\rho_g]A(x,t) \\ \partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] &= 0 \\ \partial_t[\alpha_l\rho_l u_l + \alpha_g\rho_g u_g] + \partial_x[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + P] &= -F + \partial_x[\varepsilon\partial_x u_{mix}], \end{aligned} \tag{10}$$

where $u_{mix} = \alpha_g u_g + \alpha_l u_l$ and $\varepsilon \geq 0$. This formulation allows us to study transient flows in a well together with possible flow of gas between well and surrounding reservoir represented by the rate term $A(x,t) = q_w[P_w - P(n,m)]$ given in (8). The model is supposed under isothermal conditions. The unknowns are ρ_l, ρ_g the liquid and gas densities, α_l, α_g volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$, u_l, u_g velocities of liquid and gas, P common pressure for liquid and gas, and F representing external forces like gravity and friction. Since the momentum is given only for the mixture, we need an additional closure law which connects the two phase fluid velocities. For more general information concerning two-phase flow dynamics we refer to [5] and references therein.

In this work we consider the special case where a no-slip condition is assumed, i.e.,

$$u_g = u_l = u. \tag{11}$$

We use the notation $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$. Assuming a polytropic gas law relation $P = C\rho_g^\gamma$ with $\gamma > 1$ and incompressible liquid $\rho_l = \text{Const}$, we get a pressure law of the form

$$P(n,m) = C\Big(\frac{n}{\rho_l - m}\Big)^\gamma, \tag{12}$$

since $\rho_g = n/\alpha_g = n/(1-\alpha_l) = \rho_l \cdot n/(\rho_l - m)$. In particular, we see that pressure becomes singular at transition to pure liquid phase, i.e., $\alpha_l = 1$ and $\alpha_g = 0$, which yields $m = \rho_l$ and $n = 0$. Another possibility is that the gas density ρ_g vanishes which implies vacuum, i.e., $P = 0$. In order to treat this difficulty we shall consider (10) in a free boundary problem setting where the masses m and n initially occupy only a finite interval $[a,b] \subset \mathbb{R}$. That is,

$$n(x,0) = n_0(x) > 0, \qquad m(x,0) = m_0(x) > 0, \qquad u(x,0) = u_0(x), \qquad x \in [a,b],$$

and $n_0 = m_0 = 0$ outside $[a,b]$. The viscosity coefficient ε is assumed to be a functional of the masses m and n, i.e. $\varepsilon = \varepsilon(n,m)$. More precisely, we assume that

$$\varepsilon(n,m) = D\frac{(n/m+1)(n+m)^\beta}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0, 1/3), \tag{13}$$

for a constant D, which is a natural generalization of the viscosity coefficient that was used in [9, 20] to the case where we consider the full momentum equation. We refer to [6] for more information concerning the choice of the viscosity coefficient.

We neglect external force terms (friction and gravity). We then rewrite the model (10) slightly by adding the two continuity equations and introducing the total mass ρ given by

$$\rho = n + m. \tag{14}$$

Hence, we consider the compressible gas-incompressible liquid two-phase model written in the following form:

$$\begin{aligned} \partial_t n + \partial_x[nu] &= nA \\ \partial_t \rho + \partial_x[\rho u] &= nA \\ \partial_t[\rho u] + \partial_x[\rho u^2] + \partial_x P(n,\rho) &= \partial_x[\varepsilon(n,\rho)\partial_x u], \end{aligned} \tag{15}$$

with A given by (8). Note that this system also takes the form

$$\begin{aligned} \partial_t n + \partial_x[nu] &= nA \\ \partial_t \rho + \partial_x[\rho u] &= nA, \\ u(\partial_t \rho + \partial_x[\rho u]) + \rho(\partial_t u + u\partial_x u) + \partial_x P(n,\rho) &= \partial_x[\varepsilon(n,\rho)\partial_x u], \end{aligned} \tag{16}$$

which corresponds to

$$\begin{aligned} (\partial_t n + u\partial_x n) + n\partial_x u &= nA \\ (\partial_t \rho + u\partial_x \rho) + \rho\partial_x u &= nA, \\ \rho(\partial_t u + u\partial_x u) + \partial_x P(n,\rho) &= -unA + \partial_x[\varepsilon(n,\rho)\partial_x u]. \end{aligned} \tag{17}$$

Setting the constants C and D appearing, respectively, in (12) and (13), to one, we get

$$P(n,\rho) = \Big(\frac{n}{\rho_l - m}\Big)^{\gamma} = \Big(\frac{n}{\rho_l - [\rho - n]}\Big)^{\gamma}, \qquad \gamma > 1, \tag{18}$$

$$\varepsilon(n,\rho) = \frac{(n/m+1)(n+m)^{\beta}}{(\rho_l - m)^{\beta+1}} = \frac{1}{[\rho - n]}\Big(\frac{\rho}{(\rho_l - [\rho - n])}\Big)^{\beta+1}, \qquad \beta \in (0, 1/3). \tag{19}$$

As indicated above, motivated by previous studies of the single-phase Navier-Stokes model [16, 14, 19, 17, 12], we study (15) in a free-boundary setting where the total mass ρ and gas mass n are of compact support initially and connect to the vacuum regions (where $n = \rho = 0$) discontinuously. In other words, we shall study the Cauchy problem (15) with initial data

$$(n,\rho,\rho u)(x,0) = \begin{cases} (n_0, \rho_0, \rho_0 u_0) & x \in [a,b], \\ (0,0,0) & \text{otherwise}, \end{cases}$$

where $\min_{x\in[a,b]} n_0 > 0$, $\min_{x\in[a,b]} \rho_0 > 0$, and $n_0(x), \rho_0(x)$ are in H^1. Letting $a(t)$ and $b(t)$ denote the particle paths initiating from $(a,0)$ and $(b,0)$, respectively, in the x-t coordinate system, these paths represent free boundaries, i.e., the interface of the gas-liquid mixture and the vacuum. These are determined by the equations

$$\begin{aligned} &\frac{d}{dt}a(t) = u(a(t),t), && \frac{d}{dt}b(t) = u(b(t),t), \\ &\left(-P(n,\rho) + \varepsilon(n,\rho)u_x\right)(a(t)^+, t) = 0, && \left(-P(n,\rho) + \varepsilon(n,\rho)u_x\right)(b(t)^-, t) = 0. \end{aligned} \tag{20}$$

We introduce a new set of variables (ξ, τ) by using the coordinate transformation

$$\xi = \int_{a(t)}^{x} m(y,t)\,dy, \qquad \tau = t. \tag{21}$$

Thus, ξ represents a convenient rescaling of x. In particular, the free boundaries $x = a(t)$ and $x = b(t)$, in terms of the new variables ξ and τ, take the form

$$\tilde{a}(\tau) = 0, \qquad \tilde{b}(\tau) = \int_{a(t)}^{b(t)} m(y,t)\,dy = \text{const}, \tag{22}$$

where $\int_a^b m_0(y)\,dy$ is the total liquid mass initially, which we normalize to 1. In other words, the interval $[a,b]$ in the x-t system appears as the interval $[0,1]$ in the ξ-τ system.

Remark 2.1. *Note that we avoid imposing any conditions on the well-formation term A by making use of the liquid mass in* (22)*, which indeed is a conserved mass in our system as described by the model* (10)*. The price to pay is that the resulting model takes a more complicated form as we will see below. In the work [5] we had to impose a constraint of the form $\int_{a(t)}^{b(t)} [nA](y,t)\,dy = 0$ to ensure that the total mass ρ is conserved.*

Next, we rewrite the model itself (15) in the new variables (ξ,τ). First, in view of the particle paths $X_\tau(x)$ given by

$$\frac{dX_\tau(x)}{d\tau} = u(X_\tau(x),\tau), \qquad X_0(x) = x,$$

the system (17) now takes the form

$$\begin{aligned}
\frac{dn}{d\tau} + nu_x &= nq_w[P_w - P(n,\rho)]\\
\frac{d\rho}{d\tau} + \rho u_x &= nq_w[P_w - P(n,\rho)]\\
\rho\frac{du}{d\tau} + P(n,\rho)_x &= -unq_w[P_w - P(n,\rho)] + (\varepsilon(n,\rho)u_x)_x.
\end{aligned}$$

Applying (21) to shift from (x,t) to (ξ,τ) we get

$$\begin{aligned}
n_\tau + (n[\rho - n])u_\xi &= nq_w[P_w - P(n,\rho)]\\
\rho_\tau + (\rho[\rho - n])u_\xi &= nq_w[P_w - P(n,\rho)]\\
\Big(\frac{\rho}{\rho - n}\Big)u_\tau + P(n,\rho)_\xi &= -u\Big(\frac{n}{\rho - n}\Big)q_w[P_w - P(n,\rho)] + (\varepsilon(n,\rho)[\rho - n]u_\xi)_\xi,
\end{aligned}$$

for $(\xi,\tau) \in (0,1)\times[0,\infty)$ with boundary conditions, in view of (20), given by

$$P(n,\rho) = \varepsilon(n,\rho)[\rho - n]u_\xi, \qquad \text{at } \xi = 0,1, \quad \tau \geq 0.$$

In addition, we have the initial data

$$n(\xi,0) = n_0(\xi), \quad \rho(\xi,0) = \rho_0(\xi), \quad u(\xi,0) = u_0(\xi), \qquad \xi \in [0,1].$$

In the following we replace the coordinates (ξ,τ) by (x,t) such that the model now takes the form

$$\begin{aligned}
\partial_t n + (n[\rho - n])\partial_x u &= nq_w[P_w - P(n,\rho)]\\
\partial_t \rho + (\rho[\rho - n])\partial_x u &= nq_w[P_w - P(n,\rho)]\\
g(n,\rho)\partial_t u + \partial_x P(n,\rho) &= -uh(n,\rho)q_w[P_w - P(n,\rho)] + \partial_x(E(n,\rho)\partial_x u),
\end{aligned} \tag{23}$$

for $x \in (0,1)$ where $g(n,\rho) = \frac{\rho}{\rho - n}$, $h(\rho) = \frac{n}{\rho - n}$, and $E(n,\rho) = \varepsilon(n,\rho)[\rho - n]$.

Typically, $n \ll \rho$ (if α_g is not very close to 1) since the relation between gas density ρ_g and liquid density ρ_l is of the order $\rho_l/\rho_g = O(1000)$. Hence, for many cases $h(c)$ is close to 0, and we may neglect the term $-uh(n,\rho)q_w[P_w - P(n,\rho)]$, which introduces a minor change of the mixture momentum due to the gas flow between well and formation. For the applications we have in mind where the gas volume fraction does not get too close to 1 since gas is dispersed in liquid, this approximation is indeed reasonable. In other words, we consider the following model:

$$\begin{aligned} \partial_t n + (n[\rho - n])\partial_x u &= q_w n[P_w - P(n,\rho)] \\ \partial_t \rho + (\rho[\rho - n])\partial_x u &= q_w n[P_w - P(n,\rho)] \\ g(n,\rho)\partial_t u + \partial_x P(n,\rho) &= \partial_x(E(n,\rho)\partial_x u), \qquad x \in (0,1). \end{aligned} \tag{24}$$

Here

$$P(n,\rho) = \Big(\frac{n}{\rho_l - [\rho - n]}\Big)^{\gamma}, \qquad \gamma > 1, \tag{25}$$

and

$$E(n,\rho) = \Big(\frac{\rho}{\rho_l - [\rho - n]}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{26}$$

Moreover, boundary conditions are given by

$$P(n,\rho) = E(n,\rho)u_x, \qquad \text{at } x = 0,1, \quad t \geq 0, \tag{27}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{28}$$

We observe that the model problem (24)–(28) coincides with the model (1)–(6) stated in the introduction part.

3. A global existence result. Before we state the main result for the model (24)–(28), we describe the notation we apply throughout the paper. $W^{1,2}(I) = H^1(I)$ represents the usual Sobolev space defined over $I = (0,1)$ with norm $\|\cdot\|_{W^{1,2}}$. Moreover, $L^p(K,B)$ with norm $\|\cdot\|_{L^p(K,B)}$ denotes the space of all strongly measurable, pth-power integrable functions from K to B where K typically is subset of $\mathbb{R}$ and B is a Banach space.

THEOREM 3.1 (Main Result). *Assume that* $\gamma > 1$ *and* $\beta \in (0,1/3)$ *respectively in* (25) *and* (26), *and that the initial data* (n_0, m_0, u_0) *satisfy (note that corresponding constraint on* $\rho_0 = n_0 + m_0$ *can be obtained from this)*

(i) $\inf_{[0,1]} n_0 > 0$, $\sup_{[0,1]} n_0 < \infty$, $\inf_{[0,1]} m_0 > 0$, *and* $\sup_{[0,1]} m_0 < \rho_l$*;*

(ii) $n_0, m_0 \in W^{1,2}(I)$*;*

(iii) $u_0 \in L^{2q}(I)$, *for* $q \in \mathbb{N}$.

As a consequence, the function $c_0 = \frac{n_0}{n_0 + m_0}$ *satisfies that*

$$\inf_{[0,1]} c_0 > 0, \qquad \sup_{[0,1]} c_0 < 1, \qquad c_0 \in W^{1,2}(I). \tag{29}$$

Moreover, the function $Q_0 = \frac{n_0 + m_0}{\rho_l - m_0}$ *satisfies that*

$$\inf_{[0,1]} Q_0 > 0, \qquad \sup_{[0,1]} Q_0 < \infty, \qquad Q_0 \in W^{1,2}(I). \tag{30}$$

In addition, we assume that $Q_0(x=0)$ and $c_0(x=0)$ are chosen such that

$$P_0(0) = [c_0 Q_0]^\gamma(0) > P_w, \tag{31}$$

where P_w is the reference pressure which controls whether there is inflow or efflux of gas at $x=0$ at initial time. In other words, we assume efflux of gas at $x=0$ at initial time. Then the initial-boundary problem (24)–(28) possesses a global weak solution (n,ρ,u) in the sense that for any $T>0$, the following holds:

(A) We have the estimates:

$$n,\rho \in L^\infty([0,T],W^{1,2}(I)), \qquad n_t,\rho_t \in L^2([0,T],L^2(I)),$$
$$u \in L^\infty([0,T],L^{2q}(I)) \cap L^2([0,T],H^1(I)).$$

More precisely, we have $\forall (x,t)\in[0,1]\times[0,T]$ that

$$\begin{aligned} &0<\inf_{x\in[0,1]} c(x,t), \qquad \sup_{x\in[0,1]} c(x,t)<1, \qquad c:=\frac{n}{\rho},\\ &0<\mu \inf_{x\in[0,1]}(c) \le n(x,t) \le \Big(\frac{\rho_l-\mu}{1-\sup_{x\in[0,1]}(c)}\Big)\sup_{x\in[0,1]}(c)<\infty,\\ &0<\mu\le\rho\le \frac{\rho_l-\mu}{1-\sup_{x\in[0,1]}(c)}<\infty, \end{aligned} \tag{32}$$

for a positive constant $\mu=\mu(\|c_0\|_{W^{1,2}(I)}, \|Q_0^\beta\|_{W^{1,2}(I)}, \|u_0\|_{L^{2q}(I)}, \inf_{[0,1]} c_0, \sup_{[0,1]} c_0, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, T)>0$.

(B) Moreover, the following equations hold,

(33)
$$\begin{aligned} &n_t + n[\rho-n]u_x = q_w n[P_w - P(n,\rho)],\\ &\rho_t + \rho[\rho-n]u_x = q_w n[P_w - P(n,\rho)], \end{aligned}$$

with $(n,\rho)(x,0)=(n_0(x),\rho_0(x))$, for a.e. $x\in(0,1)$ and

$$\int_0^\infty\int_0^1 \Big[ug(n,\rho)\phi_t + [P(n,\rho)-E(n,\rho)u_x]\phi_x + q_w u h(n,\rho)[P_w-P(n,\rho)]\phi\Big]dxdt + \int_0^1 u_0(x)g(n_0(x),\rho_0(x))\phi(x,0)\,dx = 0,$$

for any test function $\phi(x,t)\in C_0^\infty(D)$, with $D:=\{(x,t)\,|\,0\le x\le 1,\ t\ge 0\}$ and where $g(n,\rho)$ and $h(n,\rho)$ are defined as

$$g(n,\rho)=\frac{\rho}{\rho-n}, \qquad h(n,\rho)=\frac{n}{\rho-n}. \tag{34}$$

Note that g and h do not blow up due to the estimates in (32).

4. Estimates. Below we derive a priori estimates for (n,ρ,u) which are assumed to be a smooth solution of (24)–(28). We then construct the approximate solutions of (24) in Section 5 by mollifying the initial data n_0,ρ_0,u_0 and obtain global existence by taking the limit. More precisely, similar to [12, 9] we first assume that (n,ρ,u) is a solution of (24)–(28) on $[0,T]$ satisfying

$$\begin{aligned} &n,n_t,n_x,n_{tx},\rho,\rho_x,\rho_t,\rho_{tx},u,u_x,u_t,u_{xx}\in C^{\alpha,\alpha/2}(D_T) \qquad \text{for some } \alpha\in(0,1),\\ &n(x,t)>0, \qquad \rho(x,t)>0, \qquad [\rho-n](x,t)<\rho_l \quad \text{on } D_T=[0,1]\times[0,T]. \end{aligned} \tag{35}$$

In the following we will frequently take advantage of the fact that the model (24) can be rewritten in a form convenient for deriving various estimates. We first describe this reformulation, and then present a number of a priori estimates.

4.1. A reformulation of the model (24)**.** We introduce the variable

$$c = \frac{n}{\rho}, \tag{36}$$

and see that (24) corresponds to

$$\begin{aligned} \rho\partial_t c + c\partial_t\rho + (\rho^2 c[1-c])\partial_x u &= q_w[c\rho][P_w - P(c,\rho)] \\ \partial_t\rho + (\rho^2[1-c])\partial_x u &= q_w[c\rho][P_w - P(c,\rho)] \\ \Big(\frac{1}{1-c}\Big)\partial_t u + \partial_x P(c,\rho) &= \partial_x(E(c,\rho)\partial_x u), \qquad x \in (0,1), \end{aligned}$$

that is,

$$\begin{aligned} \partial_t c &= q_w c[1-c][P_w - P(c,\rho)] \\ \partial_t\rho + \rho^2[1-c]\partial_x u &= q_w c\rho[P_w - P(c,\rho)] \\ \Big(\frac{1}{1-c}\Big)\partial_t u + \partial_x P(c,\rho) &= \partial_x(E(c,\rho)\partial_x u), \qquad x \in (0,1), \end{aligned}$$

which, in turn can be reformulated as

$$\begin{aligned} \partial_t c &= c(1-c)A, \\ \partial_t\rho + \rho^2[1-c]\partial_x u &= c\rho A \\ g(c)\partial_t u + \partial_x P(c,\rho) &= \partial_x(E(c,\rho)\partial_x u), \end{aligned} \tag{37}$$

where

$$A = q_w[P_w - P(c,\rho)], \qquad g(c) = \frac{1}{1-c}, \tag{38}$$

and

$$P(c,\rho) = c^\gamma\Big(\frac{\rho}{\rho_l - [1-c]\rho}\Big)^\gamma, \qquad \gamma > 1, \tag{39}$$

and

$$E(c,\rho) = \Big(\frac{\rho}{\rho_l - [1-c]\rho}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{40}$$

Moreover, boundary conditions are given by

$$P(c,\rho) = E(c,\rho)u_x, \qquad \text{at } x = 0,1, \quad t \geq 0, \tag{41}$$

whereas initial data are

$$c(x,0) = c_0(x), \quad \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{42}$$

Corollary 4.1. *Under the assumptions of Theorem 3.1, it follows that for* $t \in [0,T]$ *for a given time* $T > 0$

$$0 \leq \inf_{x\in[0,1]} c(x,t), \qquad \sup_{x\in[0,1]} c(x,t) < 1. \tag{43}$$

Consequently, we have that

$$1 \le \inf_{x\in[0,1]} g(c) \le \sup_{x\in[0,1]} g(c) < \infty, \qquad 0 \le \inf_{x\in[0,1]} h(c) \le \sup_{x\in[0,1]} h(c) < \infty, \tag{44}$$

for $g(c) = \frac{1}{1-c}$ *and* $h(c) = \frac{c}{1-c}$.

Proof. Note that from (37) we have

$$c_t = c(1-c)A(x,t),$$

which corresponds to

$$\frac{1}{c(1-c)} c_t = A(x,t), \qquad c \in (0,1),$$

i.e.

$$G(c)_t = A(x,t), \qquad G(c) = \log\Big(\frac{c}{1-c}\Big).$$

This implies that

$$\frac{c(x,t)}{1-c(x,t)} = \frac{c_0(x)}{1-c_0(x)} \exp\Big(\int_0^t A(x,s)\,ds\Big).$$

Note also that the inverse of $h(c) = c/(1-c)$ is $h^{-1}(d) = d/(1+d)$, such that $h^{-1} : [0,\infty) \to [0,1)$ and is one-to-one. Consequently,

$$c(x,t) = h^{-1}\Big(\frac{c_0(x)}{1-c_0(x)} \exp\Big(\int_0^t A(x,s)\,ds\Big)\Big). \tag{45}$$

Clearly, for $A = q_w[P_w - P(c,\rho)]$ we have that

$$A \le q_w P_w,$$

since $P \ge 0$. From the assumptions on n_0, m_0 given in Theorem 3.1, it follows that

$$0 < \inf_{[0,1]} c_0(x), \qquad \sup_{[0,1]} c_0(x) < 1. \tag{46}$$

Hence, in view of (45) it follows that $\sup_{x\in[0,1]} c(x,t) < 1$. However, since we have no upper limit on $P(c,\rho)$, A can become an arbitrary large negative number which implies, in view of (45), that there is no positive lower limit for c. We can only conclude that and $0 \le \inf_{x\in[0,1]} c(x,t)$. The estimates (44) follows directly from (43). □

Remark 4.1. *Note that the consequence of* (43) *is that for a finite time* $T > 0$, *no pure gas regions* ($m = 0$) *will appear since* $\sup_{x\in[0,1]} c < 1$, *although gas will enter the well as long as well pressure* P *is lower than critical pressure* P_w. *However, at this stage we cannot conclude anything about the possibility for getting pure liquid zones* ($n = 0$ *corresponding to* $c = 0$) *due to flow of gas from well into the surrounding formation, which takes place when well pressure* P *is higher than the critical pressure* P_w.

In order to obtain the a priori estimates, it will be convenient to introduce a new reformulation of the model (37)–(42). This reformulation allows us to deal with the potential singular behavior associated with the pressure law (39) and viscosity coefficient (40). A similar approach was employed in [9, 10, 20]. However, compared to those works we now also have to take into account additional terms due to the dynamic well-formation interaction and the fact that a full momentum equation is used in the model. For that purpose, we introduce the variable

$$Q(\rho,k)=\frac{\rho}{\rho_l-k\rho},\qquad k=1-c, \tag{47}$$

and observe that

$$\rho=\frac{\rho_l Q}{1+kQ},\qquad \frac{1}{\rho}=\frac{1}{\rho_l Q}+\frac{k}{\rho_l}. \tag{48}$$

Consequently, we get

$$\begin{aligned}
Q(\rho,k)_t &= Q_\rho\rho_t+Q_k k_t\\
&=\Big(\frac{1}{\rho_l-k\rho}+\frac{\rho k}{(\rho_l-k\rho)^2}\Big)\rho_t+\frac{\rho^2}{(\rho_l-k\rho)^2}k_t\\
&=\frac{\rho_l}{(\rho_l-k\rho)^2}\rho_t+\frac{\rho^2}{(\rho_l-k\rho)^2}k_t\\
&=\frac{\rho_l}{(\rho_l-k\rho)^2}[c\rho A-(1-c)\rho^2u_x]+\frac{\rho^2}{(\rho_l-k\rho)^2}k_t \qquad \text{(using second equation of (37))}\\
&=\frac{\rho_l c\rho A}{(\rho_l-k\rho)^2}-\frac{\rho_l(1-c)\rho^2}{(\rho_l-k\rho)^2}u_x+Q^2k_t\\
&=\frac{\rho_l c\rho^2 A}{\rho(\rho_l-k\rho)^2}-\rho_l(1-c)Q^2u_x-Q^2c_t\\
&=\rho_l cA\Big(\frac{1}{\rho_l Q}+\frac{k}{\rho_l}\Big)Q^2-\rho_l(1-c)Q^2u_x-Q^2ckA \qquad \text{(using (48) and first equation of (37))}\\
&=cA\Big(Q+kQ^2\Big)-\rho_l(1-c)Q^2u_x-Q^2ckA\\
&=cAQ+cAkQ^2-\rho_l(1-c)Q^2u_x-Q^2ckA\\
&=cAQ-\rho_l(1-c)Q^2u_x.
\end{aligned}$$

Thus, we may rewrite the model (37) in the following form

$$\begin{aligned}
&\partial_t c=c(1-c)A,\qquad A=q_w[P_w-P(cQ)],\\
&\partial_t Q+\rho_l(1-c)Q^2u_x=cAQ\\
&g(c)\partial_t u+\partial_x P(cQ)=\partial_x(E(Q)\partial_x u),
\end{aligned} \tag{49}$$

with

$$P(cQ)=c^\gamma Q^\gamma,\qquad \gamma>1, \tag{50}$$

and

$$E(Q)=Q^{\beta+1},\qquad 0<\beta<1/3. \tag{51}$$

This model is then subject to the boundary conditions

$$P(cQ)=E(Q)u_x,\qquad \text{at } x=0,1,\quad t\ge 0. \tag{52}$$

In addition, we have the initial data

$$c(x,0)=c_0(x),\quad Q(x,0)=Q_0(x),\quad u(x,0)=u_0(x),\qquad x=[0,1]. \tag{53}$$

4.2. A priori estimates. Now we derive a priori estimates for (c,Q,u) by making use of the reformulated model (49)–(53).

LEMMA 4.2 (Energy estimate). *We have the basic energy estimate*

$$\begin{aligned}&\int_0^1\Big(\frac{g(c)}{2}u^2+\frac{h(c)[cQ]^{\gamma-1}}{\rho_l(\gamma-1)}\Big)\,dx+\frac{q_w}{2}\int_0^t\int_0^1 u^2h(c)[cQ]^\gamma\,dx\,ds\\&\qquad+\frac{q_w\gamma}{\rho_l(\gamma-1)}\int_0^t\int_0^1 h(c)[cQ]^{2\gamma-1}\,dx\,ds+\int_0^t\int_0^1 Q^{\beta+1}(u_x)^2\,dx\,ds\le C_1,\end{aligned}\tag{54}$$

where $C_1=C_1(\sup_{[0,1]}Q_0,\|u_0\|_{L^2(I)},\|c_0\|_{L^\gamma(I)})$. *Moreover,*

$$Q(x,t)\le C_2,\qquad \forall(x,t)\in[0,1]\times[0,T],\tag{55}$$

where $C_2=C_2(\sup_{[0,1]}Q_0,\|u_0\|_{L^2(I)},\|c_0\|_{L^\gamma(I)},T)$. *Moreover, for any positive integer* q*,*

$$\int_0^1 u^{2q}(x,t)\,dx+q(2q-1)\int_0^t\int_0^1 u^{2q-2}Q^{1+\beta}(u_x)^2\,dx\,ds\le C_3,\tag{56}$$

where $C_3=C_3(\|u_0\|_{L^{2q}(I)},T,q,C_2)$.

Proof. We consider the proof in three steps.

Estimate (54)**:** We multiply the third equation of (49) by u and integrate over $[0,1]$ in space. Applying the boundary condition (52) and the fact that the first equation of (49) is equivalent to

$$g(c)_t=h(c)A,\tag{57}$$

we get

$$\int_0^1\Big(\frac{g(c)}{2}u^2\Big)_t\,dx-\int_0^1\frac{1}{2}u^2h(c)A\,dx-\int_0^1 P(cQ)u_x\,dx=-\int_0^1 E(Q)(u_x)^2\,dx\tag{58}$$

Moreover, from the second equation of (49) we get

$$\frac{g(c)c^\gamma}{\rho_l(\gamma-1)}(Q^{\gamma-1})_t+c^\gamma Q^\gamma u_x=\frac{1}{\rho_l}h(c)c^\gamma Q^{\gamma-1}A,\tag{59}$$

by multiplying with $\frac{1}{\rho_l(1-c)}c^\gamma Q^{\gamma-2}$. This equation also corresponds to

$$\frac{1}{\rho_l(\gamma-1)}(g(c)c^\gamma Q^{\gamma-1})_t-\frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}(g(c)c^\gamma)_t+c^\gamma Q^\gamma u_x=\frac{1}{\rho_l}h(c)c^\gamma Q^{\gamma-1}A,\tag{60}$$

which in turn can be rewritten as

$$\frac{1}{\rho_l(\gamma-1)}(g(c)c^\gamma Q^{\gamma-1})_t+P(cQ)u_x=\frac{\gamma}{\rho_l(\gamma-1)}(g(c)c^\gamma Q^{\gamma-1})A,\tag{61}$$

where we have used the first equation of (49) as well as (57). Integrating (61) over $[0,1]$ and combining it with (58), we get

$$\begin{aligned}&\frac{d}{dt}\int_0^1\Big(\frac{g(c)}{2}u^2+\frac{g(c)c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\Big)\,dx-\frac{q_w}{2}\int_0^1 u^2h(c)[P_w-P(cQ)]\,dx\\&\qquad+\int_0^1 E(Q)(u_x)^2\,dx=\frac{q_w\gamma}{\rho_l(\gamma-1)}\int_0^1 g(c)c^\gamma Q^{\gamma-1}[P_w-P(cQ)]\,dx.\end{aligned}$$

In other words, we obtain the following integral equality

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \Big(\frac{g(c)}{2}u^2 + \frac{g(c)c^\gamma Q^{\gamma-1}}{\rho_l(\gamma-1)}\Big)\,dx + \frac{q_w}{2}\int_0^1 u^2 h(c)P(cQ)\,dx \\
&\qquad + \frac{q_w\gamma}{\rho_l(\gamma-1)}\int_0^1 g(c)c^\gamma Q^{\gamma-1}P(cQ)\,dx + \int_0^1 E(Q)(u_x)^2\,dx \\
&\qquad = \frac{q_w P_w}{2}\int_0^1 u^2 h(c)\,dx + \frac{q_w P_w\gamma}{\rho_l(\gamma-1)}\int_0^1 g(c)c^\gamma Q^{\gamma-1}\,dx.
\end{aligned}
$$

Using that $\sup c < 1$, $1 \le g(c) < \infty$, and $0 \le h(c) < \infty$, in view of Corollary 4.1, application of Gronwall's inequality, respectively, for the term $\int_0^1 u^2 h(c)\,dx \le \int_0^1 g(c)u^2\,dx$ and $\int_0^1 g(c)c^\gamma Q^{\gamma-1}\,dx$ appearing on the right hand side, gives (54).

Estimate (55)**:** In order to obtain a pointwise upper bound for Q we will need the boundedness of the (new) higher order terms $\iint u^2 h(c)[cQ]^\gamma dxds$ and $\iint h(c)[cQ]^{2\gamma-1}dxds$ obtained from (54). From the second equation of (49) we deduce the equation

$$
\frac{g(c)}{\rho_l}(Q^\beta)_t + \beta Q^{\beta+1}u_x = \frac{\beta}{\rho_l}h(c)Q^\beta A. \tag{62}
$$

In view of (57) this corresponds to

$$
(g(c)Q^\beta)_t + \beta\rho_l Q^{\beta+1}u_x = (\beta+1)h(c)Q^\beta A. \tag{63}
$$

Integrating over $[0,t]$, we get

$$
g(c)Q^\beta(x,t) = g(c_0)Q^\beta(x,0) - \beta\rho_l\int_0^t Q^{\beta+1}u_x\,ds + (\beta+1)\int_0^t h(c)Q^\beta A\,ds. \tag{64}
$$

Then, we integrate the third equation of (49) over $[0,x]$ and get

$$
\int_0^x g(c)u_t(y,t)\,dy + P(cQ) - P(cQ(0,t)) + (E(Q)u_x)(0,t) = E(Q)u_x = Q^{\beta+1}u_x.
$$

Using the boundary condition (52) and inserting the above relation into the right hand side of (64), we get after an application of (57)

$$
\begin{aligned}
&g(c)Q^\beta(x,t) - g(c)Q^\beta(x,0) \\
&= -\beta\rho_l\int_0^t\Big(\int_0^x g(c)u_t(y,t)\,dy + P(cQ)\Big)\,ds + (\beta+1)\int_0^t h(c)Q^\beta A\,ds \\
&= -\beta\rho_l\int_0^x (g(c)u(y,t) - g(c_0)u_0(y))\,dy + \beta\rho_l\int_0^t\int_0^x uh(c)A\,dy\,ds \\
&\qquad\qquad - \beta\rho_l\int_0^t P(cQ)\,ds + (\beta+1)\int_0^t h(c)Q^\beta A\,ds.
\end{aligned} \tag{65}
$$

Consequently, since $P(cQ) \geq 0$ and using that $A = q_w[P_w - P(cQ)]$, we get

$$\begin{aligned}
&g(c)Q^\beta(x,t) \\
&\leq g(c_0)Q_0^\beta(x) + \beta\rho_l \int_0^1 |g(c)u(y,t)|\,dy + \beta\rho_l \int_0^1 |g(c_0)u_0(y)|\,dy \\
&\quad + \beta\rho_l q_w \int_0^t \int_0^x uh(c)[P_w - P(cQ)]\,dy\,ds + (\beta+1)q_w \int_0^t h(c)Q^\beta[P_w - P(cQ)]\,ds \\
&\leq g(c_0)Q_0^\beta(x) + \beta\rho_l \sup_{x\in[0,1]} g(c) \int_0^1 |u(y,t)|\,dy + \beta\rho_l \sup_{x\in[0,1]} g(c_0) \int_0^1 |u_0(y)|\,dy \\
&\quad + \beta\rho_l q_w P_w \sup_{x\in[0,1]} h(c) \int_0^t \int_0^x |u|\,dy\,ds + \beta\rho_l q_w \int_0^t \int_0^x |u|h(c)P(cQ)\,dy\,ds \\
&\qquad\qquad + (\beta+1)q_w P_w \sup_{x\in[0,1]}(c) \int_0^t g(c)Q^\beta\,ds.
\end{aligned}$$

Applying Hölder's inequality and (54) as well as assumptions on initial data u_0 we can bound $\int_0^1 |u|\,dy$ and $\int_0^1 |u_0|\,dy$. Moreover, the term $\int_0^t g(c)Q^\beta\,ds$ can be controlled by means of Gronwall's inequality.

Consequently, the upper bound (55) then follows if we can show that $\int_0^t \int_0^x |u|h(c)P(cQ)\,dy\,ds$ is bounded. For that purpose we introduce the splitting $|u|h(c)P(cQ) = |u|h(c)^{1/2}[cQ]^{\gamma/2}\cdot h(c)^{1/2}[cQ]^{\gamma/2}$ in combination with Young's inequality:

$$\begin{aligned}
&\int_0^t \int_0^1 |u|h(c)P(cQ)\,dx\,ds \\
&\leq \frac{1}{2}\int_0^t \int_0^1 |u|^2 h(c)[cQ]^\gamma\,dx\,ds + \frac{1}{2}\int_0^t \int_0^1 h(c)[cQ]^\gamma\,dx\,ds \\
&\leq C_1 + \frac{1}{2}\int_0^t \int_0^1 h(c)[cQ]^\gamma\,dx\,ds,
\end{aligned} \tag{66}$$

where we have used (54). To estimate the last term we see that

$$h(c)[cQ]^\gamma = h(c)^{1/p}[cQ]^\gamma \cdot h(c)^{1-1/p}, \qquad p > 1.$$

By choosing $p = \frac{2\gamma-1}{\gamma} = 2 - \frac{1}{\gamma} > 1$, that is, $q = \frac{2\gamma-1}{\gamma-1}$ (such that $\frac{1}{p} + \frac{1}{q} = 1$) we see that application of Young's inequality allows us to estimate as follows:

$$\begin{aligned}
\int_0^t \int_0^1 h(c)[cQ]^\gamma\,dx\,ds &\leq \frac{1}{p}\int_0^t \int_0^1 h(c)[cQ]^{\gamma p}\,dx\,ds + \frac{1}{q}\int_0^t \int_0^1 h(c)\,dx\,ds \\
&= \frac{\gamma}{2\gamma-1}\int_0^t \int_0^1 h(c)[cQ]^{2\gamma-1}\,dx\,ds + \frac{\gamma-1}{2\gamma-1}\int_0^t \int_0^1 h(c)\,dx\,ds \leq C,
\end{aligned}$$

in view of (54) and Corollary 4.1 and for an appropriate choice of C. Thus, the estimate (55) has been proved. In particular, the following estimate holds:

$$|A| \leq q_w(P_w + C_2^\gamma) := M. \tag{67}$$

Estimate (56)**:** Multiplying the third equation of (49) by $2qu^{2q-1}$, integrating over

$[0,1]\times[0,t]$ and integration by parts together with application of the boundary conditions (52) and the equation (57), we get

(68)
$$\int_0^1 g(c)u^{2q}\,dx + 2q(2q-1)\int_0^t\int_0^1 Q^{\beta+1}(u_x)^2u^{2q-2}\,dx\,ds$$
$$= \int_0^1 g(c_0)u_0^{2q}\,dx + 2q(2q-1)\int_0^t\int_0^1 [cQ]^\gamma u^{2q-2}u_x\,dx\,ds + \int_0^t\int_0^1 h(c)u^{2q}A\,dx\,ds.$$

For the second term on the right hand side of (68) we apply Cauchy's inequality with ε,

$$ab \le (1/4\varepsilon)a^2 + \varepsilon b^2, \tag{69}$$

and get

$$\int_0^t\int_0^1 [cQ]^\gamma u^{2q-2}u_x\,dx\,ds$$
$$\le \frac{1}{4\varepsilon}\int_0^t\int_0^1 c^{2\gamma}Q^{2\gamma-\beta-1}u^{2q-2}\,dx\,ds + \varepsilon\int_0^t\int_0^1 Q^{\beta+1}u^{2q-2}(u_x)^2\,dx\,ds$$
$$\le \frac{1}{4\varepsilon}\sup_{x\in[0,1]}(c^{2\gamma})\int_0^t\int_0^1 Q^{2\gamma-\beta-1}u^{2q-2}\,dx\,ds + \varepsilon\int_0^t\int_0^1 Q^{\beta+1}u^{2q-2}(u_x)^2\,dx\,ds.$$

The last term clearly can be absorbed in the second term of the left-hand side of (68) by the choice $\varepsilon = 1/2$. Finally, let us see how we can bound the term $\int_0^t\int_0^1 u^{2q-2}Q^{2\gamma-1-\beta}\,dx\,ds$. In view of Young's inequality $ab \le (1/p)a^p + (1/r)b^r$ where $1/p + 1/r = 1$, we get for the choice $p = q$ and $r = q/(q-1)$

$$\int_0^t\int_0^1 u^{2q-2}Q^{2\gamma-1-\beta}\,dx\,ds \le \frac{1}{q}\int_0^t\int_0^1 Q^{(2\gamma-1-\beta)q}\,dx\,ds + \frac{q-1}{q}\int_0^t\int_0^1 u^{2q}\,dx\,ds$$
$$\le \frac{C_2^{(2\gamma-1-\beta)q}}{q}t + \frac{q-1}{q}\int_0^t\int_0^1 u^{2q}\,dx\,ds,$$

by using (55). To sum up, we get

(70)
$$\int_0^1 g(c)u^{2q}\,dx + q(2q-1)\int_0^t\int_0^1 Q^{\beta+1}(u_x)^2u^{2q-2}\,dx\,ds - \int_0^1 g(c_0)u_0^{2q}\,dx$$
$$\le 2q(2q-1)\frac{1}{4\varepsilon}\Big[\frac{C_2^{(2\gamma-1-\beta)q}}{q}t + \frac{q-1}{q}\int_0^t\int_0^1 u^{2q}\,dx\,ds\Big] + M\int_0^t\int_0^1 g(c)u^{2q}\,dx\,ds$$
$$= (2q-1)\Big[C_2^{(2\gamma-1-\beta)q}t + (q-1)\int_0^t\int_0^1 u^{2q}\,dx\,ds\Big] + M\int_0^t\int_0^1 g(c)u^{2q}\,dx\,ds,$$

where we have used (67) and $c \le 1$. In view of Corollary 4.1, application of Gronwall's inequality then allows us to handle the term $\int_0^t\int_0^1 u^{2q}\,dx\,ds$ appearing twice on the right hand side of (70). Here we also use that $\frac{1}{g(c)} \le 1$ and $\sup g(c) < \infty$. Hence, the estimate (56) follows. □

Remark 4.2. *As a consequence of estimate* (55)*, we can conclude that* $|A| \leq M$ *as described by* (67)*. A revisit of Corollary 4.1, see* (45) *and* (29)*, then implies that*

$$0 < \inf_{x\in[0,1]} c(x,t), \tag{71}$$

as stated in Theorem 3.1, Part (A). In other words, when the well initially is filled with a mixture of gas and liquid as described by the assumptions of Theorem 3.1, there exists no points in the well where all the gas will disappear in finite time, despite the fact that the gas is "free" to flow into the surrounding formation as long as the well pressure P *is higher than the critical pressure* P_w*.*

The next lemma represents a first step toward an estimate of $c(x,t)$ in $W^{1,2}(I)$.

LEMMA 4.3. *We have the estimate*

$$\int_0^1 (c_x)^2\,dx \leq \int_0^1 (c_{0,x})^2\,dx + C_4 \int_0^t \int_0^1 \Big[(c_x)^2 + (Q^\beta)_x^2\Big]\,dx\,ds, \tag{72}$$

for a constant $C_4 = C_4(C_2, T)$*.*

Proof. We set $w = c_x$ and differentiate the first equation of (49) with respect to x which yields

$$\begin{aligned} w_t &= w(1-c)A - cwA + c(1-c)A_x \\ &= w(1-2c)q_w[P_w - P(cQ)] - c(1-c)q_w\gamma(cQ)^{\gamma-1}[wQ + cQ_x] \\ &= \Big((1-2c)q_w[P_w - P(cQ)] - (1-c)q_w\gamma(cQ)^\gamma\Big)w - c(1-c)c^\gamma Q^{\gamma-\beta}\frac{q_w\gamma}{\beta}(Q^\beta)_x \\ &= C(c,Q)w + D(c,Q)(Q^\beta)_x, \end{aligned}$$

for appropriate choices of the constants C and D and where we have used the fact that $\frac{1}{\beta}Q^{1-\beta}(Q^\beta)_x = Q_x$. Hence, multiplying by w and integrating over $[0,1]$ we get

$$\int_0^1 (\frac{1}{2}w^2)_t\,dx = \int_0^1 Cw^2\,dx + \int_0^1 Dw(Q^\beta)_x\,dx. \tag{73}$$

Clearly, in view of the pointwise upper bound on Q given by (55) and the bound on c from Corollary 4.1, we see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 w^2\,dx &\leq \sup_{x\in[0,1]}|C|\int_0^1 w^2\,dx + \frac{1}{2}\sup_{x\in[0,1]}|D|\int_0^1 w^2\,dx + \frac{1}{2}\sup_{x\in[0,1]}|D|\int_0^1 (Q^\beta)_x^2\,dx \\ &\leq \frac{C_4}{2}\int_0^1 [w^2 + (Q^\beta)_x^2]\,dx. \end{aligned}$$

where we have used Cauchy's inequality and an appropriate choice of the constant C_4. □

In the following lemma, whose proof is along the line of previous works [9, 20], the estimate of $(Q^\beta)_x$ and c_x in $L^2(I)$ is coupled together by exploiting the result of Lemma 4.3.

LEMMA 4.4. *We have the estimate*

$$\int_0^1 \Big[(c_x)^2 + (Q^\beta)_x^2\Big]\,dx \leq C_5, \tag{74}$$

for a constant $C_5 = C_5(\|Q_0^\beta\|_{W^{1,2}(I)}, \|c_0\|_{W^{1,2}(I)}, \|u_0\|_{L^2(I)}, \|u_0\|_{L^4(I)}, C_1, C_2, C_4, T)$*.*

Proof. From (63) we get

$$(g(c)Q^\beta)_t + \beta\rho_l Q^{\beta+1}u_x = (\beta+1)h(c)Q^\beta A. \tag{75}$$

Using (75) in the third equation of (49) and integrating in time over $[0,t]$ we arrive at

$$\begin{aligned}\int_0^t g(c)u_t\,ds + \int_0^t P(cQ)_x\,ds &= \int_0^t (E(Q)u_x)_x\,ds \\ &= \frac{1}{\beta\rho_l}\int_0^t \Big((\beta+1)h(c)Q^\beta A - (g(c)Q^\beta)_t\Big)_x\,ds.\end{aligned} \tag{76}$$

This corresponds to

$$\begin{aligned}&\beta\rho_l[g(c)u - g(c_0)u_0] - \beta\rho_l\int_0^t uh(c)A\,ds + \beta\rho_l\int_0^t P(cQ)_x\,ds \\ &\qquad = \int_0^t \Big((\beta+1)h(c)Q^\beta A\Big)_x\,ds - (g(c)Q^\beta)_x + (g(c_0)Q_0^\beta)_x,\end{aligned} \tag{77}$$

where we have used (57). Dividing on $g(c)\geq 1$ we arrive at

$$\begin{aligned}&\beta\rho_l[u - \frac{g(c_0)}{g(c)}u_0] - \frac{\beta\rho_l}{g(c)}\int_0^t uh(c)A\,ds + \frac{\beta\rho_l}{g(c)}\int_0^t P(cQ)_x\,ds \\ &= \frac{(\beta+1)}{g(c)}\int_0^t \Big(h(c)Q^\beta A\Big)_x\,ds - \frac{g(c)_x}{g(c)}Q^\beta - (Q^\beta)_x + \frac{g(c_0)_x}{g(c)}Q_0^\beta + \frac{g(c_0)}{g(c)}(Q_0^\beta)_x.\end{aligned} \tag{78}$$

That is,

$$\begin{aligned}(Q^\beta)_x &= -\frac{g(c)_x}{g(c)}Q^\beta + \frac{g(c_0)_x}{g(c)}Q_0^\beta + \frac{g(c_0)}{g(c)}(Q_0^\beta)_x \\ &\quad - \beta\rho_l\Big[u - \frac{g(c_0)}{g(c)}u_0\Big] + \frac{\beta\rho_l q_w}{g(c)}\int_0^t uh(c)[P_w - P(cQ)]\,ds - \frac{\beta\rho_l}{g(c)}\int_0^t P(cQ)_x\,ds \\ &\quad + \frac{q_w(\beta+1)}{g(c)}\int_0^t \Big(h(c)Q^\beta[P_w - P(cQ)]\Big)_x\,ds.\end{aligned} \tag{79}$$

Multiplying (79) by $(Q^\beta)_x$ and integrating over $[0,1]$ in x, we get

$$
\begin{aligned}
&\int_0^1 (Q^\beta)_x^2\,dx \\
&= -\int_0^1 (g(c)c_x Q^\beta)(Q^\beta)_x\,dx + \int_0^1 \Big(\frac{g(c_0)^2 c_{0,x}}{g(c)} Q_0^\beta\Big)(Q^\beta)_x\,dx + \int_0^1 \Big(\frac{g(c_0)}{g(c)}(Q_0^\beta)_x\Big)(Q^\beta)_x\,dx \\
&- \beta\rho_l \int_0^1 (Q^\beta)_x \Big[(u - \frac{g(c_0)}{g(c)}u_0) + \frac{1}{g(c)}\int_0^t P(cQ)_x\,ds - \frac{q_w}{g(c)}\int_0^t uh(c)[P_w - P(cQ)]\,ds\Big]\,dx \\
&\qquad\qquad + \frac{q_w(\beta+1)}{g(c)}\int_0^1 (Q^\beta)_x \Big[\int_0^t \Big(h(c)Q^\beta[P_w - P(cQ)]\Big)_x\,ds\Big]\,dx \\
&\le \Big(\int_0^1 (Q^\beta)_x^2\,dx\Big)^{1/2}\Big(\|g(c)Q^\beta c_x\|_{L^2(I)} + \Big\|\frac{g(c_0)^2 c_{0,x}}{g(c)}Q_0^\beta\Big\|_{L^2(I)} + \Big\|\frac{g(c_0)}{g(c)}(Q_0^\beta)_x\Big\|_{L^2(I)} \\
&\qquad + \beta\rho_l\Big\|u - \frac{g(c_0)}{g(c)}u_0\Big\|_{L^2(I)} + \beta\rho_l\Big\|\frac{1}{g(c)}\int_0^t P(cQ)_x\,ds\Big\|_{L^2(I)} \\
&\qquad + \beta\rho_l q_w\Big\|\frac{1}{g(c)}\int_0^t uh(c)[P_w - P(cQ)]\,ds\Big\|_{L^2(I)} \\
&\qquad + (\beta+1)q_w\Big\|\frac{1}{g(c)}\int_0^t \Big(h(c)Q^\beta[P_w - P(cQ)]\Big)_x\,ds\Big\|_{L^2(I)}\Big) := ab,
\end{aligned}
\tag{80}
$$

where we have used Hölder's inequality and $g'(c) = g(c)^2$. Cauchy's inequality $ab \le a^2/2 + b^2/2$ then gives

$$
\begin{aligned}
&\int_0^1 (Q^\beta)_x^2\,dx \le \frac{1}{2}\int_0^1 (Q^\beta)_x^2\,dx \\
&+ \frac{1}{2}\Big(\|g(c)Q^\beta c_x\|_{L^2(I)} + \Big\|\frac{g(c_0)^2 c_{0,x}}{g(c)}Q_0^\beta\Big\|_{L^2(I)} + \Big\|\frac{g(c_0)}{g(c)}(Q_0^\beta)_x\Big\|_{L^2(I)} \\
&\qquad + \beta\rho_l\Big\|u - \frac{g(c_0)}{g(c)}u_0\Big\|_{L^2(I)} + \beta\rho_l\Big\|\frac{1}{g(c)}\int_0^t P(cQ)_x\,ds\Big\|_{L^2(I)} \\
&\qquad + \beta\rho_l q_w\Big\|\frac{1}{g(c)}\int_0^t uh(c)[P_w - P(cQ)]\,ds\Big\|_{L^2(I)} \\
&\qquad + (\beta+1)q_w\Big\|\frac{1}{g(c)}\int_0^t \Big(h(c)Q^\beta[P_w - P(cQ)]\Big)_x\,ds\Big\|_{L^2(I)}\Big)^2.
\end{aligned}
\tag{81}
$$

The following estimates can be obtained where the constants A_i, $i = 0, \dots, 9$ only depend on the constants C_1, C_2, C_3, C_4, T, and initial data :

$$\|g(c)Q^\beta c_x\|^2_{L^2(I)} \le A_0 + A_1 \int_0^t \int_0^1 \Big[(c_x)^2 + (Q^\beta)^2_x\Big]\, dx\, ds, \tag{82}$$

$$\Big\| \frac{g(c_0)^2 c_{0,x}}{g(c)} Q_0^\beta \Big\|^2_{L^2(I)} \le A_2, \tag{83}$$

$$\Big\| \frac{g(c_0)}{g(c)} (Q_0^\beta)_x \Big\|^2_{L^2(I)} \le A_3, \tag{84}$$

$$\Big\| u - \frac{g(c_0)}{g(c)} u_0 \Big\|^2_{L^2(I)} \le A_4, \tag{85}$$

$$\Big\| \frac{1}{g(c)} \int_0^t P(cQ)_x\, ds \Big\|^2_{L^2(I)} \le A_5 \int_0^t \int_0^1 (Q^\beta)^2_x\, dx\, ds + A_6 \int_0^t \int_0^1 (c_x)^2\, dx\, ds, \tag{86}$$

$$\Big\| \frac{1}{g(c)} \int_0^t u h(c)[P_w - P(cQ)]\, ds \Big\|^2_{L^2(I)} \le A_7, \tag{87}$$

$$\begin{aligned} &\Big\| \frac{1}{g(c)} \int_0^t \Big(h(c) Q^\beta [P_w - P(cQ)] \Big)_x\, ds \Big\|^2_{L^2(I)} \\ &\qquad \le A_8 \int_0^t \int_0^1 (Q^\beta)^2_x\, dx\, ds + A_9 \int_0^t \int_0^1 (c_x)^2\, dx\, ds. \end{aligned} \tag{88}$$

For estimate (82) we have used (72) of Lemma 4.3 together with estimate (55) of Lemma 4.2 and Corollary 4.1. Estimates (83) and (84) follow from Corollary 4.1 and assumptions on initial data c_0 and Q_0. Moreover, estimate (85) is obtained by application of (54) of Lemma 4.2. Similarly, estimate (87) follows by first using Hölder's inequality, followed by application of Cauchy's inequality

$$(uh(c)[P_w - P(cQ)])^2 \le \frac{1}{2} u^4 + \frac{1}{2} h(c)^4 [P_w - P(cQ)]^4,$$

in combination with estimates (56) and (55) of Lemma 4.2, as well as the pointwise upper bound on $h(c)$. Estimate (86) is obtained as follows:

$$\begin{aligned} \int_0^t \int_0^1 P(cQ)^2_x\, dx\, ds &= \int_0^t \int_0^1 \Big(Q^\gamma (c^\gamma)_x + c^\gamma (Q^\gamma)_x \Big)^2\, dx\, ds \\ &\le 2 \Big(\int_0^t \int_0^1 Q^{2\gamma} (c^\gamma)^2_x\, dx\, ds + \int_0^t \int_0^1 c^{2\gamma} (Q^\gamma)^2_x\, dx\, ds \Big) \\ &\le 2 (\sup_{x\in[0,1]} Q)^{2\gamma} \int_0^t \int_0^1 (c^\gamma)^2_x\, dx\, ds + 2 (\sup_{x\in[0,1]} c)^{2\gamma} \int_0^t \int_0^1 (Q^\gamma)^2_x\, dx\, ds \\ &\le 2 C_2^{2\gamma} \int_0^t \int_0^1 (c^\gamma)^2_x\, dx\, ds + 2 \int_0^t \int_0^1 (Q^\gamma)^2_x\, dx\, ds, \end{aligned} \tag{89}$$

in view of estimate (55) and Corollary 4.1. Moreover,

$$\begin{aligned} \int_0^t \int_0^1 (Q^\gamma)^2_x\, dx\, ds &= \Big(\frac{\gamma}{\beta}\Big)^2 \int_0^t \int_0^1 Q^{2(\gamma-\beta)} (Q^\beta)^2_x\, dx\, ds \\ &\le \Big(\frac{\gamma}{\beta}\Big)^2 C_2^{2(\gamma-\beta)} \int_0^t \int_0^1 (Q^\beta)^2_x\, dx\, ds \end{aligned} \tag{90}$$

and

$$\int_0^t \int_0^1 (c^\gamma)_x^2 \, dx \, ds = \gamma^2 \int_0^t \int_0^1 c^{2(\gamma-1)} (c_x)^2 \, dx \, ds \le \gamma^2 \int_0^t \int_0^1 (c_x)^2 \, dx \, ds, \tag{91}$$

in light of Corollary 4.1 and Lemma 4.2. Thus, (89)–(91) implies estimate (86). Furthermore, as a consequence of the well-reservoir interaction we must also estimate the following term

$$\begin{aligned} &\int_0^t \int_0^1 \Big(h(c) Q^\beta [P_w - P(cQ)] \Big)_x^2 \, dx \, ds \\ &= \int_0^t \int_0^1 \Big[g(c)^2 c_x Q^\beta [P_w - P(cQ)] + h(c)[P_w - P(cQ)](Q^\beta)_x - h(c) Q^\beta P(cQ)_x \Big]^2 \, dx \, ds \\ &\le B_1 \int_0^t \int_0^1 (c_x)^2 \, dx \, ds + B_2 \int_0^t \int_0^1 (Q^\beta)_x^2 \, dx \, ds + B_3 \int_0^t \int_0^1 P(cQ)_x^2 \, dx \, ds, \end{aligned} \tag{92}$$

where we have used that $h'(c) = g(c)^2$, Corollary 4.1, and Lemma 4.2 and the constants B_1, B_2, B_3 have been chosen in a suitable manner. Now, estimate (88) follows from (92) and (89)–(91). Combining (81) with estimates (82)–(88), we get

$$\frac{1}{2} \int_0^1 (Q^\beta)_x^2 \, dx \le C + D \int_0^t \int_0^1 (Q^\beta)_x^2 \, dx \, ds + E \int_0^t \int_0^1 (c_x)^2 \, dx \, ds.$$

Adding $\frac{1}{2} \int_0^1 (c_x)^2 dx$ to both sides of the above inequality and employing estimate (72) of Lemma 4.3, we get an inequality of the following form

$$\frac{1}{2} \int_0^1 [(c_x)^2 + (Q^\beta)_x^2] \, dx \le C + C \int_0^t \int_0^1 [(c_x)^2 + (Q^\beta)_x^2] \, dx \, ds,$$

for an appropriate choice of the constant C. Thus, application of Gronwall's inequality gives the estimate (74). □

The result of Lemma 4.6 is crucial. We follow along the idea of previous works [12, 9, 20], however, the proof becomes more involved due to the appearance of additional well-formation interaction terms. Thanks to the fact that we have the estimate (67) of the well-formation term $A = q_w[P_w - P(cQ)]$, the proof can borrow arguments from the one presented in [5] with some modifications. In particular, more care is needed for the estimate of Q^{-1} at the boundary point $x = 0$. For that purpose we make use of the following lemma.

Lemma 4.5. *We consider the following ODE system for $z(t), y(t)$:*

$$\begin{aligned} \frac{dz}{dt} &= z(1-z)[P_w - (zy)^\gamma] \\ \frac{dy}{dt} &= -(1-z) z^\gamma y^{1+\gamma-\beta} + zy[P_w - (zy)^\gamma], \end{aligned} \tag{93}$$

where P_w, γ, and β are given as described in Theorem 3.1. For a given time $T > 0$, if

$$0 < \inf_{t \in [0,T]} z(t), \qquad \sup_{t \in [0,T]} z(t) < 1, \tag{94}$$

and initial data z_0, y_0 is chosen such that

$$(z_0 y_0)^\gamma > P_w, \tag{95}$$

then we can conclude that

$$(zy)(t) \geq K := \Big(\frac{P_w}{2}\Big)^{\frac{1}{\gamma}}, \qquad t \in [0,T]. \tag{96}$$

That is,

$$\frac{1}{y(t)} \leq \frac{1}{K} \sup_{t\in[0,T]} z(t) \leq \frac{1}{K}, \qquad t \in [0,T]. \tag{97}$$

Proof. First, we observe that (93) can be reformulated as

$$\begin{aligned} \frac{d\ln(z)}{dt} &= (1-z)[P_w - (zy)^\gamma] \\ \frac{d\ln(y)}{dt} &= -(1-z)z^\gamma y^{\gamma-\beta} + z[P_w - (zy)^\gamma], \end{aligned} \tag{98}$$

by multiplying the first equation by z^{-1} and the second by y^{-1}. Summing these two equations yields

$$\frac{d\ln(zy)}{dt} = -(1-z)z^\beta (zy)^{\gamma-\beta} + [P_w - (zy)^\gamma].$$

Let $v = \ln(zy)$ and write this equation in the following form

$$\frac{dv}{dt} = -a(t)e^{v(\gamma-\beta)} - e^{v\gamma} + P_w := h(t,v), \tag{99}$$

where $a(t) = (1-z(t))z(t)^\beta \in (0,1)$ in view of (94). We want to prove the following statement for a constant $M > 0$ and time interval $[0,T]$:

$$\text{If } h(t,v) \geq 0 \text{ for } v \leq M, \text{ then } v(t) \geq \min\{v(0), M\}. \tag{100}$$

For that purpose, let us assume that there is a time $t_2 \in [0,T]$ such that

$$v(t_2) < \min\{v(0), M\}. \tag{101}$$

Due to continuity of $v(t)$ it follows that there must be a time $t_1 \in [0,t_2)$ such that

$$\begin{aligned} &v(t) \leq \min\{v(0), M\} \leq M, \qquad t \in (t_1, t_2] \\ \text{and}\ \ &v(t_1) = \min\{v(0), M\}. \end{aligned}$$

It follows from the assumption of statement (100) that $h(t,v) \geq 0$ for $t \in (t_1, t_2)$. Now we integrate (99) over (t_1, t_2) and get

$$v(t_2) = v(t_1) + \int_{t_1}^{t_2} h(s,v)\,ds \geq v(t_1) = \min\{v(0), M\}.$$

This contradicts (101) which ensures that (100) is true. The final step is to find an appropriate choice of $M > 0$ such that $h(t,v) \geq 0$ for $v \leq M$. Clearly, we have that

$$h(t,v) \geq P_w - 2e^{v\gamma} \geq 0,$$

if $e^{v\gamma} \le 1/2P_w$, that is, $v \le (1/\gamma)\ln(P_w/2) := M$. Then, we conclude from (100) and (95) that

$$v(t) \ge \min\{v(0), M\} = M, \qquad \text{since } v(0) = \ln(z_0 y_0) > \frac{1}{\gamma}\ln(P_w) > M,$$

from which (96) follows. ∎

LEMMA 4.6 (Pointwise lower limit). *Let $0 < \beta < 1/3$. Then we have a pointwise lower limit on $Q(x,t)$ of the form*

$$Q(x,t) \ge C_6, \qquad \forall (x,t) \in [0,1]\times[0,T], \tag{102}$$

where the constant $C_6 = C_6(C_2, C_3, C_5, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, \|u_0\|_{L^2(I)}, \inf_{[0,1]} c_0, T)$.

Proof. We first define

$$v(x,t) = \frac{1}{Q(x,t)}, \qquad V(t) = \max_{[0,1]\times[0,t]} v(x,s).$$

We calculate as follows:

$$\begin{aligned} v(x,t) - v(0,t) &= \int_0^x \partial_x v\,dx \le \int_0^1 |\partial_x Q| v^2\,dx = \frac{1}{\beta}\int_0^1 v^{\beta+1}|\partial_x Q^\beta|\,dx \\ &\le \frac{1}{\beta}\Big(\int_0^1 |\partial_x Q^\beta|^2\,dx\Big)^{1/2}\Big(\int_0^1 v^{2(\beta+1)}\,dx\Big)^{1/2} \\ &\le \frac{C_5^{1/2}}{\beta}\Big(\int_0^1 v\,dx\Big)^{1/2}\Big((\max_{[0,1]} v(\cdot,t))^{2\beta+1}\Big)^{1/2} \\ &\le \frac{C_5^{1/2}}{\beta}\Big(\int_0^1 v\,dx\Big)^{1/2}\Big(\max_{[0,1]} v(\cdot,t)\Big)^{\beta+1/2}, \end{aligned} \tag{103}$$

where we have used (74). Next, we focus on how to estimate $\int_0^1 v\,dx$. The starting point is the observation that the second equation of (49) can be written as

$$v_t - \rho_l([1-c]u)_x - \rho_l c_x u = -[cA]v.$$

Integrating over $[0,1]\times[0,t]$ we get

(104)

$$\begin{aligned} \int_0^1 v(x,t)\,dx &= \int_0^1 v(x,0)\,dx + \rho_l \int_0^t [(1-c)u(1,s) - (1-c)u(0,s)]\,ds \\ &\quad + \rho_l \int_0^t\int_0^1 c_x u\,dx\,ds - \int_0^t\int_0^1 [cA]v\,dx\,ds \\ &\le (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l \int_0^t \max_{[0,1]}|u(\cdot,s)|\,ds + \frac{\rho_l}{2}\int_0^t\int_0^1 [c_x^2 + u^2]\,dx\,ds + M\int_0^t\int_0^1 v\,dx\,ds \\ &\le (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l\sqrt{t}\Big(\int_0^t \|u^2(s)\|_{L^\infty(I)}\,ds\Big)^{1/2} + \frac{\rho_l t}{2}(C_5 + 2C_1) + M\int_0^t\int_0^1 v\,dx\,ds, \end{aligned}$$

where we have used Hölder's inequality, Cauchy's inequality, and the results of Lemma 4.2 and Lemma 4.4, as well as estimate (67). In light of Sobolev's inequality $\|f\|_{L^\infty(I)} \le$

$C\|f\|_{W^{1,1}(I)}$ it follows that the second term on the right hand side of (104) can be estimated as follows:

$$
\begin{aligned}
&\int_0^t \|u^2(s)\|_{L^\infty(I)}\,ds \\
&\le C\int_0^t \|u^2(s)\|_{W^{1,1}(I)}\,ds = C\Big(\int_0^t\int_0^1 u^2\,dx\,ds + \int_0^t\int_0^1 |(u^2)_x|\,dx\,ds\Big) \\
&\le CtC_1 + 2C\int_0^t\int_0^1 Q^{\frac{1+\beta}{2}}|u||u_x|v^{\frac{1+\beta}{2}}\,dx\,ds \qquad (105)\\
&\le CtC_1 + 2C\Big(\int_0^t\int_0^1 Q^{1+\beta}u_x^2u^2\,dx\,ds\Big)^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2} \\
&\le CtC_1 + 2CC_3^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2},
\end{aligned}
$$

where we have used (54) and (56) with $q=2$ and Hölder's inequality. Combining (104) and (105) we get

$$
\begin{aligned}
&\int_0^1 v(x,t)\,dx \\
&\le (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l\sqrt{t}\Big[CtC_1 + 2CC_3^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2}\Big]^{1/2} \\
&\qquad\qquad + \frac{\rho_l t}{2}(C_5+2C_1) + M\int_0^t\int_0^1 v\,dx\,ds \qquad (106)\\
&\le C + C\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/4} + M\int_0^t\int_0^1 v\,dx\,ds \\
&= C + C\Big(\int_0^t\int_0^1 v^{2\beta}v^{1-\beta}\,dx\,ds\Big)^{1/4} + M\int_0^t\int_0^1 v\,dx\,ds \\
&\le C + CV(t)^{2\beta/4}\Big(\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds\Big)^{1/4} + MV(t)^\beta\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds,
\end{aligned}
$$

where $C = C(\inf_{[0,1]} Q_0, C_1, T)$. Now we focus on estimating $\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds$. For that purpose, we note that the second equation of (49), by multiplying with $Q^{\frac{\beta-1}{2}-1}$, can be written as

$$(Q^{\frac{\beta-1}{2}})_t = \rho_l\frac{1-\beta}{2}(1-c)Q^{\frac{\beta+1}{2}}u_x - \frac{1-\beta}{2}[cA]Q^{\frac{\beta-1}{2}}.$$

Integrating this equation over $[0,t]$ we get

$$Q^{\frac{\beta-1}{2}}(x,t) = Q^{\frac{\beta-1}{2}}(x,0) + \rho_l\frac{1-\beta}{2}\int_0^t (1-c)Q^{\frac{\beta+1}{2}}u_x\,ds - \frac{1-\beta}{2}\int_0^t [cA]Q^{\frac{\beta-1}{2}}\,ds.$$

Consequently, using the inequality $(a+b)^2 \le 2a^2+2b^2$ we get

$$
\begin{aligned}
Q^{\beta-1}(x,t) &\le 2Q^{\beta-1}(x,0) + 4\rho_l^2\Big(\frac{1-\beta}{2}\Big)^2\Big(\int_0^t Q^{\frac{\beta+1}{2}}u_x\,ds\Big)^2 + 4\Big(\frac{1-\beta}{2}\Big)^2\Big(\int_0^t [cA]Q^{\frac{\beta-1}{2}}\,ds\Big)^2 \\
&\le 2Q^{\beta-1}(x,0) + \rho_l^2 t(1-\beta)^2\int_0^t Q^{\beta+1}u_x^2\,ds + M^2t(1-\beta)^2\int_0^t Q^{\beta-1}\,ds,
\end{aligned}
$$

by Hölder's inequality. Integrating over $[0,1]$ in space yields

(107)
$$\begin{aligned}
&\int_0^1 v^{1-\beta}\,dx = \int_0^1 Q^{\beta-1}\,dx \\
&\leq 2\int_0^1 v^{1-\beta}(x,0)\,dx + \rho_l^2 t(1-\beta)^2\int_0^1\int_0^t Q^{\beta+1}u_x^2\,ds\,dx + M^2 t(1-\beta)^2\int_0^1\int_0^t Q^{\beta-1}\,ds\,dx \\
&\leq C + M^2 t(1-\beta)^2\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds,
\end{aligned}$$

with $C = C(\inf_{[0,1]} Q_0, C_1, T)$ where we have used (54). Thus, by Gronwall's inequality we conclude that

$$\int_0^1 v^{1-\beta}\,dx \leq C(\inf_{[0,1]} Q_0, C_1, M, T). \tag{108}$$

Consequently, (106) and (108) imply that

$$\int_0^1 v(x,t)\,dx \leq C + D[V(t)^{\beta/2} + V(t)^{\beta}] \leq E[1 + V(t)^{\beta/2} + V(t)^{\beta}], \tag{109}$$

for appropriate constants C, D and E that depend essentially on $\inf_{[0,1]} Q_0, M, T, C_1$. Substituting (109) into (103) we get

$$\begin{aligned}
v(x,t) - v(0,t) &\leq \frac{C_5^{1/2}}{\beta}\Big(\int_0^1 v\,dx\Big)^{1/2}\Big(\max_{[0,1]} v(\cdot,t)\Big)^{\beta+1/2} \\
&\leq \frac{(C_5E)^{1/2}}{\beta}[1 + V(t)^{\beta/2} + V(t)^{\beta}]^{1/2}V(t)^{\beta+1/2} \\
&\leq F[1 + V(t)^{\beta/4} + V(t)^{\beta/2}]V(t)^{\beta+1/2} \\
&\leq F\max(CV(t)^{(3/2)\beta+1/2}, 3),
\end{aligned} \tag{110}$$

for $F = F(C_5, E)$. Here we have used the inequality $(1 + x^{\beta/4} + x^{\beta/2})x^{\beta+1/2} \leq Cx^{(3/2)\beta+1/2}$ which holds for $x \geq 1$ and an appropriate constant $C \geq 3$. This follows by observing that

$$\begin{aligned}
f(x) &= Cx^{(3/2)\beta+1/2} - x^{\beta+1/2}(1 + x^{\beta/4} + x^{\beta/2}) = x^{\beta+1/2}((C-1)x^{\beta/2} - 1 - x^{\beta/4}) \\
&\geq x^{\beta+1/2}((C-1)x^{\beta/2} - 1 - x^{\beta/2}) = x^{\beta+1/2}((C-2)x^{\beta/2} - 1) \geq 0,
\end{aligned}$$

for $x \geq 1$ and $C \geq 3$.

We must check that $v(0,t)$ remains bounded in $[0,T]$. From the boundary condition (52) we have

$$(cQ)^{\gamma} - Q^{\beta+1}u_x\Big|_{x=0} = 0.$$

Using this in combination with the second equation of (49) gives us the following equation at $x = 0$:

$$Q_t + \rho_l(1-c)Q^{1-\beta}(cQ)^{\gamma} = cQ[P_w - (cQ)^{\gamma}], \tag{111}$$

whereas the first equation of (49) corresponds to

$$c_t = c(1-c)[P_w - (cQ)^\gamma], \qquad \text{for } x = 0. \tag{112}$$

Setting that

$$z(t) = c(x=0,t), \qquad y(t) = Q(x=0,t),$$

and without loss of generality we may set $\rho_l = 1$, then we see that the ODE system (111) and (112) corresponds to the ODE system of Lemma 4.5. In view of Corollary 4.1 we see that the assumption (94) is fulfilled. In view of assumption (31) of Theorem 3.1, it is also clear that assumption (95) is fulfilled. Consequently, we can conclude that

$$v(0,t) \le K^{-1}, \qquad t \in [0,T].$$

In conclusion, from (110) we have

$$V(T) \le K^{-1} + 3F \max\Big(V(T)^{(3/2)\beta+1/2}, 1\Big).$$

Since $\beta < 1/3$ we see that $(3/2)\beta + 1/2 < 1$. Therefore, it is clear from the inequality $x \le C(1+x^\xi)$ with $0 < \xi < 1$, that $x \le G$ for some constant G. Consequently, $V(T) \le G$ where (in view of the above estimates)

$$G = G(C_2, C_3, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, \|u_0\|_{L^2(I)}, T).$$

Thus, the result (102) follows. □

Now, we can directly deduce the following pointwise estimates which ensure that no transition to single-phase flow occurs.

Corollary 4.7. *There is a constant $\mu = \mu(C_2, C_6) > 0$ such that for $(x,t) \in [0,1] \times [0,T]$, we have*

$$0 < \mu \le [1-c]\rho(x,t), \qquad [1-c]\rho(x,t) \le \rho_l - \mu < \rho_l \tag{113}$$

$$0 < \mu \inf_{x\in[0,1]}(c) \le n(x,t) \le \Big(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\Big) \sup_{x\in[0,1]}(c) < \infty, \tag{114}$$

for $c = n/\rho$.

Proof. In view of (47) and the bounds (55) and (102) it is clear that there is a $\mu > 0$ such that (113) holds. Consequently,

$$0 < \mu \inf_{x\in[0,1]}(c) \le n = c\rho \le \Big(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\Big) \sup_{x\in[0,1]}(c) < \infty,$$

where we have used the estimates (43) of Corollary 4.1 as well as the refined lower limit (71). □

Corollary 4.8. *We have the estimates*

$$\int_0^1 (\partial_x \rho)^2\, dx \le C_7, \qquad \int_0^1 (\partial_x n)^2\, dx \le C_8, \tag{115}$$

for a constant $C_7 = C_7(C_2, C_4, C_5, C_6)$ and $C_8 = C_8(C_2, C_4, C_5, C_6)$.

Proof. It follows that

$$\partial_x Q(\rho,k)^\beta = \beta Q(\rho,k)^{\beta-1}[Q_\rho \partial_x \rho + Q_k \partial_x k] = \beta Q(\rho,k)^{\beta+1}\Big[\frac{\rho_l}{\rho^2}\partial_x \rho + \partial_x k\Big].$$

In view of this calculation and the pointwise upper and lower limits for $Q(\rho,k)$, as well as ρ, given by (55), (102), and Corollary 4.7, it follows by application of Lemma 4.4 that the first estimate of (115) holds. The second follows directly from the relation

$$\partial_x n = \rho\partial_x c + c\partial_x\rho, \qquad \text{since } n = c\rho,$$

and the corresponding estimate

$$\int_0^1 (\partial_x n)^2\,dx \le 2(\sup_{x\in[0,1]}\rho)^2\int_0^1(\partial_x c)^2 + 2(\sup_{x\in[0,1]} c)^2\int_0^1(\partial_x\rho)^2\,dx \le C_8,$$

where we use the first estimate of (115), Lemma 4.4 and Corollary 4.7. ☐

5. Proof of existence result. Equipped with the estimates of Section 4 we apply arguments similar to those used in [12, 11, 13, 9] to show compactness, i.e., convergence of a sequence of approximate solutions of (24) (obtained by regularization of initial data) to limit functions (n,ρ,u). The final step is to show that these are solutions in the sense of (33) of Theorem 3.1.

First, we introduce the Friedrichs mollifier $j_\delta(x)$. Let $\psi(x)\in C_0^\infty(\mathbb{R})$ satisfy $\psi(x)=1$ when $|x|\le 1/2$ and $\psi(x)=0$ when $|x|\ge 1$, and define $\psi_\delta := \psi(x/\delta)$.

Mollifying. We extend n_0,ρ_0,u_0 to $\mathbb{R}$ by using

$$n_0(x) := \begin{cases} n_0(1), & x\in(1,\infty),\\ n_0(x), & x\in[0,1],\\ n_0(0), & x\in(-\infty,0),\end{cases} \qquad \rho_0(x) := \begin{cases} \rho_0(1), & x\in(1,\infty),\\ \rho_0(x), & x\in[0,1],\\ \rho_0(0), & x\in(-\infty,0),\end{cases}$$

whereas we extend $u_0(x)$ to $\mathbb{R}$ by defining it to be zero outside the interval $[0,1]$. Approximate initial data $(n_0^\delta,\rho_0^\delta,u_0^\delta)$ to (n_0,ρ_0,u_0) are now defined as follows:

(116)
$$\begin{aligned} n_0^\delta(x) &= (n_0 * j_\delta)(x), \qquad \rho_0^\delta(x) = (\rho_0 * j_\delta)(x),\\ u_0^\delta &= (u_0 * j_\delta)(x)[1-\psi_\delta(x)-\psi_\delta(1-x)] + (u_0*j_\delta)(0)\psi_\delta(x) + (u_0*j_\delta)(1)\psi_\delta(1-x)\\ &\quad + (c_0^\delta)^\gamma Q(\rho_0^\delta)^{\gamma-\beta-1}(0)\int_0^x \psi_\delta(y)\,dy - (c_0^\delta)^\gamma Q(\rho_0^\delta)^{\gamma-\beta-1}(1)\int_x^1\psi_\delta(1-y)\,dy. \end{aligned}$$

Then it follows that $n_0^\delta,\rho_0^\delta\in C^{1+s}[0,1]$, $u_0^\delta\in C^{2+s}[0,1]$ for any $0<s<1$, and n_0^δ,ρ_0^δ and u_0^δ are compatible with the boundary conditions (27). Moreover, it follows that

$$\begin{aligned} |(u_0*j_\delta)(0)|^{2q}\int_0^1\psi_\delta^{2q}\,dx &\le C\delta\Big(\int_0^\delta u_0(x)j_\delta(x)\,dx\Big)^{2q}\\ &\le C\delta\int_0^\delta u_0^{2q}\,dx\Big(\int_0^\delta j_\delta^{2q/(2q-1)}(x)\,dx\Big)^{2q-1}\\ &\le C\int_0^\delta u_0^{2q}(x)\,dx \to 0 \quad \text{as } \delta\to 0. \end{aligned}$$

Similarly, it follows that $|(u_0*j_\delta(1)|^{2q}\int_0^1\psi_\delta^{2q}(1-x)\,dx\to 0$. Therefore, recalling the definition of $u_0^\delta(x)$ we see that as $\delta\to 0$,

$$u_0^\delta \to u_0 \ \text{ in } L^{2q}(I). \tag{117}$$

In addition,

$$n_0^\delta \to n_0, \quad \rho_0^\delta \to \rho_0 \qquad \text{uniformly in } [0,1], \tag{118}$$

as $\delta \to 0$.

Now, we consider the initial boundary value problem (24)–(28) with the initial data (n_0, ρ_0, u_0) replaced by $(n_0^\delta, \rho_0^\delta, u_0^\delta)$. For this problem standard arguments can be used (the energy estimates and the contraction mapping theorem) to obtain the existence of a unique *local* solution $(n^\delta, \rho^\delta, u^\delta)$ with $n^\delta, n_t^\delta, n_x^\delta, n_{tx}^\delta, \rho^\delta, \rho_x^\delta, \rho_t^\delta, \rho_{tx}^\delta, u^\delta, u_x^\delta, u_t^\delta, u_{xx}^\delta \in C^{\alpha,\alpha/2}([0,1]\times[0,T^*])$ for some $T^* > 0$.

In view of the estimates of Section 4.2, obtained by relying on the reformulated model (49)–(53), it follows that n^δ and ρ^δ are pointwise bounded from above and below, $(u^\delta)^q$, n_x^δ, and ρ_x^δ are bounded in $L^\infty([0,T], L^2(I))$ and u_x^δ is bounded in $L^2((0,T), L^2(I))$ for any $T > 0$. Furthermore, we can differentiate the equations in (49) and apply the energy method to derive bounds of high-order derivatives of $(n^\delta, \rho^\delta, u^\delta)$. Then the Schauder theory for linear parabolic equations can be applied to conclude that the $C^{\alpha,\alpha/2}(D_T)$-norm of $n^\delta, n_t^\delta, n_x^\delta, n_{tx}^\delta, \rho^\delta, \rho_x^\delta, \rho_t^\delta, \rho_{tx}^\delta, u^\delta, u_x^\delta, u_t^\delta, u_{xx}^\delta$ is a priori bounded. Therefore, we can continue the local solution globally in time and obtain that there exists a unique global solution $(n^\delta, \rho^\delta, u^\delta)$ of (24)–(28) with initial data $(n_0^\delta, \rho_0^\delta, u_0^\delta)$ such that for any $T > 0$, the regularity of (35) holds.

Estimates and Compactness. Clearly, in view of the estimates of Section 4 and the model itself (24), we have

(119)
$$\begin{aligned}
&\int_0^1 (u^\delta)^{2q}(x,t)\,dx + \int_0^1 (n_x^\delta)^2(x,t)\,dx + \int_0^1 (\rho_x^\delta)^2(x,t)\,dx \le C, \quad t\in[0,T],\ q\in\mathbb{N},\\
&\qquad 0 < \mu \le \rho^\delta(x,t) \le \Big(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\Big),\\
&\qquad 0 < \mu \inf_{x\in[0,1]}(c) \le n^\delta(x,t) \le \Big(\frac{\rho_l - \mu}{1 - \sup_{x\in[0,1]}(c)}\Big) \sup_{x\in[0,1]}(c), \quad (x,t)\in[0,1]\times[0,T],\\
&\int_0^T \int_0^1 \Big[(u_x^\delta)^2 + (n_t^\delta)^2 + (\rho_t^\delta)^2\Big](x,s)\,dx\,ds \le C,
\end{aligned}$$

where the constants $C, \mu > 0$ do not depend on δ. Note that the boundedness of ρ_t^δ and n_t^δ in $L^2([0,T], L^2(I))$ follows in view of the equation $\rho_t^\delta + (\rho^\delta[\rho^\delta - n^\delta])u_x^\delta = nA$ and $n_t^\delta + (n^\delta[\rho^\delta - n^\delta])u_x^\delta = nA$, the estimates of Corollary 4.7, and the energy estimate (54) of Lemma 4.2. Hence, we can extract a subsequence of $(n^\delta, \rho^\delta, u^\delta)$, still denoted by $(n^\delta, \rho^\delta, u^\delta)$, such that as $\delta \to 0$,

$$\begin{aligned}
&u^\delta \rightharpoonup u \text{ weak-* in } L^\infty([0,T], L^{2q}(I)),\\
&n^\delta \rightharpoonup n \text{ weak-* in } L^\infty([0,T], W^{1,2}(I)),\\
&\rho^\delta \rightharpoonup \rho \text{ weak-* in } L^\infty([0,T], W^{1,2}(I)),\\
&(n_t^\delta, \rho_t^\delta, u_x^\delta) \rightharpoonup (n_t, \rho_t, u_x) \text{ weakly in } L^2([0,T], L^2(I)).
\end{aligned} \tag{120}$$

Next, we show that (n, ρ, u) obtained in (120) in fact is a weak solution of (24)–(28). The classical Sobolev imbedding (Morrey's inequality) $W^{1,2q}(0,1) \hookrightarrow C^{1-1/(2q)}[0,1]$ applied with $q = 1$ gives that for any $x_1, x_2 \in (0,1)$ and $t \in [0,T]$

$$|\rho^\delta(x_1,t) - \rho^\delta(x_2,t)| \le C|x_1 - x_2|^{1/2}. \tag{121}$$

To control continuity in time, in view of the sequence of imbeddings $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1) \hookrightarrow L^2(0,1)$, we can apply Lions-Aubin lemma (see for example [15], Section 1.3.12) for a constant $\nu > 0$ (arbitrary small) to find a constant C_ν such that

$$
\begin{aligned}
\|\rho^\delta(t_1) - \rho^\delta(t_2)\|_{L^\infty(I)} &\le \nu\|\rho^\delta(t_1) - \rho^\delta(t_2)\|_{W^{1,2}(I)} + C_\nu\|\rho^\delta(t_1) - \rho^\delta(t_2)\|_{L^2(I)} \\
&\le 2\nu\|\rho^\delta(t)\|_{W^{1,2}(I)} + C_\nu|t_1 - t_2|^{1/2}\|\rho_t^\delta\|_{L^2([0,T],L^2(I))} \\
&\le C\nu + C_\nu C|t_1 - t_2|^{1/2},
\end{aligned} \tag{122}
$$

where we have used (119) to derive the last two inequalities. Consequently, (121) and (122) together with the triangle inequality show that $\{\rho^\delta\}$ is equi-continuous on $D_T = [0,1] \times [0,T]$. Hence, by Arzela-Ascoli's theorem and a diagonal process for t, we can extract a subsequence of $\{\rho^\delta\}$, such that

$$\rho^\delta(x,t) \to \rho(x,t) \text{ strongly in } C^0(D_T). \tag{123}$$

The same arguments apply to n yielding

$$n^\delta(x,t) \to n(x,t) \text{ strongly in } C^0(D_T). \tag{124}$$

Clearly, ρ_t is also bounded in $L^2([0,T],L^2(I))$ and from the estimate

$$
\begin{aligned}
\|\rho(t_1) - \rho(t_2)\|^2_{L^2(I)} = \int_0^1 |\rho(t_1) - \rho(t_2)|^2\,dx = \int_0^1 \Big|\int_{t_1}^{t_2} \rho_t\,ds\Big|^2 dx \le \int_0^1 \Big(\int_{t_1}^{t_2} |\rho_t|\,ds\Big)^2 dx \\
\le |t_1 - t_2| \int_0^T \int_0^1 \rho_t^2\,dx\,ds,
\end{aligned}
$$

where we have used Hölder's inequality, we may also conclude that

$$\rho \in C^{1/2}([0,T],L^2(I)). \tag{125}$$

Similarly, the same arguments apply to n. Thus, we conclude that the limit functions (n,ρ,u) from (120) satisfy the first two equations $n_t + n[\rho - n]u_x = q_w n[P_w - P(n,\rho)]$ and $\rho_t + \rho[\rho - n]u_x = q_w n[P_w - P(n,\rho)]$ of (33) for a.e. $x \in (0,1)$ and any $t \ge 0$. To show that the last integral equality holds, we multiply the third equation of (24) by $\phi \in C_0^\infty(D)$ with $D = [0,1] \times [0,\infty)$ and integrate over $(0,T) \times (0,1)$, followed by integration by parts with respect to x and t. Taking the limit as $\delta \to 0$, we see that (n,ρ,u) also must satisfy weakly the third equation of (33).

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