GLOBAL WEAK SOLUTIONS FOR A GAS-LIQUID MODEL WITH EXTERNAL FORCES AND GENERAL PRESSURE LAW

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Abstract. In this work we show existence of global weak solutions for a two-phase gas-liquid model where the gas phase is represented by a general isothermal pressure law whereas the liquid is assumed to be incompressible. To make the model relevant for pipe and well-flow applications we have included external forces in the momentum equation representing respectively wall friction forces and gravity forces. The analysis relies on a proper combination of the methods introduced in [9, 10] where a two-phase gas-liquid model without external forces was studied for the first time, and techniques that have been developed for the single-phase gas model. As a motivation for further research, some numerical examples are also included demonstrating the ability of the model to describe the ascent of a gas slug due to buoyancy forces in a vertical well. Characteristic features like expansion of the moving gas slug as well as counter-current flow mechanisms (i.e., liquid is moving downward due to gravity and gas is displaced upward) are highlighted. These examples are highly relevant for modeling of gas-kick flow scenarios which represent a major concern in the context of oil and gas well control operations.

Key words. two-phase flow, well model, gas-kick, weak solutions, Lagrangian coordinates, free boundary problem

AMS subject classifications. 76T10, 76N10, 65M12, 35L60

1. Introduction and examples. This work is devoted to a study of a one-dimensional two-phase model of the drift-flux type. The model is frequently used in industry simulators to simulate unsteady, compressible flow of liquid and gas in pipes and wells [1, 4, 5, 7, 19, 23, 26, 31]. The model consists of two mass conservation equations corresponding to each of the two phases gas (g) and liquid (l) and one equation for the conservation of the momentum of the mixture and is given in the following form:

$$
\begin{aligned}
&\partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] = 0\\
&\partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] = 0 \qquad (1.1)\\
&\partial_t[\alpha_g\rho_g u_g + \alpha_l\rho_l u_l] + \partial_x[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + p] = q + \partial_x[\varepsilon\partial_x u_{mix}], \quad u_{mix} = \alpha_g u_g + \alpha_l u_l,
\end{aligned}
$$

where $\varepsilon \geq 0$. The model is supposed under isothermal conditions. The unknowns are: ρ_l, ρ_g the liquid and gas densities; α_l, α_g volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$; u_l, u_g fluid velocities of liquid and gas; p common pressure for liquid and gas; and q representing external forces like gravity and friction. Since the momentum is given only for the mixture, we need an additional closure law, a so-called hydrodynamical closure law, which connects the two phase velocities. More generally, this law should be able to take into account the different flow regimes. In addition, we need a thermodynamical equilibrium model which specifies the fluid properties. More details will be given in the next section. Otherwise, we refer to [6, 7, 13, 23, 25, 26, 28, 31] for various numerical schemes which have been developed for the study of the drift-flux model. See also [8] for a study of the relation between the drift-flux model and the more general two-fluid model where two separate momentum equations are used instead of a mixture momentum equation [4, 19].

In [9, 10] we studied a simplified version of the model (1.1) obtained by assuming that fluid velocities are equal, $u_g = u_l = u$, and by neglecting the external forces, i.e., $q = 0$. In addition, we neglected certain gas effects by considering a simplified momentum equation where acceleration terms depend solely on the liquid phase. This is motivated by the fact that liquid phase density

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typically is much higher than gas phase density. Consequently, we considered a model in the form

$$\begin{aligned} &\partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u] = 0 \\ &\partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u] = 0 \\ &\partial_t[\alpha_l\rho_l u] + \partial_x[\alpha_l\rho_l u^2] + \partial_x p = \partial_x[\varepsilon\partial_x u], \qquad p,\varepsilon \geq 0. \end{aligned} \tag{1.2}$$

Assuming a polytropic gas law relation

$$p = C\rho_g^{\gamma}, \tag{1.3}$$

with $\gamma > 1$ for the gas phase whereas the liquid phase is treated as an incompressible fluid, i.e., $\rho_l = \text{Const}$, we get a pressure law of the form

$$p(n,m) = C\Big(\frac{n}{\rho_l - m}\Big)^{\gamma}, \tag{1.4}$$

where we use the notation $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$. In particular, we see that there is a possibly singular behavior associated with pressure at transition to pure liquid phase, i.e., $\alpha_l = 1$, which yields $m = \rho_l$ and $n = 0$. In addition, we have the possibility for vacuum as in the single-phase gas model, i.e., that $\rho_g = 0$ which implies that $n = 0$ and $p = 0$.

Different forms for the viscosity function ε have been considered. In [9] we used

$$\varepsilon = \varepsilon(m) = \frac{m^{\theta}}{(\rho_l - m)^{\theta+1}}, \qquad \theta \in (0, 1/3), \tag{1.5}$$

whereas in [10] we considered

$$\varepsilon = \varepsilon(n,m) = \frac{n^{\theta}}{(\rho_l - m)^{\theta+1}}, \qquad \theta \in (0, 1/3). \tag{1.6}$$

More recently, Yao and Zhu [34] also studied the model (1.2) in a flow regime where the viscosity coefficient $\varepsilon > 0$ was assumed to take the form (1.5). They gave a proof of the global existence and uniqueness of weak solutions when θ is in $(0, 1]$ and thereby improved the result of [9]. They also gave an interesting asymptotic behavior result, and obtained the regularity of the solutions by the energy method. The same authors also presented a nice result for the same gas-liquid model (but constant viscosity term) when the masses m, n connected continuously to a vacuum state $m = n = 0$ [35]. A key point in the analysis of the model (1.2) and exploited in the above mentioned works, is to rewrite it in terms of Lagrangian coordinates (ξ, τ). This gives us a model of the following form:

$$\begin{aligned} &\partial_\tau n + (nm)\partial_\xi u = 0 \\ &\partial_\tau m + m^2\partial_\xi u = 0 \\ &\partial_\tau u + \partial_\xi p(n,m) = \partial_\xi(\varepsilon(m)m\partial_\xi u), \end{aligned} \tag{1.7}$$

which also clearly can be written as

$$\begin{aligned} &\partial_\tau c = 0 \\ &\partial_\tau m + m^2\partial_\xi u = 0 \\ &\partial_\tau u + \partial_\xi p(c,m) = \partial_\xi(\varepsilon(m)m\partial_\xi u), \qquad c = \frac{n}{m}. \end{aligned} \tag{1.8}$$

A motivation for the form of the viscosity term ε. Motivated by lab experiments different examples of a mixture viscosity term μ_m, where the gas-liquid mixture is considered as a single-phase fluid, have been proposed. One of them is the following correlation [27, 32]:

$$\frac{1}{\mu_m} = \frac{y}{\mu_g} + \frac{1-y}{\mu_l}, \qquad \text{(McAdams et al.'s model)}. \tag{1.9}$$

Here y is defined as mass flux fraction

$$y = \frac{\alpha_g \rho_g u_g}{\alpha_g \rho_g u_g + \alpha_l \rho_l u_l}. \tag{1.10}$$

For equal fluid velocities $u_l = u_g$ this corresponds to $y = \frac{n}{n+m}$.

If we assume that $n << m$ (i.e., the liquid phase is dominating), then $y = \frac{n}{n+m} \approx \frac{n}{m} := c$ for $0 \le y \le 1$. Moreover, typically the liquid viscosity μ_l is considerable larger than the gas viscosity μ_g, see (2.6). Consequently, $\mu_l >> \mu_g$ and we may approximate as follows by using the viscosity model of McAdams et al (1.9):

$$\frac{1}{\mu_m} = \frac{y}{\mu_g} + \frac{1-y}{\mu_l} \approx \frac{y}{\mu_g} = \frac{c}{\mu_g}. \tag{1.11}$$

Directly motivated by the traditional single-phase viscosity term of the form $E = (\mu\rho)^{\theta+1} = C\rho^{\theta+1}$ in Lagrangian coordinates, see for example [29, 22, 24, 36, 30, 21], we may propose a similar viscosity coefficient $E = (\mu_m \rho_m)^{\theta+1}$ for the gas-liquid mixture model (1.1) where μ_m is a mixture viscosity defined by, e.g., (1.9), and ρ_m is a suitable mixture density. If we define a mixture density ρ_m as

$$\rho_m = [(\alpha_g \rho_g)^{\theta+1} + (\alpha_l \rho_l)^{\theta+1}]^{\frac{1}{\theta+1}}, \tag{1.12}$$

and combine it with the approximation (1.11), then $E = (\mu_m \rho_m)^{\theta+1}$ corresponds to

$$\begin{aligned} E = (\mu_m \rho_m)^{\theta+1} &= \mu_m^{\theta+1}[(\alpha_g \rho_g)^{\theta+1} + (\alpha_l \rho_l)^{\theta+1}] = (\mu_m \alpha_g \rho_g)^{\theta+1} + (\mu_m \alpha_l \rho_l)^{\theta+1} \\ &= (\alpha_g \rho_l \mu_g)^{\theta+1} \Big(\frac{1}{c} \frac{n}{\rho_l - m} \Big)^{\theta+1} + (\alpha_l \mu_g)^{\theta+1} \Big(\frac{\rho_l}{c} \Big)^{\theta+1} \\ &= (\alpha_g \rho_l \mu_g)^{\theta+1} \Big(\frac{m}{\rho_l - m} \Big)^{\theta+1} + (\alpha_l \mu_g)^{\theta+1} \Big(\frac{\rho_l}{c} \Big)^{\theta+1} \\ &:= E_1 + E_2, \end{aligned}$$

where we have used the fact that $\rho_g = \rho_l \frac{n}{\rho_l - m}$, see (3.4). Recalling that ρ_l is constant and that $c = \frac{n}{m} = c(x)$ is constant in time according to the first equation of (1.8), the most "dynamic" part of this viscosity term is the first part

$$E_1 = (\alpha_g \rho_l \mu_g)^{\theta+1} \Big(\frac{m}{\rho_l - m} \Big)^{\theta+1}. \tag{1.13}$$

Comparing (1.13) with (1.5) and taking into account that the viscosity term in terms of the Lagrangian description takes the form $E = \varepsilon(m)m$, see (1.8), we see that E_1 coincides with the one that is studied in [9] except that the coefficient $(\alpha_g \rho_l \mu_g)^{\theta+1}$ has been replaced by a constant.

Purpose of this work. The objective of this work is two-fold.

A) Firstly, we demonstrate some simple but highly relevant flow cases from an engineering point of view. More precisely, we illustrate by numerical calculations that the drift-flux model (1.1) can be used to study how a gas slug, initially located at the bottom of a vertical well, will ascend driven by the buoyancy forces. The dynamics are determined by a relatively complicated interplay between friction forces, gravity, and slip relation. Strong gas slug expansion is possible near the surface and transition between two-phase and single-phase regions typically will occur. This type of flow is highly relevant for gas kick scenarios, that ultimately can lead to blowout [1], as well as for the study of volcanic eruption mechanisms [20].

B) Secondly, we provide mathematical analysis of a simplified gas-liquid model similar to (1.2) but with two important extensions relevant for the simulation cases demonstrated in A): (i) inclusion of a frictional force term and gravity term, compare (1.14) with (1.7);

and (ii) use of a general equation of state for the gas phase. In particular, we derive an existence result for a class of weak solutions by employing a proper combination of the techniques introduced in [9, 10] for the study of (1.2) and single-phase analysis as described, e.g. in [36, 37, 38, 33]. We refer to the remark after Theorem 3.1 for more details concerning additional difficulties due to the new terms and how these terms are handled within the chosen mathematical framework.

To be precise, we study the following gas-liquid model described in terms of Lagrangian variables where we replace (ξ, τ) by (x, t):

$$\begin{aligned} &\partial_t n + (nm)\partial_x u = 0 \\ &\partial_t m + m^2 \partial_x u = 0 \\ &\partial_t u + \partial_x p(n,m) = -f m^2 u|u| + g + \partial_x (E(m)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{1.14}$$

with

$$p(n,m) = P\Big(\frac{n}{\rho_l - m}\Big), \tag{1.15}$$

where P is a general pressure function whose properties are specified in Section 3.2, see (3.27)–(3.29). Moreover, the viscosity term is the same as studied in [9, 34]

$$E(m) := \varepsilon(m) m = \Big(\frac{m}{\rho_l - m}\Big)^{\theta+1}, \qquad 0 < \theta < 1/2. \tag{1.16}$$

We here note that θ is allowed to be in a larger interval compared to the works [9, 34]. Boundary conditions are given by

$$[p(n,m) - E(m)\partial_x u](0,t) = 0, \qquad u(1,t) = 0, \tag{1.17}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x \in (0,1). \tag{1.18}$$

Hopefully the combination of (A) and (B) can serve as a motivation for other researchers to deepen the insight into the mathematical properties of the general drift-flux model (1.1) as well as bring forth further development of the drift-flux model itself to make it more applicable for various large scale multiphase flow scenarios. Before we end this section we recall that the key results leading to Theorem 3.1 is the fact that we can derive a series of a priori estimates for approximate solutions of (1.14)–(1.18) and a corresponding limit procedure. The main estimates are to obtain appropriate upper and lower pointwise bounds on the masses m and n. These estimates must be sharp enough to handle the potential singular behavior associated with pressure $p(n,m)$ and viscosity $E(m)$. For that purpose we introduce a transformed version of the model, see (4.8)–(4.11), described in terms of new variables (c, Q, u). One crucial step is to obtain the result of Lemma 4.6 which allows us to derive more control on Q, as expressed by Lemma 4.7. Ultimately, this gives us the pointwise lower bound on Q as described by Lemma 4.8. These bounds can then be transferred back to sufficient control on the masses m and n. Finally, equipped with the regularity results of Lemma 4.12–4.14 we can apply standard compactness arguments to derive convergence of the approximate solutions to limit functions that can be shown to be a weak solution of (1.14)–(1.18).

The rest of the paper is organized as follows: In Section 2 we present more details for the gas-liquid model we study in a setting relevant for well control operations. We also present numerical calculations of two characteristic gas slug flow examples where gravity and frictional forces play an important role. In Section 3, motivated by the numerical examples we derive the simplified version of the full model (1.1) with inclusion of friction and gravity, as given by (1.14). We present the model in appropriate Lagrangian variables and give the main assumptions as well as the main existence result, Theorem 3.1. Section 4 is devoted to the various a priori estimates which in turn imply compactness and convergence to weak solutions.

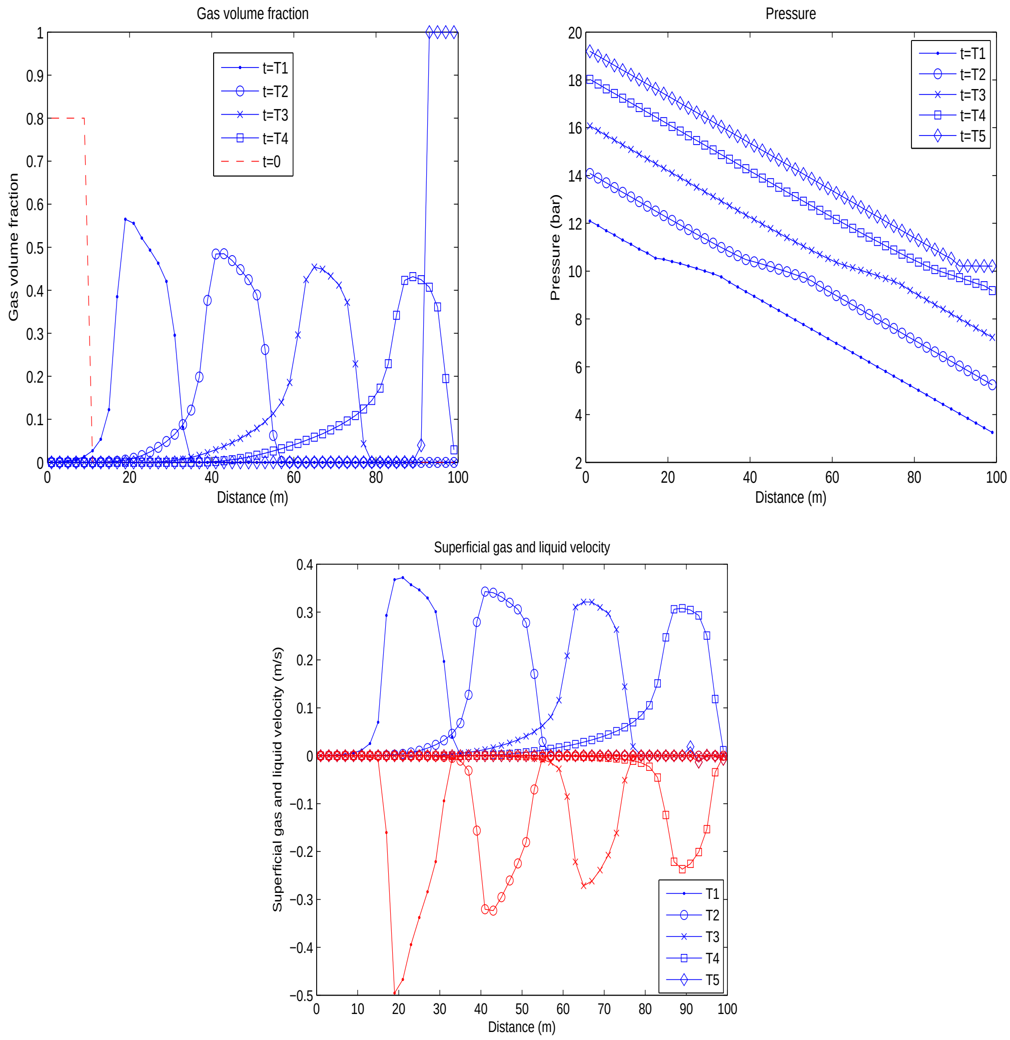

FIG. 2.1. **Top:** *The behavior of the gas slug (left) and corresponding pressure (right) as the gas slug is moving upwards. Note the increase in pressure as the gas slug is ascending towards the top.* **Bottom:** *The superficial velocity of gas ($\alpha_g v_g$) (blue) and liquid ($\alpha_l v_l$) (red), respectively, reflect the upward movement of the gas slug and the downward behavior of the surrounding liquid. Note that this problem is relatively complicated to solve as it involves strong nonlinear phenomena associated with counter-current flow and challenges associated with transition from two-phase to single-phase flow.*

2. Application of the drift-flux model for well control operations.

2.1. Specification of the model (1.1)**.** To close the system (1.1), we need to include the following additional equations: The volume fractions are related by

$$\alpha_l + \alpha_g = 1. \tag{2.1}$$

Thermodynamical laws specify fluid properties such as densities ρ_l, ρ_g and viscosities μ_l, μ_g. In particular we will assume that the liquid density has the following form

$$\rho_l = \rho_{l,0} + \frac{p - p_{l,0}}{a_l^2}, \tag{2.2}$$

where $a_l = 1000$ [m/s] is the velocity of sound in the liquid phase and $\rho_{l,0}$ and $p_{l,0}$ are given constants. Here we will assume that $\rho_{l,0} = 1000$ [kg/m^3] and $p_{l,0} = 1$ [bar]. It is often assumed

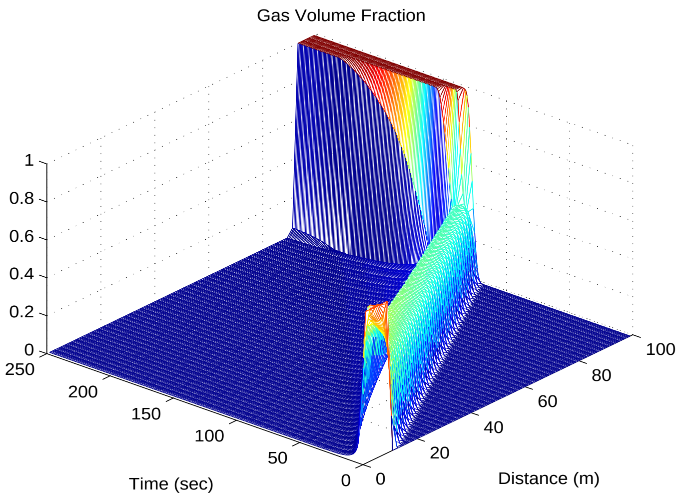

FIG. 2.2. *A visualization of the gas volume fraction in x-t plan. The plot shows the linear trend associated with the ascension velocity of the gas slug. Ultimately, all the gas will be localized in a pure single-phase gas region at the top since the well is closed.*

that the liquid is incompressible, i.e.

$$\rho_l = \rho_{l,0}. \tag{2.3}$$

Typically, we assume that we consider a polytropic, isentropic ideal gas characterized by

$$p(\rho_g) = a_g^2 \rho_g^\gamma, \qquad \gamma \geq 1. \tag{2.4}$$

In other words, we have

$$\rho_g = \left(\frac{p}{a_g^2}\right)^{1/\gamma}, \qquad \gamma \geq 1, \tag{2.5}$$

where $a_g = 316$ [m/s] is the velocity of sound in the gas phase. Furthermore, the viscosity for liquid and gas are assumed to be

$$\mu_l = 5 \cdot 10^{-2} \text{ [Pa s]}, \qquad \mu_g = 5 \cdot 10^{-6} \text{ [Pa s]}. \tag{2.6}$$

Since we only have one momentum equation for the mixture of the two phases, the model must be supplemented with an additional hydrodynamical closure law whose purpose is to determine the fluid velocities u_l, u_g through a so-called slip relation. We may assume that the slip relation can be expressed by a general relation

$$f(\alpha_g, u_l, u_g, \rho_g, \rho_l) = 0. \tag{2.7}$$

A commonly used slip relation, see for example [1, 7], is given by

$$f(\alpha_g, u_l, u_g, \rho_g, \rho_l) = u_g - c_0 u_{mix} - c_1 = 0, \tag{2.8}$$

where

$$u_{mix} = \alpha_l u_l + \alpha_g u_g,$$

and c_0, c_1 are flow dependent coefficients. c_0 is the so-called profile parameter (or distribution coefficient) whereas c_1 is the drift velocity. The gas concentration tends to be highest in the center of the well for many flow scenarios, where the local mixture velocity is also fastest. Thus, when integrated across the area of the the well, the average velocity of the gas tends to be greater than that of the liquid. This effect is represented by the c_0 parameter. c_1, on the other hand, represents the buoyancy effect. Important characteristics of the different flow patterns can be captured through appropriate choices for these two parameters. For the source term q we have two components

$$q = F_f + F_g,$$

where

$$F_g = g(\alpha_l \rho_l + \alpha_g \rho_g)\sin\theta \tag{2.9}$$

represents the gravity force where g is the gravitational constant and θ is the inclination. Moreover, F_f represents friction forces between the wall and the fluids. Typically, see for example [7] and references therein, the following simple expression for F_f is assumed

$$F_f = -\frac{32 u_{mix}|u_{mix}|\mu_{mix}}{d^2}, \tag{2.10}$$

where d is the inner diameter and the mixed viscosity μ_{mix} is given by

$$\mu_{mix} = \alpha_l \mu_l + \alpha_g \mu_g,$$

where the viscosity μ_l, μ_g are given by (2.6). In order to see how pressure p is related to the masses $m = \alpha_l \rho_l$ and $n = \alpha_g \rho_g$ we observe that the relation (2.1) can be written as

$$\frac{n}{\rho_g(p)} + \frac{m}{\rho_l(p)} = 1. \tag{2.11}$$

Using this, we can express the pressure p as a function P of n and m, i.e.

$$p = P(n, m). \tag{2.12}$$

In particular, for the choices (2.2) and (2.5) with $\gamma = 1$, we see that (2.12) corresponds to solving a second order polynomial which has a unique physical relevant solution. More generally, for $\gamma > 1$ the existence and uniqueness of solutions leading to a well-defined pressure p require finer investigations. See for example [14, 15] and references therein for more information.

2.2. The ascent of a gas slug in the context of well control operations. Various gas kick simulators have been developed for the purpose of studying well control aspects during exploratory and development drilling subject to high pressure and temperature bottomhole conditions. Precise predictions of wellbore pressures, liquid/gas volumes as well as flow rates at the top of the well represent central issues. The Deepwater Horizon oil spill that took place in 2010 is a strong reminder of the need of sufficient well control. Clearly, the possibility of blowout occurrences needs to be mitigated in order to avoid human casualties, financial losses (production stop and equipment losses), and finally but not least, environmental damage. We refer to [1] and references therein for more information pertaining to this subject. In particular, in [1] the simulations are based on the drift-flux model (1.1) equipped with density-pressure relations similar to those used in the present work as well as a slip law that is based on the formulation (2.8).

In the following we consider two different examples which involve the ascent of a gas slug initially located at the bottom of a 100 m deep well; the first example assumes that the well is closed at the top, whereas the second example assumes that the well is open at the top. The wellbore has a diameter of $d = 0.06$ cm, otherwise we use the data as described in Section 2.1. In particular, we use a slip relation (2.8) with c_0 and c_1 defined as

$$c_0 = 1.2 - 0.2\alpha_g, \qquad c_1 = 2(0.2 + \alpha_g)(1 - \alpha_g). \tag{2.13}$$

We have also used $\gamma = 1$ in (2.4) which implies that the pressure (2.12) is obtained as the solution of a second order polynomial. We rely on the numerical methods presented in [6, 7] for the following numerical examples.

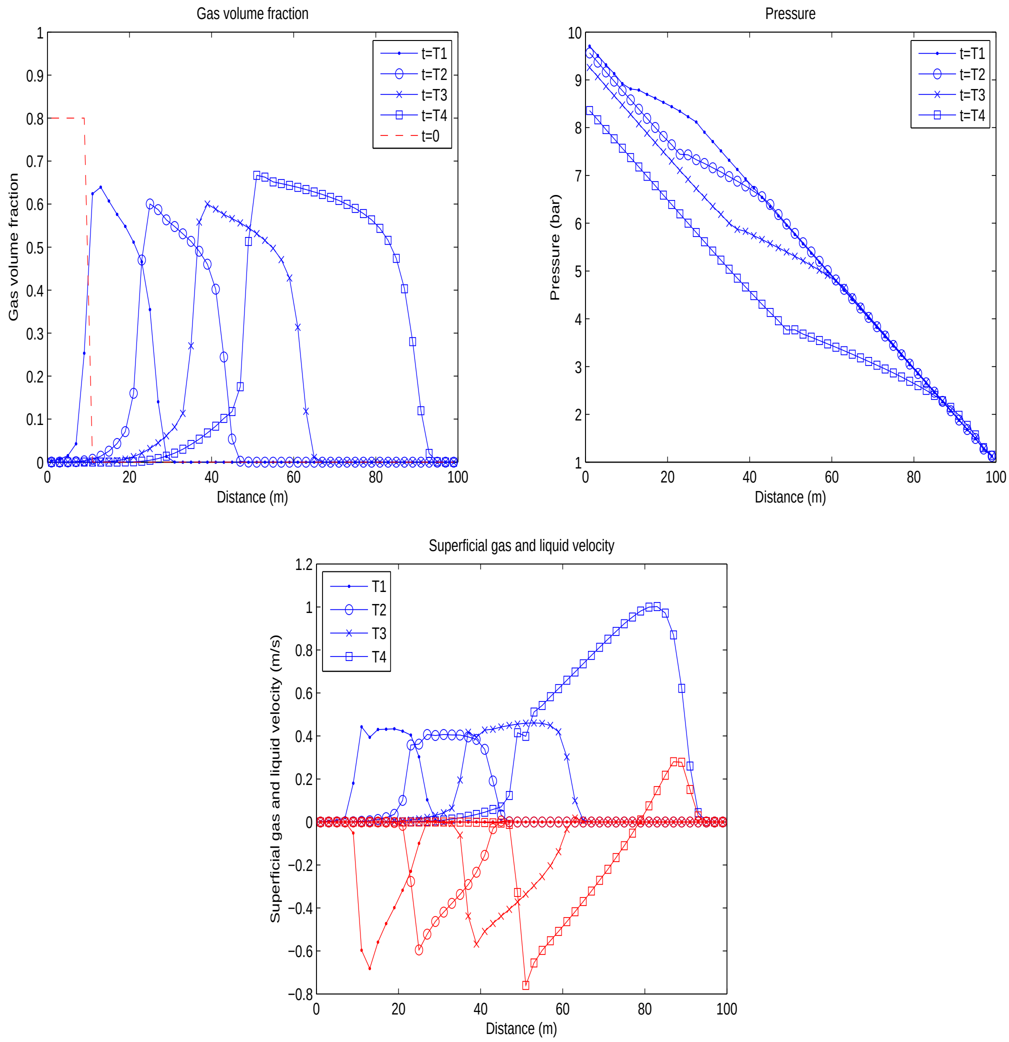

FIG. 2.3. **Top:** *The gas volume fraction (left plot) reflects the strong expansion effect as the gas slug is approaching the surface where the pressure is equal to ambient pressure. Note the drop in pressure (right plot) as the gas slug is approaching the top. Note also the viscous effect associated with the falling film of liquid that surrounds the gas slug that leads to a lower pressure gradient locally in the slug region.* **Bottom:** *The strong expansion of gas close to the open top leads to a strong increase in the gas and liquid superficial velocities.*

Example 1: Gas slug in a closed well. In this example we consider the ascent of a gas slug initially located at the bottom of a 100 m deep well. The well is closed at the bottom as well as at the top. Due to the slip law that is used, the gas slug will immediately start ascending due to the fact that the heavy liquid falls towards the bottom. We refer to Fig. 2.1 for a visualization of the gas volume fraction α_g, pressure p and superficial velocities $(\alpha_g v_g)$ and $(\alpha_l v_l)$ for different times. Finally, the gas will be accumulated at the top of the closed well. The form of the gas slug as it ascends (length, height, and shape) is strongly related to the slip coefficients c_0 and c_1 given by (2.13). See also Fig. 2.2 for a plot of the gas volume fraction in space and time.

Example 2: Gas slug in an open well. In Fig. 2.3 we consider the same flow case as in Example 1, except that the well now is open at the top. This implies that the pressure is kept fixed at the ambient pressure (1 bar) at the top. As the gas slug ascends the drop in pressure results in an expansion effect clearly seen from the plot of the gas volume fraction (top, left). At the same time there will be a rather strong increase in fluid velocities (bottom). This simulation case gives

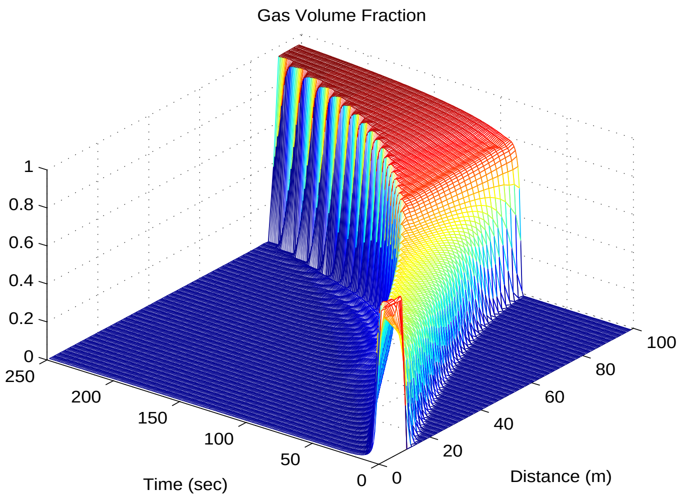

FIG. 2.4. *A visualization of the gas volume fraction in x-t plan. The plot shows the strong expansion effect as the slug is approaching the surface.*

an indication of the driving mechanisms as a gas slug approaches the surface and expands. See also Fig. 2.4 for a plot of the gas volume fraction in space and time.

3. An existence result for a viscous gas-liquid model relevant for well operations. Development of accurate and robust discretization techniques for solving the system (1.1) is naturally related to a good understanding of its mathematical features (long-time behavior, estimates of various quantities, compactness, etc.). In particular, clearly it is of interest to obtain existence, stability, and uniqueness results of various versions of the model (1.1).

3.1. The gas-liquid model. In this work we apply the same simplifying assumptions as used in the previous works [9, 10, 34].

(i) We use a simplified momentum equation by neglecting the gas-related terms. This is motivated by the fact that the liquid density ρ_l is much higher than the gas density ρ_g, typically, $\rho_l/\rho_g = O(1000)$. The mixture momentum equation we consider is then in the form

$$\partial_t[mu_l] + \partial_x[mu_l^2 + p(n,m)] = F_f + F_g + \partial_x[\varepsilon(m)\partial_x u_l] \tag{3.1}$$

where, in view of (2.9) and (2.10),

$$F_f = -\frac{32(\alpha_l\mu_l)}{d^2}(\alpha_l u_l)|\alpha_l u_l| := -fm^3 u_l|u_l|, \qquad F_g = g(\alpha_l\rho_l)\sin\theta := gm,$$

for appropriate constants $f, g > 0$. Here we use the fact that the liquid density ρ_l is constant. We consider the model in a domain $[a(t), b]$ such that the positive direction coincides with the direction of the gravity force. The left point $x = a(t)$ is moving whereas the right point $x = b$ is fixed. This is mostly motivated by the fact that we want to make use of a mathematical framework similar to that employed in [33].

(ii) We assume that the model is used for a no-slip flow regime, i.e., $u_g = u_l = u$, which corresponds to the choice $c_0 = 1$ and $c_1 = 0$ in (2.8).

Hence, we consider the model

$$\begin{aligned} \partial_t n + \partial_x[nu] &= 0 \\ \partial_t m + \partial_x[mu] &= 0 \\ \partial_t[mu] + \partial_x[mu^2 + p(n,m)] &= -fm^3 u|u| + gm + \partial_x[\varepsilon(m)\partial_x u], \qquad x \in (a(t), b) \end{aligned} \tag{3.2}$$

together with the constitutive relations

$$\alpha_l + \alpha_g = 1, \quad \rho_l = \text{Constant}, \quad P = P(\rho_g), \tag{3.3}$$

where P represents a general pressure law for the gas phase whose properties are specified in Section 3.2, see (3.27)–(3.29). Clearly, P becomes a function of the masses n and m by observing that

$$\rho_g = \frac{n}{\alpha_g} = \frac{n}{1-\alpha_l} = \rho_l \frac{n}{\rho_l - m} = k_1 \frac{n}{\rho_l - m}, \qquad k_1 = \rho_l. \tag{3.4}$$

Here we again take advantage of the fact that the liquid is assumed to be incompressible. Consequently,

$$p(n,m) = P\Big(k_1 \frac{n}{\rho_l - m}\Big). \tag{3.5}$$

To conclude, in view of (3.2)–(3.5), we shall in the rest of this paper deal with the following compressible gas-incompressible liquid two-phase model:

$$\begin{aligned} &\partial_t n + \partial_x[nu] = 0 \\ &\partial_t m + \partial_x[mu] = 0 \\ &\partial_t[mu] + \partial_x[mu^2] + \partial_x p(n,m) = -fm^3u|u| + gm + \partial_x[\varepsilon(m)\partial_x u], \qquad x \in (a(t), b), \end{aligned} \tag{3.6}$$

where

$$p(n,m) = P\Big(k_1 \frac{n}{\rho_l - m}\Big), \tag{3.7}$$

$$\varepsilon(m) = k_2 \frac{m^\theta}{(\rho_l - m)^{\theta+1}}, \qquad \theta \in (0, 1/2), \tag{3.8}$$

where k_1 and k_2 are constants. One special feature of the above two-phase model (3.6)–(3.8) is the possible singularity associated with the pressure law at transition to pure liquid flow, that is, when $m = \rho_l \alpha_l = \rho_l$, or vacuum in the gas phase corresponding to $\rho_g = 0$.

As already mentioned, we here propose to study the model (3.6) in a free boundary setting where the top point (relatively the gravity force) $x = a(t)$ is moving whereas the bottom point $x = b$ is fixed. Note that $x = a(t)$ is the particle path separating the two-phase mixture and the vacuum state $n = m = 0$ and is characterized as follows:

$$\frac{d}{dt}a(t) = u(a(t), t), \qquad \text{and} \qquad [p(n,m) - \varepsilon(m)\partial_x u](a(t), t) = 0. \tag{3.9}$$

Furthermore, the initial data is specified as follows

$$n(x,0) = n_0(x), \qquad m(x,0) = m_0(x), \qquad u(x,0) = u_0(x), \qquad x \in (a_0, b), \tag{3.10}$$

where $a_0 = a(0)$. The boundary condition is set as follows:

$$[p(n,m) - \varepsilon(m)\partial_x u]|_{x=a(t)} = 0, \qquad u|_{x=b} = 0. \tag{3.11}$$

In this work we shall assume that the initial masses $n_0(x), m_0(x)$ connect to vacuum discontinuously, i.e., $\inf_{[0,1]} n_0(x), \inf_{[0,1]} m_0(x) \geq C_0 > 0$ for a positive constant C_0.

Following along the line of previous studies for the single-phase Navier-Stokes equations [29, 22, 24], it is convenient to replace the moving domain $[a(t), b]$ by a fixed domain by introducing suitable Lagrangian coordinates. First, in view of the particle paths $X_t(x)$ given by

$$\frac{dX_t(x)}{dt} = u(X_t(x), t), \qquad X_0(x) = x,$$

the system (3.6) takes the form

$$\begin{aligned} \frac{dn}{dt} + nu_x &= 0 \\ \frac{dm}{dt} + mu_x &= 0 \\ m\frac{du}{dt} + p(n,m)_x &= -fm^3u|u| + gm + (\varepsilon(m)u_x)_x. \end{aligned} \tag{3.12}$$

Next, we introduce the coordinate transformation

$$\xi = \int_{a(t)}^{x} m(y,t)\,dy, \qquad \tau = t, \tag{3.13}$$

such that the free boundary $x = a(t)$ and the fixed boundary $x = b$, in terms of the (ξ, τ) coordinate system, are given by

$$\xi_{a(t)}(\tau) = 0, \qquad \xi_b(\tau) = \int_{a(t)}^{b} m(y,t)\,dy = \int_{a_0}^{b} m_0(y)\,dy = \text{const}, \tag{3.14}$$

where $\int_{a_0}^{b} m_0(y)\,dy$ is the total liquid mass initially, which we normalize to 1. Applying (3.13) to shift from (x,t) to (ξ,τ) in (3.12), we get

$$\begin{aligned} n_\tau + (nm)u_\xi &= 0 \\ m_\tau + (m^2)u_\xi &= 0 \\ u_\tau + p(n,m)_\xi &= -fm^2u|u| + g + (\varepsilon(m)mu_\xi)_\xi, \qquad \xi \in (0,1), \quad \tau \geq 0, \end{aligned}$$

where boundary conditions, in light of (3.11), are given by

$$[p(n,m) - \varepsilon(m)m\partial_\xi u](0,\tau) = 0, \qquad u(1,\tau) = 0.$$

In addition, we have the initial data

$$n(\xi,0) = n_0(\xi), \quad m(\xi,0) = m_0(\xi), \quad u(\xi,0) = u_0(\xi), \qquad \xi \in (0,1).$$

In the following, we find it convenient to replace the coordinates (ξ,τ) by (x,t) such that the model we shall work with in the rest of this paper is given in the form

$$\begin{aligned} \partial_t n + (nm)\partial_x u &= 0 \\ \partial_t m + m^2\partial_x u &= 0 \\ \partial_t u + \partial_x p(n,m) &= -fm^2u|u| + g + \partial_x(E(m)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{3.15}$$

with

$$p(n,m) = P\Big(\frac{n}{\rho_l - m}\Big) \tag{3.16}$$

and

$$E(m) := \varepsilon(m)m = \Big(\frac{m}{\rho_l - m}\Big)^{\theta+1}, \qquad 0 < \theta < 1/2, \tag{3.17}$$

where we, for simplicity, have set the constants k_1, k_2 associated with p and ε to be $k_1 = k_2 = 1$. Moreover, boundary conditions are given by

$$[p(n,m) - E(m)\partial_x u](0,t) = 0, \qquad u(1,t) = 0, \tag{3.18}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x \in (0,1). \tag{3.19}$$

3.2. Main result. Before we state the main result for the model (1.14)–(1.18), we describe the notation we apply throughout the paper. $W^{1,2}(I) = H^1(I)$ represents the usual Sobolev space defined over $I = (0,1)$ with norm $\|\cdot\|_{W^{1,2}}$. Moreover, $L^p(K,B)$ with norm $\|\cdot\|_{L^p(K,B)}$ denotes the space of all strongly measurable, pth-power integrable functions from K to B where K typically is subset of $\mathbb{R}$ and B is a Banach space. In addition, let $C^\alpha[0,1]$ for $\alpha \in (0,1)$ denote the Banach space of functions on $[0,1]$ which are uniformly Hölder continuous with exponent α. Similarly, let $C^{\alpha,\alpha/2}(D_T)$ represent the Banach space of functions on $D_T = [0,1] \times [0,T]$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t.

Assumptions. The above model is subject to the following assumptions.

$$0 < \theta < \frac{1}{2}, \tag{3.20}$$

$$c_0(x) \in L^\infty([0,1]), \qquad \inf_{x\in[0,1]}[c_0(x)] > 0, \qquad (c_0)_x \in L^\infty([0,1]), \tag{3.21}$$

$$Q_0(x) \in L^\infty([0,1]), \quad \inf_{x\in[0,1]}[Q_0(x)] > 0, \quad (Q_0^\theta)_x \in L^2([0,1]), \tag{3.22}$$

$$u_0(x) \in L^\infty([0,1]), \tag{3.23}$$

$$P(c_0Q_0)_x \in L^2([0,1]), \qquad (Q_0^{\theta+1}u_{0,x})_x \in L^2([0,1]). \tag{3.24}$$

The function Q_0 is given by $Q_0 = Q(m_0)$ where $Q(s) = \frac{s}{\rho_l - s}$, and $c_0 = \frac{n_0}{m_0}$. The role of these functions are explained in Section 4.1. As a consequence of assumption (3.22) it is clear that

$$Q_0^{-1}(x) \le C,$$

and consequently, $\int_0^1 Q_0^p dx < \infty$ for $p < 0$. This is used repeatedly in Section 4. Note that the lower and upper bounds of c_0 and Q_0 (as well as bounds on $c_{0,x}$ and $Q_{0,x}$) formulated in (3.21) and (3.22) are satisfied by assuming

$$\inf_{[0,1]} n_0 > 0, \ \sup_{[0,1]} n_0 < \infty, \quad \text{and} \quad \inf_{[0,1]} m_0 > 0, \ \sup_{[0,1]} m_0 < \rho_l, \tag{3.25}$$

and

$$(n_0)_x, (m_0)_x \in L^\infty([0,1]). \tag{3.26}$$

The general pressure function P is assumed to satisfy general conditions similar to those assumed in the single-phase work [33]. More precisely, we assume that P as a function of $s \in \mathbb{R}_0^\infty = [0,\infty)$ satisfies:

$$\int_0^1 \frac{P(s)}{s^2} ds < \infty, \tag{3.27}$$

$$P(0) = 0, \quad P'(0) = 0; \qquad P(s), \quad P'(s), \quad P''(s) > 0, \quad \forall s \in \mathbb{R}^\infty = (0,\infty), \tag{3.28}$$

$$P(s),\ P(s)^2 s^{-1-\theta},\ \frac{P(s)}{s},\ P'(s)s^{1-\theta} \in L^\infty_{\text{loc}}(\mathbb{R}_0^\infty). \tag{3.29}$$

Then we can state the main theorem.

Theorem 3.1 (Main Result). *Under the assumptions* (3.20)–(3.29) *the initial-boundary problem* (1.14)–(1.18) *possesses a global weak solution* (n,m,u) *in the sense that for any* $T > 0$, *(A) we have the following regularity:*

$$
\begin{aligned}
n, m &\in L^\infty([0,1]\times[0,T]) \cap C^1([0,T]; L^2([0,1])),\\
u &\in L^\infty([0,1]\times[0,T]) \cap C^1([0,T]; H^1([0,1])),\\
E(m)u_x &\in L^\infty([0,1]\times[0,T]) \cap C^{\frac{1}{2}}([0,T]; L^2([0,1])).
\end{aligned}
$$

In particular, the following pointwise estimates holds for $\mu > 0$*:*

$$
\begin{aligned}
&\mu^{-1}\inf_{[0,1]}(c_0) \le n(x,t) \le \mu \sup_{[0,1]}(c_0), \qquad c_0 := \frac{n_0}{m_0},\\
&0 < \mu^{-1} \le m(x,t) \le \mu < \rho_l, \qquad \forall (x,t) \in [0,1]\times[0,T],
\end{aligned}
$$

where the positive constant μ *only depends on time* T *and the regularity of the initial data as stated in the assumptions.*

(B) Moreover, the following weak formulation of (1.14) *hold:*

$$
\begin{aligned}
&\int_0^\infty\int_0^1 \Big[n\phi_t - nmu_x\phi\Big]\,dx\,dt + \int_0^1 n_0(x)\phi(x,0)\,dx = 0\\
&\int_0^\infty\int_0^1 \Big[m\varphi_t - m^2u_x\varphi\Big]\,dx\,dt + \int_0^1 m_0(x)\varphi(x,0)\,dx = 0\\
&\int_0^\infty\int_0^1 \Big[u\psi_t + (p(n,m) - E(m)u_x)\psi_x - (fm^2u|u| - g)\psi\Big]\,dx\,dt\\
&\qquad\qquad + \int_0^1 u_0(x)\psi(x,0)\,dx = 0,
\end{aligned}
\tag{3.30}
$$

for any test function $\phi, \varphi, \psi \in C_0^\infty(D)$*, with* $D := \{(x,t)\,|\,0 \le x \le 1,\ t \ge 0\}$.

The proof of Theorem 3.1 is based on a priori estimates for the approximate solutions of (1.14)–(1.18) and a corresponding limit procedure. As in [9, 10] we can obtain pointwise upper and lower limits for m that is transferred also to n. This in turn opens up for all the Lemmas 4.11–4.14, which allow use of standard compactness arguments.

The main idea in the following analysis, which also was employed in the works [9, 10, 34], is to focus on the quantity $Q(m) = m/(\rho_l - m)$ which connects pressure $P(n,m)$ and viscosity coefficient $E(m)$. It turns out that we naturally can reformulate the initial boundary value problem (IBVP) (1.14)–(1.18) described in terms of the variables (n,m,u) into a corresponding IBVP (4.8)–(4.11) described in terms of the variables (c,Q,u) where $c = n/m$. However, the appearance of the friction term $-fm^2u|u|$ requires special care. The following new aspects compared to the works [9, 10, 34] are highlighted:

- Thanks to the fact that $m = \rho_l Q/(1+Q) < \rho_l$ for all $Q \ge 0$, we can directly get the upper bound on Q as described in Lemma 4.2 by means of the energy estimate of Lemma 4.1.
- As far as Lemma 4.3 and Lemma 4.6 are concerned, it turns out that the friction term appears as a non-negative term on the left hand side of the inequality, similar to the energy estimate of Lemma 4.1. The higher order estimate of the velocity u as given by Lemma 4.3 is then employed to control the friction term in Lemma 4.4.
- New arguments must be introduced due to the frictional term to obtain the result of Lemma 4.11. In particular, we must show that $W(t) = \int_0^1 |(h(Q)u)_x|dx$ is in $L^1([0,T])$ for $h(Q)$ given by (4.9).
- The analysis demonstrates that the gravity term and the general pressure function P for the gas-liquid model are handled by techniques similar to those used in [33] for the single-phase Navier-Stokes model.

4. A priori estimates. In this section we first describe how to obtain a more convenient representation of our model. Then we give a series of a priori estimates that will imply existence of weak solutions.

4.1. Transformed models. We introduce the variable

$$c = \frac{n}{m}, \tag{4.1}$$

and see from the first two equations of (1.14) that

$$\partial_t c = \frac{1}{m} n_t - \frac{n}{m^2} m_t = -\frac{nm}{m} u_x + \frac{nm^2}{m^2} u_x = 0.$$

Consequently, the model (1.14)–(1.18) then can be written in terms of the variables (c, m, u) in the form

$$\begin{aligned} &\partial_t c = 0 \\ &\partial_t m + m^2 \partial_x u = 0 \\ &\partial_t u + \partial_x p(c, m) = -f m^2 u|u| + g + \partial_x (E(m)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{4.2}$$

with

$$p(c, m) = P\Big(\frac{mc}{\rho_l - m}\Big) \tag{4.3}$$

and

$$E(m) = \Big(\frac{m}{\rho_l - m}\Big)^{\theta+1}, \qquad 0 < \theta < 1/2. \tag{4.4}$$

Moreover, boundary conditions are given by

$$[p(c, m) - E(m) u_x](0, t) = 0, \qquad u(1, t) = 0, \quad t \geq 0, \tag{4.5}$$

whereas initial data are

$$c(x, 0) = c_0(x), \quad m(x, 0) = m_0(x), \quad u(x, 0) = u_0(x), \qquad x \in (0, 1). \tag{4.6}$$

It is clear from the functions P and E that m must obey an upper limit strong enough to ensure that these functions do not blow up. For that purpose we introduce the quantity $Q(m) = m/(\rho_l - m)$ and deduce a reformulated model in terms of the variables (c, Q, u). That is, we introduce the variable

$$Q(m) = \frac{m}{\rho_l - m} = \frac{\alpha_l}{1 - \alpha_l} > 0, \qquad \text{(which implies that } m = \rho_l \frac{Q}{1+Q}), \tag{4.7}$$

implicitly assuming $m > 0$ and $m < \rho_l$, and observe that

$$\begin{aligned} Q(m)_t = \Big(\frac{m}{\rho_l - m}\Big)_t &= \Big(\frac{1}{\rho_l - m} + \frac{m}{(\rho_l - m)^2}\Big) m_t \\ &= \frac{\rho_l}{(\rho_l - m)^2} m_t = -\rho_l \frac{m^2}{(\rho_l - m)^2} u_x = -\rho_l Q(m)^2 u_x, \end{aligned}$$

in view of the second equation of (4.2). Consequently, we rewrite the model (4.2) in the form

$$\begin{aligned} &\partial_t c = 0 \\ &\partial_t Q + \rho_l Q^2 u_x = 0 \\ &\partial_t u + \partial_x p(c, Q) = -h(Q) u|u| + g + \partial_x (E(Q)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{4.8}$$

with

$$p(c, Q) = P(cQ),$$

and

$$h(Q) = f\rho_l^2\Big(\frac{Q}{1+Q}\Big)^2, \tag{4.9}$$

and

$$E(Q) = Q^{\theta+1}, \qquad 0 < \theta < 1/2.$$

This model is then subject to the boundary conditions

$$[p(c,Q) - E(Q)u_x](0,t) = 0, \qquad u(1,t) = 0, \quad t \geq 0. \tag{4.10}$$

In addition, we have the corresponding initial data

$$c(x,0) = c_0(x), \quad Q(x,0) = \frac{m_0(x)}{\rho_l - m_0(x)}, \quad u(x,0) = u_0(x), \qquad x \in (0,1). \tag{4.11}$$

In particular, the first equation of (4.8) gives that

$$c(x,t) = c_0(x) = \frac{n_0}{m_0}(x) > 0, \qquad t > 0. \tag{4.12}$$

4.2. A priori estimates. We are now ready to establish some important estimates. We let C and $C(T)$ denote generic positive constants depending only on the intial data and the given time T, respectively. In particular, we note from (4.9) that

$$h(Q) \leq C. \tag{4.13}$$

This estimate plays an important role in Lemma 4.2.

Lemma 4.1 (Energy estimate). *Under the assumptions of Theorem 3.1 we have the basic energy estimate*

$$\begin{aligned}&\int_0^1\Big(\frac{1}{2}u^2 + \int_0^Q \frac{P(cs)}{s^2}ds + \frac{gx}{Q}\Big)(x,t)\,dx + \int_0^t\int_0^1 Q^{1+\theta}u_x^2\,dx\,ds + \int_0^t\int_0^1 h(Q)u^2|u|dxds\\ &= \int_0^1\Big(\frac{1}{2}u_0^2 + \int_0^{Q_0}\frac{P(cs)}{s^2}ds + \frac{gx}{Q_0}\Big)\,dx \leq C, \qquad \forall t\in[0,T].\end{aligned} \tag{4.14}$$

Proof. Start by summing equation (4.8)(b) multiplied by $(\frac{P(cQ)}{\rho_l Q^2} - \frac{gx}{\rho_l Q^2})$ with equation (4.8)(c) multiplied by u to obtain

$$\frac{P(cQ)Q_t}{\rho_l Q^2} + P(cQ)u_x - Q_t\frac{gx}{\rho_l Q^2} - gxu_x + uu_t + uP(cQ)_x = u(Q^{1+\theta}u_x)_x + gu - h(Q)u^2|u|. \tag{4.15}$$

Then rewrite equation (4.15) as

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}u^2 + \int_0^Q\frac{P(cs)}{\rho_l s^2}ds + \frac{gx}{\rho_l Q}\Big) + (P(cQ)u)_x - gxu_x = u(Q^{1+\theta}u_x)_x + gu - h(Q)u^2|u|, \tag{4.16}$$

and integrate it over $[0,1]\times[0,t]$ to yield

$$\begin{aligned}&\int_0^1\Big(\frac{1}{2}u^2 + \int_0^Q\frac{P(cs)}{\rho_l s^2}ds + \frac{gx}{\rho_l Q}\Big)dx + \int_0^t\int_0^1 Q^{1+\theta}u_x^2dxds\\ &= \int_0^1\Big(\frac{1}{2}u_0^2 + \int_0^{Q_0}\frac{P(cs)}{\rho_l s^2}ds + \frac{gx}{\rho_l Q_0}\Big)dx + \int_0^t (Q^{1+\theta}uu_x)|_{x=0}^{x=1}ds\\ &- \int_0^t (uP(cQ))|_{x=0}^{x=1}ds + \int_0^t (gxu)|_{x=0}^{x=1}ds - \int_0^t\int_0^1 h(Q)u^2|u|dxds.\end{aligned} \tag{4.17}$$

Now invoking the boundary conditions (4.10) and the assumptions on the initial data we arrive at the conclusion (4.14). □

LEMMA 4.2. *Under the assumptions of Theorem 3.1 we have the pointwise upper bound*

$$Q(x,t) \le C(T), \qquad \forall (x,t) \in [0,1]\times[0,T]. \tag{4.18}$$

Proof. Multiplying equation (4.8)(b) with $\theta Q^{\theta-1}$, we observe that

$$(Q^\theta)_t = -\rho_l \theta Q^{1+\theta} u_x \tag{4.19}$$

We then integrate equation (4.19) over $[0,t]$ and, moreover, equation (4.8)(c) over $[0,x]$, which gives

$$Q^\theta(x,t) = Q_0^\theta(x) - \rho_l\theta \int_0^t (Q^{1+\theta}u_x)(x,s)ds \tag{4.20}$$

and

$$\int_0^x u_t(y,t)dy + P(cQ) - P(cQ(0,t)) + (Q^{1+\theta}u_x)(0,t) = Q^{1+\theta}u_x + gx - \int_0^x h(Q)u|u|dy. \tag{4.21}$$

We further substitute equation (4.21) into equation (4.20), and exploit the boundary conditions such that

$$\begin{aligned} Q^\theta(x,t) + \rho_l\theta \int_0^t P(cQ)(x,s)ds &= Q_0^\theta(x) + \rho_l\theta\Big(\int_0^x u_0(y)dy - \int_0^x u(y,t)dy\Big) \\ &\quad + \rho_l\theta g x t - \rho_l\theta \int_0^t\int_0^x h(Q)u|u|dyds. \end{aligned} \tag{4.22}$$

Now invoking the estimate (4.13), Lemma 4.1, the assumptions (3.22) and (3.23), together with an application of the Cauchy inequality on the third term on the right hand side of (4.22), we arrive at the following estimate:

$$Q^\theta(x,t) + \rho_l\theta \int_0^t P(cQ)(x,s)ds \le C(T) \tag{4.23}$$

for $0 < x < 1$, $0 < t \le T$. But (4.23) implies (4.18), in light of assumption (3.28), and the proof is completed. □

LEMMA 4.3. *Under the assumptions of Theorem 3.1 we have the following higher order estimate for any positive integer m*

$$\int_0^1 u^{2m}dx + m(2m-1)\int_0^t\int_0^1 u^{2m-2}Q^{1+\theta}u_x^2 dxds + 2m\int_0^t\int_0^1 h(Q)u^{2m}|u|dxds \le C(T). \tag{4.24}$$

Proof. Multiply equation (4.8)(c) by u^{2m-1} and integrate it over $[0,1]\times[0,t]$. Then by using integration by parts and employing the boundary conditions, we arrive at

$$\begin{aligned} &\int_0^1 u^{2m}dx + 2m(2m-1)\int_0^t\int_0^1 u^{2m-2}Q^{1+\theta}u_x^2dxds + 2m\int_0^t\int_0^1 h(Q)u^{2m}|u|dxds \\ &= \int_0^1 u_0^{2m}dx + 2m(2m-1)\int_0^t\int_0^1 u^{2m-2}P(cQ)u_xdxds + 2mg\int_0^t\int_0^1 u^{2m-1}dxds. \end{aligned} \tag{4.25}$$

We further apply the Cauchy inequality, multiplying the second integrand on the right hand side of equation (4.25) by the identity $Q^{-\frac{1+\theta}{2}}Q^{\frac{1+\theta}{2}}$, to obtain the estimate

$$\begin{aligned} &\int_0^1 u^{2m}dx + m(2m-1)\int_0^t\int_0^1 u^{2m-2}Q^{1+\theta}u_x^2dxds + 2m\int_0^t\int_0^1 h(Q)u^{2m}|u|dxds \\ &\le \int_0^1 u_0^{2m}dx + m(2m-1)\int_0^t\int_0^1 u^{2m-2}P(cQ)^2Q^{-1-\theta}dxds + 2mg\int_0^t\int_0^1 u^{2m-1}dxds. \end{aligned} \tag{4.26}$$

Moreover, we now make use of the Young's inequality, $ab < \frac{1}{p}a^p + \frac{1}{q}b^q$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q > 1$, for the second and third terms on the right hand side of equation (4.26), respectively. More precisely, using $p = 2m-1, q = \frac{2m-1}{2m-2}$ and $p = 2m, q = \frac{2m}{2m-1}$ successively for the second term, and then $p = 2m, q = \frac{2m}{2m-1}$ for the third term, we get

$$\begin{aligned} & m(2m-1)\int_0^t\int_0^1 u^{2m-2}P(cQ)^2Q^{-1-\theta}dxds \\ & \le m(2m-1)\Big[\int_0^t\int_0^1 \frac{1}{2m-1}[P(cQ)^2Q^{-1-\theta}]^{2m-1}dxds + \int_0^t\int_0^1 \frac{2m-2}{2m-1}u^{2m-1}dxds\Big] \\ & \le C(T) + C(T) + m(2m-2)\int_0^t\int_0^1 \frac{2m-1}{2m}u^{2m}dxds, \end{aligned} \tag{4.27}$$

where we have also used Lemma 4.2 and the assumptions (3.29) on the pressure, to conclude that $P(cQ)^2Q^{-1-\theta} \le C$. Furthermore, we have

$$2mg\int_0^t\int_0^1 u^{2m-1}dxds \le \int_0^t\int_0^1 g^{2m}dxds + (2m-1)\int_0^t\int_0^1 u^{2m}dxds. \tag{4.28}$$

We can then conclude from the equations (4.26), (4.27) and (4.28) that

$$\begin{aligned} & \int_0^1 u^{2m}dx + m(2m-1)\int_0^t\int_0^1 u^{2m-2}Q^{1+\theta}u_x^2dxds + 2m\int_0^t\int_0^1 h(Q)u^{2m}|u|dxds \\ & \le C(T) + m(2m-1)\int_0^t\int_0^1 u^{2m}dxds. \end{aligned} \tag{4.29}$$

Clearly, equation (4.29) implies that

$$\int_0^1 u^{2m}dx \le C(T) + m(2m-1)\int_0^t\int_0^1 u^{2m}dxds, \tag{4.30}$$

and thus $\int_0^1 u^{2m}dx \le C(T)$ by Gronwall's lemma. The conclusion (4.24) then follows directly from equation (4.29). □

Lemma 4.4. *Under the assumptions of Theorem 3.1 we have the following upper bound*

$$\int_0^1 Q^{2\theta-2}Q_x^2\,dx \le C(T). \tag{4.31}$$

Proof. Using equation (4.19) in combination with the momentum equation (4.8)(c) we obtain

$$(Q^\theta)_{xt} = -\theta\rho_l(u_t + P(cQ)_x) + \theta\rho_l g - \theta\rho_l h(Q)u|u|. \tag{4.32}$$

Time-integration of this equation over $[0, t]$ then gives

$$(Q^\theta)_x = (Q_0^\theta)_x - \theta\rho_l(u(x,t) - u_0(x)) - \theta\rho_l\int_0^t P(cQ)_xds + \theta\rho_l gt - \theta\rho_l\int_0^t h(Q)u|u|ds. \tag{4.33}$$

Furthermore, multiply equation (4.33) with $(Q^\theta)_x$ and integrate it with respect to x over $[0,1]$ to obtain

$$\begin{aligned} \int_0^1 [(Q^\theta)_x]^2dx = & \int_0^1 (Q_0^\theta)_x(Q^\theta)_xdx - \theta\rho_l\int_0^1 (u(x,t) - u_0(x))(Q^\theta)_xdx \\ & -\theta\rho_l\int_0^1 (Q^\theta)_x\int_0^t P(cQ)_xdsdx + \theta\rho_l gt\int_0^1 (Q^\theta)_xdx \\ & -\theta\rho_l\int_0^1 (Q^\theta)_x\int_0^t h(Q)u|u|dsdx. \end{aligned} \tag{4.34}$$

We now seek to limit the term $\int_0^1 [(Q^\theta)_x]^2 dx$ on the left side of equation (4.34) by making use of the 'epsilon-version' of Youngs inequality i.e. $ab < \varepsilon a^p + C(\varepsilon) b^q$ for $a, b \geq 0$, $\varepsilon > 0$ and $C(\varepsilon) = (\varepsilon p)^{-\frac{q}{p}} q^{-1}$, on each of the four terms on the right side of (4.34). Using $p = q = \frac{1}{2}$ together with appropriate choices of ε, this leads to the following inequality

$$\begin{aligned} \int_0^1 [(Q^\theta)_x]^2 dx \leq & \frac{1}{10} \int_0^1 [(Q^\theta)_x]^2 dx + C \int_0^1 [(Q_0^\theta)_x]^2 dx \\ & + \frac{1}{10} \int_0^1 [(Q^\theta)_x]^2 dx + C \int_0^1 (u(x,t)^2 + u_0(x)^2) dx \\ & + \frac{1}{10} \int_0^1 [(Q^\theta)_x]^2 dx + C \int_0^1 \Big(\int_0^t |P(cQ)_x| ds \Big)^2 dx \\ & + \frac{1}{10} \int_0^1 [(Q^\theta)_x]^2 dx + Ct^2 + \frac{1}{10} \int_0^1 [(Q^\theta)_x]^2 dx + C \int_0^1 \Big(\int_0^t h(Q) u^2 ds \Big)^2 dx \\ & \leq \frac{1}{2} \int_0^1 [(Q^\theta)_x]^2 dx + C \int_0^1 [(Q_0^\theta)_x]^2 dx + C \int_0^1 \Big(\int_0^t h(Q) u^2 ds \Big)^2 dx \\ & \quad + C \int_0^1 (u(x,t)^2 + u_0(x)^2) dx + C \int_0^1 \Big(\int_0^t |P(cQ)_x| ds \Big)^2 dx + C(T). \end{aligned} \tag{4.35}$$

Using the assumptions on the initial data together with Lemma 4.1, (4.35) can be rephrased as

$$\int_0^1 [(Q^\theta)_x]^2 dx \leq C \int_0^1 \Big(\int_0^t |P(cQ)_x| ds \Big)^2 dx + C \int_0^1 \Big(\int_0^t h(Q) u^2 ds \Big)^2 dx + C(T). \tag{4.36}$$

However, the Hölder inequality implies that

$$\int_0^t |P(cQ)_x| ds \leq C \Big(\int_0^t (P(cQ)_x)^2 ds \Big)^{\frac{1}{2}}, \tag{4.37}$$

and likewise

$$\int_0^t h(Q) u^2 ds \leq C \Big(\int_0^t h(Q)^2 u^4 ds \Big)^{\frac{1}{2}}. \tag{4.38}$$

Moreover, noticing that

$$P(cQ)_x = P'(cQ)(c_x Q + c Q_x) = P'(cQ)(c_x Q + c \frac{1}{\theta} Q^{1-\theta} (Q^\theta)_x), \tag{4.39}$$

and using Fubini's theorem, we get from equation (4.36)–(4.39) that

$$\begin{aligned} \int_0^1 [(Q^\theta)_x]^2 dx \leq & C \int_0^t \int_0^1 [P(cQ)_x]^2 dx ds + C \int_0^t \int_0^1 h(Q)^2 u^4 dx ds + C(T) \\ & \leq C \int_0^t \int_0^1 \Big(P'(cQ)^2 [2(c_x Q)^2 + 2 \frac{c^2}{\theta^2} Q^{2-2\theta} (Q^\theta)_x^2] \Big) dx ds + C(T) \\ & \leq C(T) \int_0^t \int_0^1 [(Q^\theta)_x]^2 dx ds + C(T), \end{aligned} \tag{4.40}$$

where we have also used (4.13), Lemma (4.3) with $m = 2$, Lemma (4.2), assumptions (3.29) and (3.21), as well as the Cauchy inequality. Equation (4.40) then calls for the application of Gronwall's lemma, and the conclusion (4.31) follows since $(Q^\theta)_x = \theta Q^{\theta-1} Q_x$. □

Lemma 4.5. *For any $l > \frac{1}{2m}$, we have the following upper bound*

$$\int_0^1 \frac{x^l}{Q} dx \leq C(T). \tag{4.41}$$

Proof. A simple manipulation of equation (4.8)(b) leads to

$$\Big(\frac{x^l}{Q(x,t)}\Big)_t = x^l \rho_l u_x(x,t). \tag{4.42}$$

We then integrate equation (4.42) over $[0,1]\times[0,t]$ to yield

$$\int_0^1 \frac{x^l}{Q(x,t)} = \int_0^1 \frac{x^l}{Q_0(x,t)} + \rho_l \int_0^t \Big(x^l u(x,s)\Big)\Big|_{x=0}^{x=1} ds - l\rho_l \int_0^t\int_0^1 x^{l-1}u(x,s)dxds. \tag{4.43}$$

Now employing Young's inequality (on the third term and with $p=2m$ and $q=\frac{2m}{2m-1}$), the boundary conditions, and (3.22) we get

$$\int_0^1 \frac{x^l}{Q(x,t)} \le C + C\int_0^t\int_0^1 u^{2m}(x,s)dxds + C\int_0^t\int_0^1 x^{\frac{2m(l-1)}{2m-1}}dxds. \tag{4.44}$$

Finally we observe that the second term is limited due to Lemma 4.3, and the last term is also limited since $m\ge 1$ and $\frac{2m(l-1)}{2m-1} > -1$ for $l > \frac{1}{2m}$. This proves the lemma. □

LEMMA 4.6. *Under the assumptions of Theorem 3.1 and for any integer $m>0$ (sufficiently large) and for $\alpha_1 = (1-\frac{1}{2^m})(\theta-1) < 0$, we have the following upper bound*

$$\int_0^1 Q^{\alpha_1}u^2dx + \int_0^t\int_0^1 Q^{1+\theta+\alpha_1}u_x^2\,dxds + \int_0^t\int_0^1 h(Q)Q^{\alpha_1}u^2|u|dxds \le C(T). \tag{4.45}$$

Proof. First let

$$\alpha_m = \frac{\theta-1}{2}, \tag{4.46}$$

and, moreover, define α_{m-1} as,

$$\alpha_{m-1} = \frac{\alpha_m}{2} + \frac{\theta-1}{2} = \frac{3}{2}\alpha_m. \tag{4.47}$$

It follows from the equations (4.8)(b) and (c) that

$$\begin{aligned}(Q^{\alpha_m}u^{2^m})_t =& -\alpha_m\rho_l Q^{1+\alpha_m}u^{2^m}u_x + 2^m Q^{\alpha_m}u^{2^m-1}(Q^{1+\theta}u_x)_x \\ &+2^m Q^{\alpha_m}u^{2^m-1}g - 2^m Q^{\alpha_m}u^{2^m-1}P(cQ)_x - 2^m h(Q)Q^{\alpha_m}u^{2^m}|u|.\end{aligned} \tag{4.48}$$

We integrate (4.48) over $[0,1]\times[0,t]$, which after application of partial integration and the boundary conditions yields

$$\begin{aligned}&\int_0^1 Q^{\alpha_m}u^{2^m}dx + 2^m(2^m-1)\int_0^t\int_0^1 Q^{1+\theta+\alpha_m}u^{2^m-2}u_x^2dxds \\ &+2^m\int_0^t\int_0^1 h(Q)Q^{\alpha_m}u^{2^m}|u|dxds = \int_0^1 Q_0^{\alpha_m}u_0^{2^m}dx - \alpha_m\rho_l\int_0^t\int_0^1 Q^{1+\alpha_m}u^{2^m}u_xdxds \\ &-2^m\alpha_m\int_0^t\int_0^1 Q^{\theta+\alpha_m}u^{2^m-1}Q_xu_xdxds + 2^m(2^m-1)\int_0^t\int_0^1 P(cQ)Q^{\alpha_m}u^{2^m-2}u_xdxds \\ &+2^m\alpha_m\int_0^t\int_0^1 P(cQ)Q^{\alpha_m-1}u^{2^m-1}Q_xdxds + 2^m g\int_0^t\int_0^1 Q^{\alpha_m}u^{2^m-1}dxds \\ &:= \sum_{i=1}^6 I_i^m \le C(T),\end{aligned} \tag{4.49}$$

where the estimation of I_i^m (for $i = 1, 2, 3, 4, 5, 6$) is given in the Appendix, see (4.89)–(4.94). Obviously, equation (4.49) is also valid for α_{m-1} and $m-1$ (instead of α_m and m and with the exception of the inequality part, which must be proved), and thus we obtain

$$
\begin{aligned}
&\int_0^1 Q^{\alpha_{m-1}} u^{2^{m-1}} dx + 2^{m-1}(2^{m-1}-1) \int_0^t \int_0^1 Q^{1+\theta+\alpha_{m-1}} u^{2^{m-1}-2} u_x^2 dxds \\
&+2^{m-1} \int_0^t \int_0^1 h(Q) Q^{\alpha_{m-1}} u^{2^{m-1}} |u| dxds = \int_0^1 Q_0^{\alpha_{m-1}} u_0^{2^{m-1}} dx \\
&-\alpha_{m-1}\rho_l \int_0^t \int_0^1 Q^{1+\alpha_{m-1}} u^{2^{m-1}} u_x dxds - 2^{m-1}\alpha_{m-1} \int_0^t \int_0^1 Q^{\theta+\alpha_{m-1}} u^{2^{m-1}-1} Q_x u_x dxds \\
&+2^{m-1}(2^{m-1}-1) \int_0^t \int_0^1 P(cQ) Q^{\alpha_{m-1}} u^{2^{m-1}-2} u_x dxds \\
&+2^{m-1}\alpha_{m-1} \int_0^t \int_0^1 P(cQ) Q^{\alpha_{m-1}-1} u^{2^{m-1}-1} Q_x dxds \\
&+2^{m-1} g \int_0^t \int_0^1 Q^{\alpha_{m-1}} u^{2^{m-1}-1} dxds := \sum_{i=1}^{6} I_i^{m-1} \le C(T),
\end{aligned} \tag{4.50}
$$

where the estimation of I_i^{m-1} (for $i = 1, 2, 3, 4, 5, 6$) follows from the estimates in the Appendix, see (4.95)–(4.102). These estimates, in turn, depend on the estimate (4.49). The recurrence relation (4.47) then implies that $\alpha_k = (2 - \frac{1}{2^{m-k}})(\frac{\theta-1}{2})$ for $k = 1, \ldots, m$. In particular, $\alpha_1 = (1 - \frac{1}{2^m})(\theta - 1)$. We can thus conclude by induction that

$$
\int_0^1 Q^{\alpha_1} u^2 dx + \int_0^t \int_0^1 Q^{1+\theta+\alpha_1} u_x^2 dxds + \int_0^t \int_0^1 h(Q) Q^{\alpha_1} u^2 |u| dxds \le C(T), \tag{4.51}
$$

and the proof is completed. □

LEMMA 4.7. *Under the assumptions of Theorem 3.1 and for any integer* $m > 0$ *(sufficiently large) and for* $\beta_1 = (2 - \frac{1}{2^m})(\theta - 1) < 0$*, we have*

$$
\int_0^1 Q^{\beta_1} dx \le C(T). \tag{4.52}
$$

Proof. From equation (4.8)(b) it follows that

$$
(Q^{\beta_1})_t = -\beta_1 \rho_l Q^{1+\beta_1} u_x. \tag{4.53}
$$

Integrate (4.53) over $[0, 1] \times [0, t]$ to obtain

$$
\int_0^1 Q^{\beta_1} dx = \int_0^1 Q_0^{\beta_1} dx - \beta_1 \rho_l \int_0^t \int_0^1 Q^{1+\beta_1} u_x dxds. \tag{4.54}
$$

Furthermore, we can obtain an estimate for $\int_0^1 Q^{\beta_1} dx$ by using the Cauchy inequality

$$
\begin{aligned}
\int_0^1 Q^{\beta_1} dx &\le \int_0^1 Q_0^{\beta_1} dx + C \int_0^t \int_0^1 Q^{\frac{1+\theta+\alpha_1}{2}} u_x Q^{1+\beta_1} Q^{\frac{-(1+\theta+\alpha_1)}{2}} dxds \\
&\le \int_0^1 Q_0^{\beta_1} dx + C \int_0^t \int_0^1 Q^{1+\theta+\alpha_1} u_x^2 dxds + C \int_0^t \int_0^1 Q^{1+2\beta_1-\theta-\alpha_1} dxds.
\end{aligned} \tag{4.55}
$$

Now notice, in view of assumption (3.22), that

$$
\int_0^1 Q_0^{\beta_1} dx \le C.
$$

Moreover,

$$\int_0^t \int_0^1 Q^{1+\theta+\alpha_1} u_x^2 dxds \le C(T)$$

due to Lemma 4.6. Thus by using these two latter facts and the fact that $1 + 2\beta_1 - \theta - \alpha_1 = \beta_1$, equation (4.55) can be written as

$$\int_0^1 Q^{\beta_1} dx \le C(T) + C \int_0^t \int_0^1 Q^{\beta_1} dxds. \tag{4.56}$$

After an application of Gronwall's lemma we arrive at the conclusion (4.52). □

Lemma 4.8. *Under the assumptions of Theorem 3.1 we have the following pointwise lower bound on Q*

$$Q(x,t) \ge C(T), \qquad \forall (x,t) \in [0,1] \times [0,T]. \tag{4.57}$$

Proof. It follows from the Sobolev inequality that

$$Q^{\beta_2}(x,t) \le C \int_0^1 Q^{\beta_2} dx + C \int_0^1 |(Q^{\beta_2})_x| dx. \tag{4.58}$$

Choosing β_2 such that $\beta_2 = \theta + (1 - \frac{1}{2^{m+1}})(\theta - 1)$, and noting that $\frac{\beta_1}{2} = (1 - \frac{1}{2^{m+1}})(\theta - 1)$ then it's clear that

$$\beta_2 = \theta + \frac{\beta_1}{2}, \qquad \beta_2 - \beta_1 = \theta - \frac{\beta_1}{2} > 0.$$

Moreover, for $0 < \theta < \frac{1}{2}$ it is also clear that by choosing m sufficiently large, $\beta_2 < 0$. Some further straightforward manipulations, including application of the Cauchy inequality and Lemma 4.2, then gives

$$\begin{aligned} Q^{\beta_2}(x,t) &\le C \int_0^1 Q^{\beta_2} dx + C \int_0^1 Q^{\beta_2 - 1} |Q_x| dx \\ &\le C \int_0^1 Q^{\beta_1} Q^{\beta_2 - \beta_1} dx + C \int_0^1 Q^{\beta_2 - 1} |Q_x| dx \\ &\le C \max_{x \in [0,1]} (Q^{\beta_2 - \beta_1}) \int_0^1 Q^{\beta_1} dx + C \int_0^1 Q^{2\beta_2 - 2\theta} dx + C \int_0^1 Q^{2\theta - 2} Q_x^2 dx \\ &\le C(T) + C \int_0^1 Q^{\beta_1} dx + C \int_0^1 Q^{2\theta - 2} Q_x^2 dx. \end{aligned} \tag{4.59}$$

Moreover, application of Lemmas 4.7 and 4.4 let us conclude that

$$Q^{\beta_2}(x,t) \le C(T). \tag{4.60}$$

But, since $\beta_2 < 0$, (4.57) follows. □

Corollary 4.9. *Under the assumptions of Theorem 3.1 there is a constant* $\mu > 0$ *such that*

$$\mu^{-1} \le m \le \mu < \rho_l, \qquad \mu^{-1} \inf_{[0,1]} (c_0) \le n \le \mu \sup_{[0,1]} (c_0), \tag{4.61}$$

for $c = c_0 = n_0/m_0$.

Proof. In view of the expression for $Q(m)$ given by (4.7) and the upper bound (4.18) and lower bound (4.57) it is clear that the first estimate of (4.61) follows. The second follows from the first and the fact that $n = c_0 m$. □

Corollary 4.10. *Under the assumptions of Theorem 3.1 we have the estimates*

$$\int_0^1 |m_x|\,dx \le C(T), \qquad \int_0^1 |n_x|\,dx \le C(T), \tag{4.62}$$

for a constant $C = C(T)$.

Proof. It follows that

$$\partial_x Q(m)^\theta = \theta Q(m)^{\theta-1} Q'(m)\partial_x m = \theta\rho_l Q(m)^{\theta-1}\frac{Q(m)^2}{m^2}\partial_x m = \theta\rho_l \frac{Q(m)^{\theta+1}}{m^2}\partial_x m,$$

since $Q'(m) = (\rho_l/m^2)Q(m)^2$. In view of this calculation and the pointwise upper and lower limits for $Q(m)$, as well as m, given respectively by (4.18), (4.57), and (4.61), it follows by application of Lemma 4.4 that the first estimate of (4.62) holds. For the second estimate of (4.62) we note that we have the relation

$$\partial_x n = m\partial_x c_0 + c_0\partial_x m, \qquad \text{since } n = c_0 m.$$

Thus, we estimate as follows:

$$\int_0^1 |\partial_x n|\,dx \le \rho_l \int_0^1 |\partial_x c_0|\,dx + \sup_{[0,1]} c_0 \int_0^1 |\partial_x m|\,dx \le C(T),$$

by the first estimate of (4.62) and the assumption (3.21). □

Lemmas 4.11–4.14 can be proved by following along the lines of [10] which in turn is strongly inspired by works like [36, 37, 38]. In particular, in the next lemmas we for the first time need the additional regularity of assumption (3.24).

Lemma 4.11. *Under the assumptions of Theorem 3.1 we can prove that*

$$\int_0^1 u_t^2\,dx + \int_0^t\int_0^1 Q(m)^{\theta+1} u_{xt}^2\,dx\,ds \le C(T). \tag{4.63}$$

Proof. We differentiate the third equation of (4.8) with respect to time t, multiply the resulting equation by $2u_t$ and integrate over $[0,1]\times[0,t]$, and obtain

$$\begin{aligned} &\int_0^1 u_t^2(x,t)\,dx + 2\int_0^t\int_0^1 [P(cQ)_{xt} - (Q^{\theta+1}u_x)_{xt}]u_t\,dx\,ds \\ &\quad = \int_0^1 u_t^2(x,0)\,dx - 2\int_0^t\int_0^1 (h(Q)u|u|)_t u_t\,dx\,ds. \end{aligned} \tag{4.64}$$

First, it follows that

$$\int_0^1 u_t^2(x,0)\,dx \le C(T), \tag{4.65}$$

by considering the momentum equation of (4.8) at time $t = 0$

$$(u_0)_t + P(cQ_0)_x = -h(Q_0)u_0|u_0| + g + (Q_0^{\theta+1}u_{0,x})_x,$$

together with assumptions (3.21)–(3.23). We also note that

$$\begin{aligned} &\int_0^t\int_0^1 [P(cQ) - (Q^{\theta+1}u_x)]_{xt} u_t\,dx\,ds \\ &= \int_0^t ([P(cQ) - (Q^{\theta+1}u_x)]_t u_t)\Big|_{x=0}^{x=1} ds - \int_0^t\int_0^1 [P(cQ) - (Q^{\theta+1}u_x)]_t u_{xt}\,dx\,ds \\ &= \int_0^t ([P(cQ) - (Q^{\theta+1}u_x)]u_t)_t\Big|_{x=0}^{x=1} ds - \int_0^t ([P(cQ) - (Q^{\theta+1}u_x)]u_{tt})\Big|_{x=0}^{x=1} ds \\ &\qquad - \int_0^t\int_0^1 [P(cQ) - (Q^{\theta+1}u_x)]_t u_{xt}\,dx\,ds \\ &= -\int_0^t\int_0^1 [P(cQ) - (Q^{\theta+1}u_x)]_t u_{xt}\,dx\,ds, \end{aligned} \tag{4.66}$$

by application of the boundary conditions (4.10). Moreover, using the second equation of (4.8) it follows that

$$\begin{aligned}\int_0^t\int_0^1 (Q^{\theta+1}u_x)_t u_{xt}\,dx\,ds &= \int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds - (\theta+1)\rho_l\int_0^t\int_0^1 Q^{\theta+2}u_x^2u_{xt}\,dx\,ds\\ &= \int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds + I_1,\end{aligned} \tag{4.67}$$

and

$$\int_0^t\int_0^1 P(cQ)_t u_{xt}\,dx\,ds = -\rho_l\int_0^t\int_0^1 P'(cQ)cQ^2u_xu_{xt}\,dx\,ds = I_2, \tag{4.68}$$

and

$$\begin{aligned}&\int_0^t\int_0^1 (h(Q)u|u|)_t u_t\,dx\,ds\\ &= -\rho_l\int_0^t\int_0^1 h'(Q)Q^2u_xu|u|u_t\,dx\,ds + 2\int_0^t\int_0^1 h(Q)|u|u_t^2\,dx\,ds = I_3+I_4.\end{aligned} \tag{4.69}$$

Now, we consider how to estimate I_1, I_2, I_3, and I_4. First, we have for I_1

$$|I_1|\le \frac12\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds + C\int_0^t\int_0^1 Q^{\theta+3}u_x^4\,dx\,ds = \frac12\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds + I_{11}, \tag{4.70}$$

where we have used $ab\le\varepsilon a^2+\frac{b^2}{4\varepsilon}$ with $\varepsilon=\frac12$. Similarly, we have for I_2

$$\begin{aligned}|I_2| &\le \frac12\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds + C\int_0^t\int_0^1 P'(cQ)^2c^2Q^{3-\theta}u_x^2\,dx\,ds\\ &= \frac12\int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds + I_{22}.\end{aligned} \tag{4.71}$$

Combining (4.64)–(4.71), we get

$$\int_0^1 u_t^2(x,t)\,dx + \int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds \le C(1+I_{11}+I_{22}+I_3+I_4). \tag{4.72}$$

We then estimate as follows:

$$I_{11} = \int_0^t\int_0^1 Q^{\theta+3}u_x^4\,dx\,ds \le \int_0^t \max(Q^2u_x^2)V(s)\,ds \tag{4.73}$$

where $V(s)=\int_0^1 Q^{\theta+1}u_x^2\,dx$. We observe that the third equation of (4.8) gives

$$Q^{1+\theta}u_x = P(cQ) + \int_0^x u_t(y,t)\,dy - g + \int_0^x h(Q)u|u|\,dy.$$

It follows that

$$\begin{aligned}Q^2u_x^2 &= [Q^{\theta+1}u_x]^2Q^{-2\theta}\\ &= Q^{-2\theta}\Big(\int_0^x u_t\,dy + P(cQ) - g + \int_0^x h(Q)u|u|\,dy\Big)^2\\ &\le C\Big(\int_0^1 u_t^2\,dx + P(cQ)^2 + g^2 + C^2\Big) \le C(T)\int_0^1 u_t^2\,dx + C(T),\end{aligned}$$

by using Lemma 4.1, 4.2, and 4.8. Consequently, we have

$$I_{11} \le C(T)\int_0^t V(s)\,ds + C(T)\int_0^t V(s)\int_0^1 u_t^2\,dx\,ds \le C(T) + C(T)\int_0^t V(s)\int_0^1 u_t^2\,dx\,ds, \tag{4.74}$$

where $V(s) \in L^1([0,T])$ in view of (4.14) of Lemma 4.1. Moreover, we have

$$\begin{aligned} I_{22} = \int_0^t\int_0^1 P'(cQ)^2c^2Q^{3-\theta}u_x^2\,dx\,ds &\le \max(c^{2\theta})\int_0^t (\max[P'(cQ)(cQ)^{1-\theta}])^2V(s)\,ds \\ &\le C(T)\int_0^t V(s)\,ds \le C(T), \end{aligned} \tag{4.75}$$

in view of assumptions (3.21) and (3.29) and Lemma 4.2. Finally, we estimate I_3 and I_4. We have

$$\begin{aligned} |I_3| &\le \max(h'(Q)^2Q^{3-\theta})\int_0^t\int_0^1 Q^{1+\theta}u^4u_x^2\,dx\,ds + \int_0^t\int_0^1 u_t^2\,dx\,ds \\ &\le C(T) + \int_0^t\int_0^1 u_t^2\,dx\,ds, \end{aligned} \tag{4.76}$$

in light of Lemma 4.3. Also

$$|I_4| \le \int_0^t \max(h(Q)|u|)\int_0^1 u_t^2\,dx\,ds. \tag{4.77}$$

Furthermore, we observe that Sobolev embedding theorem gives

$$|h(Q)u| \le \int_0^1 |h(Q)u|\,dx + \int_0^1 |(h(Q)u)_x|\,dx \le C(T) + W(s),$$

by Lemma 4.1. Next, we estimate

$$\begin{aligned} W(s) &= \int_0^1 |(h(Q)u)_x|\,dx = \int_0^1 |h'(Q)Q_xu|\,dx + \int_0^1 |h(Q)u_x|\,dx \\ &\le \int_0^1 h'(Q)^2Q^{2(1-\theta)}u^2\,dx + \int_0^1 Q^{2(\theta-1)}Q_x^2\,dx + \int_0^1 |h(Q)u_x|\,dx \\ &\le C(T) + \int_0^1 |h(Q)^2Q^{-(\theta+1)}|\,dx + \int_0^1 Q^{\theta+1}u_x^2\,dx \\ &\le C(T) + V(s), \end{aligned} \tag{4.78}$$

where we have applied Lemma 4.1, 4.2, and 4.4 and the decay properties of h. Consequently,

$$|I_4| \le \int_0^t \max(h(Q)|u|)\int_0^1 u_t^2\,dx\,ds \le \int_0^t [C(T)+V(s)]\int_0^1 u_t^2\,dx\,ds. \tag{4.79}$$

Using (4.74), (4.75), (4.76), and (4.79) in (4.72) we get

$$\int_0^1 u_t^2(x,t)\,dx + \int_0^t\int_0^1 Q^{\theta+1}u_{xt}^2\,dx\,ds \le C(T) + C(T)\int_0^t [1+V(s)]\int_0^1 u_t^2\,dx\,ds, \tag{4.80}$$

which by application of Gronwall's lemma yields

$$\int_0^1 u_t^2\,dx \le C(T)\exp\Big(C(T)\int_0^t [1+V(s)]\,ds\Big) \le C(T).$$

□

The arguments for the proof of the next lemmas can be directly adopted from [10] combined with arguments similar to those used above for the treatment of the friction term. We state the lemma and refer to [10] for further details.

Lemma 4.12. *Under the assumptions of Theorem 3.1 we have the estimates*

$$\int_0^1 |m_x|\,dx \le C(T), \qquad \int_0^1 |n_x|\,dx \le C(T), \tag{4.81}$$

$$\|Q(m(x,t))^{\theta+1}u_x(x,t)\|_{L^\infty(D_T)} \le C(T), \tag{4.82}$$

$$\int_0^1 |(Q(m)^{\theta+1}u_x)_x(x,t)|\,dx \le C(T), \tag{4.83}$$

for a suitable constant $C(T)$ and where $D_T = [0,1]\times[0,T]$.

The following lemma, which gives the pointwise control on the velocity u, follows essentially by employing the strict lower limit of Q and the estimate of Lemma 4.11. We refer to [10] for details.

Lemma 4.13. *Under the assumptions of Theorem 3.1 it follows that we have the estimates*

$$\int_0^1 |u_x(x,t)|\,dx \le C(T), \qquad \|u(x,t)\|_{L^\infty(D_T)} \le C(T). \tag{4.84}$$

What remains is continuity in time type of estimates in L^2 norm. We just observe that the arguments directly carry over to our model and again refer to [10] for details.

Lemma 4.14. *Under the assumptions of Theorem 3.1, we have for $0 < s \le t \le T$ that*

$$\int_0^1 |m(x,t)-m(x,s)|^2\,dx \le C(T)|t-s|, \tag{4.85}$$

$$\int_0^1 |n(x,t)-n(x,s)|^2\,dx \le C(T)|t-s|, \tag{4.86}$$

$$\int_0^1 |u(x,t)-u(x,s)|^2\,dx \le C(T)|t-s|, \tag{4.87}$$

$$\int_0^1 |Q(m)^{\theta+1}u_x(x,t)-Q(m)^{\theta+1}u_x(x,s)|^2\,dx \le C(T)|t-s|. \tag{4.88}$$

Remark 4.1. *Note that due to the strict estimates of Corollary 4.9 we could get compactness and existence of weak solution without the estimates of Lemmas 4.11–4.14 by using similar arguments as those in [9]. However, the above approach opens for the possibility to treat the case when masses degenerate at the boundaries. In this sense the chosen approach is more general and therefore potentially more interesting. This case is left to another work.*

4.3. Proof of Theorem 3.1. Following standard arguments we can apply the line method as in [10], and formulate a semi-discrete version of the initial-boundary problem (1.14)–(1.18). Semi-discrete version of the various lemmas can be obtained, and in combination with Helly's theorem, the result of Theorem 3.1 follows, see [17, 18, 36, 37, 38, 33] and references therein for details.

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Appendix. In this Appendix we estimate the quantities I_i^m and I_i^{m-1} (for $i = 1, 2, 3, 4, 5, 6$), which are used in the proof of Lemma 4.6. The arguments goes along the line of e.g. [33], which in turn build upon central works like [36, 37, 38]. However, for completeness we include the proof. Note that the equations (4.46) and (4.47) are extensively used throughout these proofs. We start by estimating I_i^m (for $i = 1, 2, 3, 4, 5, 6$).

Estimate for I_1^m. Using the Cauchy inequality, $0 < \theta < \frac{1}{2}$, the relation (4.46), assumptions (3.22) and (3.23), we get

$$I_1^m = \int_0^1 Q_0^{\alpha_m} u_0^{2^m} dx \le C \int_0^1 Q_0^{2\alpha_m} dx + C \int_0^1 u_0^{2^{m+1}} dx \le C(T). \tag{4.89}$$

Estimate for I_2^m. Using the Cauchy inequality, relation (4.46), Lemma 4.1, and Lemma 4.3, we have

$$\begin{aligned} I_2^m &= -\alpha_m \rho_l \int_0^t \int_0^1 Q^{1+\alpha_m} u^{2^m} u_x dxds \\ &\le C \int_0^t \int_0^1 u^{2^{m+1}} dxds + C \int_0^t \int_0^1 Q^{2+2\alpha_m} u_x^2 dxds \\ &\le C \int_0^t \int_0^1 u^{2^{m+1}} dxds + C \int_0^t \int_0^1 Q^{1+\theta} u_x^2 dxds \\ &\le C \int_0^t C(T) ds + C(T) \le C(T). \end{aligned} \tag{4.90}$$

Estimate for I_3^m. Using the Cauchy inequality, relation (4.46), Lemma 4.3, and Lemma 4.4, we have

$$\begin{aligned} I_3^m &= -2^m \alpha_m \int_0^t \int_0^1 Q^{\theta+\alpha_m} u^{2^m-1} Q_x u_x dxds \\ &\le -2^m \alpha_m \int_0^t \int_0^1 |Q^{\frac{\theta}{2}+\frac{1}{2}} u^{2^m-1} u_x Q^{\frac{\theta}{2}-\frac{1}{2}} Q^{\alpha_m} Q_x| dxds \\ &\le C \int_0^t \int_0^1 Q^{\theta+1} u^{2^{m+1}-2} u_x^2 dxds + C \int_0^t \int_0^1 Q^{\theta+2\alpha_m-1} Q_x^2 dxds \\ &\le C \int_0^t \int_0^1 Q^{\theta+1} u^{2^{m+1}-2} u_x^2 dxds + C \int_0^t \int_0^1 Q^{2\theta-2} Q_x^2 dxds \le C(T). \end{aligned} \tag{4.91}$$

Estimate for I_4^m. Using the Cauchy inequality, relation (4.46), Lemma 4.1, Lemma 4.2, assumptions (3.21) and (3.29), and Lemma 4.3, we have

$$\begin{aligned} I_4^m &= 2^m (2^m - 1) \int_0^t \int_0^1 P(cQ) Q^{\alpha_m} u^{2^m-2} u_x dxds \\ &\le C \int_0^t \int_0^1 u^{2^{m+1}-4} dxds + C \int_0^t \int_0^1 [P(cQ)]^2 Q^{2\alpha_m} u_x^2 dxds \\ &\le C \int_0^t \int_0^1 u^{2^{m+1}-4} dxds + C \int_0^t \Big[\Big(\frac{P(cQ)}{Q}\Big)^2\Big]_{\max_{x\in[0,1]}} \int_0^1 Q^{\theta+1} u_x^2 dxds \\ &\le C \int_0^t C(T) ds + C(T) \le C(T). \end{aligned} \tag{4.92}$$

Estimate for I_5^m. Using the Cauchy inequality, relation (4.46), assumptions (3.21) and (3.29), Lemma 4.2, Lemma 4.3 and Lemma 4.4, we have

$$
\begin{aligned}
I_5^m &= 2^m \alpha_m \int_0^t \int_0^1 P(cQ) Q^{\alpha_m - 1} u^{2^m - 1} Q_x \, dxds \\
&\le C \int_0^t \int_0^1 u^{2^{m+1}-2} dxds + C \int_0^t \int_0^1 [P(cQ)]^2 Q^{2\alpha_m - 2} Q_x^2 \, dxds \\
&\le C \int_0^t \int_0^1 u^{2^{m+1}-2} dxds + C \int_0^t \int_0^1 [P(cQ)]^2 Q^{-1-\theta} Q^{2\theta-2} Q_x^2 \, dxds \\
&\le C \int_0^t \int_0^1 u^{2^{m+1}-2} dxds + C \int_0^t \Big([P(cQ)]^2 Q^{-1-\theta} \Big)_{\max_{x\in[0,1]}} \int_0^1 Q^{2\theta-2} Q_x^2 \, dxds \\
&\le C \int_0^t C(T) ds + C \int_0^t C(T) ds \le C(T).
\end{aligned}
\tag{4.93}
$$

Estimate for I_6^m. Using the Cauchy inequality, relation (4.46), Lemma 4.3, Lemma 4.5, and Young's inequality (with $p = \frac{1}{1-\theta}$ and $q = \frac{1}{\theta}$), we have

$$
\begin{aligned}
I_6^m &= 2^m g \int_0^t \int_0^1 Q^{\alpha_m} u^{2^m - 1} dxds \\
&\le C \int_0^t \int_0^1 Q^{2\alpha_m} dxds + C \int_0^t \int_0^1 u^{2^{m+1}-2} dxds \\
&\le C \int_0^t \int_0^1 Q^{\theta - 1} dxds + C \int_0^t \int_0^1 u^{2^{m+1}-2} dxds \\
&\le C \int_0^t \int_0^1 \Big(\frac{x^l}{Q} \Big)^{1-\theta} x^{(\theta-1)l} dxds + C \int_0^t \int_0^1 u^{2^{m+1}-2} dxds \\
&\le C \int_0^t \int_0^1 \Big(\frac{x^l}{Q} \Big) dxds + C \int_0^t \int_0^1 x^{\frac{(\theta-1)l}{\theta}} dxds + C \int_0^t \int_0^1 u^{2^{m+1}-2} dxds \le C(T).
\end{aligned}
\tag{4.94}
$$

Note that for the estimate of the term $\int_0^t \int_0^1 x^{\frac{(\theta-1)l}{\theta}} dxds$, implicitly we assume that for a given $\theta \in (0, \frac{1}{2})$ we set l small enough to ensure that $\theta + (\theta - 1)l > 0$. At the same time l must satisfy the condition $l > 1/(2m)$ of Lemma 4.5. In other words, m must be chosen large enough.

Next we estimate I_i^{m-1} (for $i = 1, 2, 3, 4, 5, 6$). In particular, we shall make use of the estimate (4.49) corresponding to $\sum_{i=1}^6 I_i^m \le C$.

Estimate for I_1^{m-1}. This follows by the same arguments as for I_1^m.

$$
I_1^{m-1} \le C(T).
\tag{4.95}
$$

Estimate for I_2^{m-1}. Using the Cauchy inequality, relation (4.46) and (4.47), Lemma 4.1 and the inequality (4.49), we have

$$
\begin{aligned}
I_2^{m-1} &= -\alpha_{m-1} \rho_l \int_0^t \int_0^1 Q^{1+\alpha_{m-1}} u^{2^{m-1}} u_x \, dxds \\
&= -\alpha_{m-1} \rho_l \int_0^t \int_0^1 |Q^{1+\alpha_{m-1} - \frac{\alpha_m}{2}} u^{2^{m-1}} u_x Q^{\frac{\alpha_m}{2}}| dxds \\
&\le C \int_0^t \int_0^1 Q^{\alpha_m} u^{2^m} dxds + C \int_0^t \int_0^1 Q^{1+\theta} u_x^2 \, dxds \le C(T).
\end{aligned}
\tag{4.96}
$$

Estimate for I_3^{m-1}. Using the Cauchy inequality, relation (4.46) and (4.47), Lemma 4.4, and the inequality (4.49), we have

$$
\begin{aligned}
I_3^{m-1} &= -2^{m-1}\alpha_{m-1}\int_0^t\int_0^1 Q^{\theta+\alpha_{m-1}}u^{2^{m-1}-1}Q_xu_xdxds \\
&= -2^{m-1}\alpha_{m-1}\int_0^t\int_0^1 \Big|Q^{\frac{\theta}{2}+\frac{1}{2}+\frac{\alpha_m}{2}}u^{2^{m-1}-1}Q_xu_xQ^{\alpha_{m-1}+\frac{\theta}{2}-\frac{1}{2}-\frac{\alpha_m}{2}}\Big|dxds \\
&\leq C\int_0^t\int_0^1 Q^{1+\theta+\alpha_m}u^{2^m-2}u_x^2dxds + C\int_0^t\int_0^1 Q^{2\theta-2}Q_x^2dxds \leq C(T).
\end{aligned}
\tag{4.97}
$$

Estimate for I_4^{m-1}. Using the Cauchy inequality and relation (4.47) we obtain

$$
\begin{aligned}
I_4^{m-1} &= 2^{m-1}(2^{m-1}-1)\int_0^t\int_0^1 P(cQ)Q^{\alpha_{m-1}}u^{2^{m-1}-2}u_xdxds \\
&\leq 2^{m-1}(2^{m-1}-1)\int_0^t\int_0^1 \Big|Q^{\frac{\alpha_m}{2}}P(cQ)Q^{\alpha_{m-1}}Q^{\frac{-\alpha_m}{2}}u^{2^{m-1}-2}u_x\Big|dxds \\
&\leq C\int_0^t\int_0^1 Q^{\alpha_m}dxds + C\int_0^t\Big[\frac{P(cQ)}{Q}\Big]^2_{\max_{x\in[0,1]}}\int_0^1 u^{2^m-4}Q^{\theta+1}u_x^2dxds \\
&:= C_1 + C_2 \leq C(T).
\end{aligned}
\tag{4.98}
$$

The argument for the last line in (4.98) goes as follows. Considering C_1 first and using Young's inequality (with $p=\frac{2}{1-\theta}$ and $q=\frac{2}{1+\theta}$), Lemma 4.5 and the assumption that $0<\theta<\frac{1}{2}$, we see that

$$
\begin{aligned}
C_1 &= C\int_0^t\int_0^1 Q^{\alpha_m}dxds = C\int_0^t\int_0^1 Q^{\frac{\theta-1}{2}}dxds = C\int_0^t\int_0^1 \Big(\frac{x^l}{Q}\Big)^{\frac{1-\theta}{2}}x^{\frac{\theta-1}{2}l}dxds \\
&\leq C\int_0^t\int_0^1 \frac{x^l}{Q}dxds + C\int_0^t\int_0^1 x^{(\frac{\theta-1}{\theta+1})l}dxds \leq C(T),
\end{aligned}
\tag{4.99}
$$

for an appropriate choice of l. The estimate of C_2 follows directly from assumption (3.21) and (3.29) and Lemma 4.3 for $m\geq 2$.

Estimate for I_5^{m-1}. After multiplying the integrand with the identity $1=Q^{-\frac{\alpha_m}{2}}Q^{\frac{\alpha_m}{2}}$ and applying of the Cauchy inequality twice we obtain

$$
\begin{aligned}
I_5^{m-1} &= 2^{m-1}\alpha_{m-1}\int_0^t\int_0^1 P(cQ)Q^{\alpha_{m-1}-1}u^{2^{m-1}-1}Q_xdxds \\
&\leq C\int_0^t\int_0^1 Q^{\alpha_m}u^{2^m-2}dxds + C\int_0^t\int_0^1 P(cQ)^2Q^{2\alpha_{m-1}-2-\alpha_m}Q_x^2dxds \\
&\leq C\int_0^t\int_0^1 Q^{\alpha_m}u^{2^m-2}dxds + C\int_0^t\Big[P(cQ)^2Q^{-1-\theta}\Big]_{\max_{x\in[0,1]}}\int_0^1 Q^{2\theta-2}Q_x^2dxds \\
&\leq C\int_0^t\int_0^1 Q^{2\alpha_m}dxds + C\int_0^t\int_0^1 u^{2^{m+1}-4}dxds \\
&\quad + C\int_0^t\Big[P(cQ)^2Q^{-1-\theta}\Big]_{\max_{x\in[0,1]}}\int_0^1 Q^{2\theta-2}Q_x^2dxds \leq C(T).
\end{aligned}
\tag{4.100}
$$

The last inequality in (4.100) is explained as follows. Using Young's inequality (with $p=\frac{1}{1-\theta}$ and $q=\frac{1}{\theta}$), Lemma 4.5 and the assumption that $0<\theta<\frac{1}{2}$, we can estimate the first term as follows

$$
\begin{aligned}
C\int_0^t\int_0^1 Q^{2\alpha_m}dxds &= C\int_0^t\int_0^1 Q^{\theta-1}dxds = C\int_0^t\int_0^1 \Big(\frac{x^l}{Q}\Big)^{1-\theta}x^{(\theta-1)l}dxds \\
&\leq C\int_0^t\int_0^1 \frac{x^l}{Q}dxds + C\int_0^t\int_0^1 x^{(\frac{\theta-1}{\theta})l}dxds \leq C(T),
\end{aligned}
\tag{4.101}
$$

for an appropriate choice of l. Moreover, using assumption (3.29), the two last terms in (4.100) is limited by Lemma 4.3 and Lemma 4.4, respectively.

Estimate for I_6^{m-1}. Using Lemma 4.3, Lemma 4.5, and Young's inequality twice (with $p = \frac{2n}{2n-1}$ and $q = 2n$, where n is an integer and $p = \frac{-(2n-1)}{2n\alpha_{m-1}}$ and $q = \frac{1}{1+\frac{2n\alpha_{m-1}}{2n-1}}$, respectively), we have

$$
\begin{aligned}
& I_6^{m-1} \\
&= 2^{m-1} g \int_0^t \int_0^1 Q^{\alpha_{m-1}} u^{2^{m-1}-1} dxds \\
&\le C \int_0^t \int_0^1 u^{2n(2^{m-1}-1)} dxds + C \int_0^t \int_0^1 Q^{\frac{2n\alpha_{m-1}}{2n-1}} dxds \qquad (4.102) \\
&= C \int_0^t \int_0^1 u^{2n(2^{m-1}-1)} dxds + C \int_0^t \int_0^1 \left(\frac{x^l}{Q}\right)^{\frac{-2n\alpha_{m-1}}{2n-1}} x^{\frac{2n\alpha_{m-1} l}{2n-1}} dxds \\
&\le C \int_0^t \int_0^1 u^{2n(2^{m-1}-1)} dxds + C \int_0^t \int_0^1 x^{\frac{\frac{2n}{2n-1}\alpha_{m-1} l}{1+\frac{2n\alpha_{m-1}}{2n-1}}} dxds + C \int_0^t \int_0^1 \frac{x^l}{Q} dxds \le C(T),
\end{aligned}
$$

where n is chosen large enough and $l > \frac{1}{2m}$ is chosen small enough such that $\frac{\frac{2n}{2n-1}\alpha_{m-1} l}{1+\frac{2n\alpha_{m-1}}{2n-1}} > -1$.