A hyperbolic-elliptic model for coupled well-porous media flow

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1 Introduction

A natural model for well-porous media flow is obtained by coupling a hyperbolic system of two conservation laws corresponding to the isothermal Euler equations with source terms, and an integral equation. In dimensionless form it is given by

$$\begin{aligned} &\partial_t(\rho) + \partial_x(\rho u) = \frac{1}{\eta}\rho q_{\mathrm{V}}, \qquad \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p(\rho) = 0, \qquad \eta > 0, \\ &p_0 - p(x,t) = \int_0^t \int_0^1 H^r(x,x',t-t') q_{\mathrm{V}}(x',t')\, dx' dt', \end{aligned} \tag{1}$$

for $x \in [0,1]$. ρ, u, and $p(\rho)$ are, respectively, density, velocity, and pressure, whereas q_{V} represents volumetric flow rate. p_0 is initial pressure (assumed to be constant) and η is a small parameter. The kernel $H^r(x,x',t-t')$ is characteristic for the porous media under consideration.

In order to get some understanding of basic underlying mechanisms present in the model (1), we assume that the fluid is incompressible. We then get a scalar conservation law on the form

$$\partial_t u + \partial_x(u^2) = -\partial_x p, \qquad p_0 - p(x,t) = \varepsilon \int_{-\infty}^{+\infty} G^r(x,x') u_{x'}(x',t)\, dx', \tag{2}$$

$$\text{with} \qquad \varepsilon = \frac{\mu D}{4\rho k}, \qquad G^r(x,x') = \frac{r^2}{\sqrt{(x-x')^2 + r^2}}, \qquad r > 0, \tag{3}$$

where μ is fluid viscosity, k is permeability, D is a characteristic time, r the well radius (which typically is small relatively the size of the porous media). We may write (2) on the form

$$\partial_t u + \partial_x(u^2) = \varepsilon G^r_x * u_x = \varepsilon G^r_{xx} * u, \qquad \varepsilon, r > 0. \tag{4}$$

Various properties of the model (4) was studied in [EK06]. In particular, wellposedness was demonstrated, respectively, in a L^∞ and L^2 setting. The purpose of this note is to replace the kernel $G^r(x,x')$ with an approximation $\widetilde{G}^r(x,x')$ such that $\|G^r_x - \widetilde{G}^r_x\|_{L^1(\mathbb{R})} = O(r)$, and then derive various estimates for the *approximate* well-reservoir model

$$\partial_t u + \partial_x(u^2) = \varepsilon \widetilde{G}^r_x * u_x, \qquad \varepsilon, r > 0. \tag{5}$$

These estimates, which imply existence and uniqueness of entropy weak solutions, are sharper than those presented in [EK06]. This is essentially due to the fact that the approximate kernel function $\widetilde{G}^r$ leads to a source term possessing a dissipative nature. In that respect, the approximate well-reservoir model (5) bears a clear link to the so-called radiating gas model, studied by many researchers more lately [I97, KN99, N00, LT01, LM03, S03]. This model can be written on the form

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = \partial_x p, \quad p(x,t) = \int_{-\infty}^{+\infty} H(x,x') u_{x'}(x',t)\,dx' = H * u_x, \tag{6}$$

with $H(x,x') = \frac{1}{2}e^{-|x-x'|}$. Alternatively, one can express the model on the form $\partial_t u + \partial_x(\frac{1}{2}u^2) = H_{xx} * u = [H-\delta] * u = H * u - u$, where δ represents the Dirac delta function. In this note we shall see that the approximate model (5) possesses a similar formulation. Nevertheless, there is also a clear difference between (5) and (6) since the kernel H corresponding to the latter is associated with the differential operator $(1-\partial^2_{xx})$. Consequently, (6) can be written on the form

$$u_t + uu_x = -p_x, \qquad -p_{xx} + p = -u_x. \tag{7}$$

The fact that (6) can be written as a hyperbolic-elliptic coupled system on the form (7), is explicitly used, for example in traveling wave analysis [KN99, N00, S03].

2 Mathematical models

Porous media flow. Darcy's law and the continuity equation for flow in porous medium can be combined to give a transient pressure equation [B88]

$$c\phi \frac{\partial p}{\partial t} - \nabla \cdot \left[\frac{k}{\mu}\nabla p\right] = Q_{\mathrm{vol}}(\mathbf{x},t). \tag{8}$$

Here p is pressure, ϕ porosity, μ viscosity whereas $Q_{\mathrm{vol}}(\mathbf{x},t)$ accounts for the mass flow between well and porous media. We assume that the the porosity ϕ and compressibility c is constant. Furthermore, let $\mathbf{X}_w(s) = (x_w(s), y_w(s), z_w(s))$ with $s \in [0,1]$ (dimensionless) be a parametrization of the line Γ_w describing the well path. The source term $Q_{\mathrm{vol}}(\mathbf{x},t)$ represents a delta function singularity along the well path Γ_w given by

$$Q_{\mathrm{vol}}(\mathbf{x},t) = \int_{\Gamma_w} q_{\mathrm{V}}(\alpha,t)\delta(\mathbf{x}-\mathbf{X}_w(\alpha))\,d\alpha, \tag{9}$$

where $\delta(\mathbf{x})$ is a three-dimensional Dirac function $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$, $q_{\mathrm{V}}(\alpha,t)$ the volumetric influx or efflux rate per unit wellbore length, and α denotes the arc-length function. In the following we restrict ourselves to a straight line well geometry of length L_w. We also assume that Ω is a cube of length L.

In terms of dimensionless variables the pressure equation (8) takes the form (see [EK06] for details)

$$\frac{\partial p}{\partial t} - \Big[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\Big] = Q_{\mathrm{vol}}(\mathbf{x},t), \tag{10}$$

where $(\mathbf{x},t) = (x,y,z,t) \in \Omega \times [0,T]$. In the following we shall apply the integral formulation of (10).

$$p_0(\mathbf{x}) - p(\mathbf{x},t) = \int_0^t \int_0^1 G(\mathbf{x},\mathbf{X}_w(s'),t-t')q_{\mathrm{V}}(s',t')\,ds'dt', \tag{11}$$

where G is the free-space kernel $G(\mathbf{x},\mathbf{x}',t-t') = \frac{1}{[4\pi(t-t')]^{3/2}}\exp\Big[-\frac{\|\mathbf{x}-\mathbf{x}'\|^2}{4(t-t')}\Big]$. By setting $\mathbf{x} = \mathbf{X}_w(s)+\mathbf{r}_w$ for $s \in [0,1]$ in (11), we note that $q_{\mathrm{V}}(s',t')$ satisfies the integral equation

$$\Delta p(\mathbf{X}_w(s)+\mathbf{r}_w,t) = \int_0^t \int_0^1 G(s,s',t-t')q_{\mathrm{V}}(s',t')\,ds'dt'. \tag{12}$$

Here $\Delta p(\mathbf{X}_w(s)+\mathbf{r}_w,t) = p_0(\mathbf{X}_w(s)+\mathbf{r}_w) - p(\mathbf{X}_w(s)+\mathbf{r}_w,t)$ represents the change in pressure at the well boundary. Equation (12) is an integral equation of first kind, Fredholm in space and Volterra in time.

A simplified model is obtained by assuming that the fluid in the porous media is incompressible. Then the pressure is given by

$$-\nabla\cdot\Big[\frac{k}{\mu}\nabla p\Big] = Q_{\mathrm{vol}}(\mathbf{x},t), \tag{13}$$

where Q_{vol} is given by (9). Following the approach as described above, we arrive at the following integral equation

$$\Delta p(\mathbf{X}_w(s)+\mathbf{r}_w,t) = \int_0^1 G(\mathbf{X}_w(s)+\mathbf{r}_w,\mathbf{X}_w(s'))q_{\mathrm{V}}(s',t)\,ds', \tag{14}$$

where $\Delta p(\mathbf{X}_w(s)+\mathbf{r}_w,t) = p_0(\mathbf{X}_w(s)+\mathbf{r}_w) - p(\mathbf{X}_w(s)+\mathbf{r}_w,t)$. Here the kernel G is the Green's function associated with the pressure equation $-\Delta p = \delta(\mathbf{x}-\mathbf{X}_w)$. That is,

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\|\mathbf{x}-\mathbf{x}'\|}. \tag{15}$$

Well flow. A single-phase, compressible, isothermal and unsteady well flow model is given on the form

$$\begin{aligned} \partial_t(A\rho_w) + \partial_\alpha(A\rho_w u) &= \rho_w q_{\rm V} \\ \partial_t(A\rho_w u) + \partial_\alpha(A\rho_w u^2) + A\partial_\alpha p_w &= 0, \end{aligned} \tag{16}$$

where α is the arc-length variable associated with the well path Γ_w. Here ρ_w is the fluid density, u the fluid velocity, $p_w = p(\rho_w)$ the pressure, $q_{\rm V}$ represents volumetric flux per unit wellbore length. Moreover, $A = \pi r_w^2$ is the pipe cross-sectional area for a well of radius r_w. In terms of the non-dimensional variables the model (16) takes the form (see [EK06] for more details)

$$\begin{aligned} &\partial_t(\rho_w) + \partial_s(\rho_w u) = \frac{1}{\eta}\rho_w q_{\rm V}, \qquad \eta = \frac{L_w A\mu}{DLk\bar{p}}, \\ &\partial_t(\rho_w u) + \partial_s(\rho_w u^2) + h_0\partial_s p_w = 0, \quad p_w = p_w(\rho_w), \quad h_0 = \frac{\bar{p}}{\bar{\rho}\bar{u}^2}. \end{aligned} \tag{17}$$

Here $\bar{p}, \bar{\rho}, \bar{u}$ represent characteristic quantities and $\bar{u} = L_w/D$.

Coupled well-porous media flow. Equipped with the well model (17) and the porous media model (12) we now formulate a *coupled* well-porous media flow model along the line of [OA01] by imposing the coupling condition $p_w(\rho_w(s,t)) = p(\mathbf{X}_w(s) + \mathbf{r}_w, t) := p(s,t)$. This results in a model on the form (skipping the index "w")

$$\begin{aligned} &\partial_t(\rho) + \partial_s(\rho u) = \frac{1}{\eta}\rho q_{\rm V}, \qquad \partial_t(\rho u) + \partial_s(\rho u^2) + \partial_s P(\rho) = 0, \\ &P_0 - P(s,t) = h_0\int_0^t\int_0^1 G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{X}_w(s'), t-t')q_{\rm V}(s',t')\,ds'dt', \end{aligned} \tag{18}$$

where $P(\rho) = h_0 p(\rho)$ and $h_0 = \frac{\bar{p}}{\bar{\rho}\bar{u}^2}$. This model corresponds to the model problem (1).

A simplified "compressible well-incompressible porous media" model. We may treat the reservoir fluid as an incompressible fluid. In view of (14) and (15) we then obtain a well-porous media model on the form

$$\begin{aligned} &\partial_t(\rho) + \partial_s(\rho u) = \frac{1}{\eta}\rho q_{\rm V}, \qquad \partial_t(\rho u) + \partial_s(\rho u^2) + \partial_s P(\rho) = 0, \\ &P_0 - P(s,t) = \int_0^1 H^r(s,s')q_{\rm V}(s',t)\,ds', \end{aligned} \tag{19}$$

where $H^r(s,s') = h_0 G(\mathbf{X}_w(s) + \mathbf{r}, \mathbf{X}_w(s'))$ for $G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\|\mathbf{x}-\mathbf{x}'\|}$. We introduce the dimensionless radius $\bar{r} = \frac{r_w}{L_w}$ and arrive at the following expression for the kernel $H^r(s,s')$ with $\varepsilon_1 = \frac{h_0 L}{4\pi\bar{r}L_w}$ (see [EK06] for more details).

$$H^r(s,s') = h_0 G(\mathbf{X}_w(s) + \mathbf{r}, \mathbf{X}_w(s')) = \varepsilon_1\left[\left((s-s')/\bar{r}\right)^2 + 1\right]^{-1/2}. \tag{20}$$

A simplified incompressible well-porous media model. We take a step further and impose in (19) that the well fluid is incompressible, i.e. $\rho = 1$. This yields the following simplified model

$$\partial_s(u) = \frac{1}{\eta}q_{\mathrm{V}}, \qquad \partial_t(u) + \partial_s(u^2) + \partial_s P = 0,$$
$$P_0 - P(s,t) = \int_0^1 H^r(s,s')q_{\mathrm{V}}(s',t)\,ds', \tag{21}$$

where $\eta = \frac{L_w \mu A}{Lk\bar{p}D}$. In view of (20), we introduce the function $G^r(s,s')$ defined by (3) and see that $H^r(s,s') = \frac{h_0 L}{4\pi \bar{r}^2 L_w} G^r(s,s')$. Inserting this in (21), we obtain the model problem (2)–(3), where we have replaced the finite domain $[0,1]$ by the real axis.

An approximate well-porous media model relevant for (4)**.** In this section we focus on a well-reservoir model which represents an approximation to the well-porous media model (4). More precisely, we replace the kernel $G^r(x,x')$ given in (3) by an *approximate* kernel $\widetilde{G}^r(x,x')$ defined in the following. First, we observe that

$$G^r_x(x) = \frac{-r^2 x}{\left(x^2+r^2\right)^{3/2}}, \qquad G^r_{xx}(x) = \frac{r^2\left(\sqrt{2}x - r\right)\left(\sqrt{2}x + r\right)}{\left(x^2+r^2\right)^{5/2}}. \tag{22}$$

We then introduce the approximation

$$\widetilde{G}^r_x(x) = \begin{cases} G^r_x(x) & \text{if } |x| > r/\sqrt{2}, \\ c(1-2H(x)) & \text{if } |x| \le r/\sqrt{2}, \quad c = G^r_x(-r/\sqrt{2}), \end{cases} \tag{23}$$

where $H(x)$ is the Heaviside function and $c = \frac{2}{3\sqrt{3}}$. We easily see that $\|G^r_x - \widetilde{G}^r_x\|_{L^1(\mathbb{R})} = O(r)$. Next, we define $\widetilde{G}^r(x) = \int_{-\infty}^x \widetilde{G}^r_x(s)\,ds$. Moreover, it follows that $\widetilde{G}^r \in C^2(\mathbb{R}/\{0\})$ since

$$\widetilde{G}^r_{xx}(x) = \begin{cases} G^r_{xx}(x) & \text{if } |x| > r/\sqrt{2}, \\ -2c\delta(x) & \text{if } |x| \le r/\sqrt{2}, \qquad c = G^r_x(-r/\sqrt{2}) = -G^r_x(r/\sqrt{2}), \end{cases}$$

where $\delta(x)$ is the Dirac mass centred at $x = 0$ In particular, we note that $\widetilde{G}^r_{xx}$ is continuous at $\pm r/\sqrt{2}$ since $G^r_{xx}(r/\sqrt{2}) = 0$. We now consider the corresponding well-porous media model defined by

$$\begin{aligned} \partial_t \widetilde{u} + \partial_x(\widetilde{u}^2) &= \varepsilon \widetilde{G}^r_x * \widetilde{u}_x = \varepsilon \widetilde{G}^r_{xx} * \widetilde{u} \\ &= \varepsilon\Big(G^r_{xx}\chi_{|x-x'|>r/\sqrt{2}} * \widetilde{u} - 2c\delta(x-x')\chi_{|x-x'|\le r/\sqrt{2}} * \widetilde{u}\Big) \\ &= \varepsilon(G^r_{xx}\chi_{|x-x'|>r/\sqrt{2}} * \widetilde{u} - 2c\widetilde{u}), \end{aligned} \tag{24}$$

where $\chi_E(x) = 1$ for $x \in E$ and $\chi_E(x) = 0$ for $x \notin E$.

3 A well-posedness result for the model (24)

Definition 1 (Entropy weak solution). *A function* $u \in L^\infty((0,T)\times\mathbb{R}) \cap C\left([0,T]; L^1(\mathbb{R})\right)$ *for any* $T > 0$*, is called an entropy weak solution to* (24) *provided for any convex C^2 entropy $\eta : \mathbb{R} \to \mathbb{R}$ with corresponding entropy flux $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = 2u\eta'(u)$ there holds the inequality*

$$\int_0^T \int_{\mathbb{R}} [\eta(u)\phi_t + q(u)\phi_x - \eta'(u)p_x\phi]dxdt + \int_{\mathbb{R}} \eta(u_0(x))\phi(x,0)dx \geq 0, \quad (25)$$

$\forall \phi \in C_0^\infty([0,T) \times \mathbb{R})$, $\phi \geq 0$ *where* $p_0 - p(x,t) = \varepsilon \int_{\mathbb{R}} \widetilde{G}_x^r(x,x')u(x',t)\,dx'$.

We rely on the standard approach and seek for a solution to (24) by letting μ go to zero in the viscous approximation

$$\partial_t \widetilde{u} + \partial_x(\widetilde{u}^2) = \varepsilon \widetilde{G}_{xx}^r * \widetilde{u} + \mu \partial_{xx}^2 \widetilde{u}, \qquad \mu > 0. \quad (26)$$

Lemma 1. *Let u and $\bar{u}$ be solutions of* (26) *with initial data $u_0, \bar{u}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, for any $t > 0$,*

$$\int_{\mathbb{R}} [u(x,t) - \bar{u}(x,t)]^+ \, ds \leq \int_{\mathbb{R}} [u_0(x) - \bar{u}_0(x)]^+ \, ds, \quad (27)$$

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R})}, \quad (28)$$

$$\text{If } u_0(x) \leq \bar{u}_0(x) \text{ a.e. on } \mathbb{R}, \text{ then } u(x,t) \leq \bar{u}(x,t) \text{ a.e. on } \mathbb{R} \times [0,T], \quad (29)$$

$$-a \leq \|u(\cdot,t)\|_\infty, \|\bar{u}(\cdot,t)\|_\infty \leq +a, \qquad a = \max\{\|u_0\|_\infty, \|\bar{u}_0\|_\infty\}. \quad (30)$$

Proof. We know that (26) has smooth (classical) solutions. We define $\eta_\delta(\cdot)$ such that, pointwise we have $\eta_\delta(u - \bar{u}) \to [u - \bar{u}]^+$, $\eta_\delta'(u - \bar{u}) \to \operatorname{sgn}(u - \bar{u})^+$, $\eta_\delta''(u - \bar{u})[u^2 - \bar{u}^2] \to 0$, as $\delta \downarrow 0$. In view of (26) we can find an equation for $u - \bar{u}$. Multiplying this equation by $\eta_\delta'(u - \bar{u})$, we get

$$\begin{aligned} &\int_{\mathbb{R}} \eta_\delta(u - \bar{u})dx \leq \int_{\mathbb{R}} \eta_\delta(u_0 - \bar{u}_0)dx + \\ &\int_0^t \int_{\mathbb{R}} \eta_\delta''(u - \bar{u})(u^2 - \bar{u}^2)(u - \bar{u})_x dxdt + \varepsilon \int_0^t \int_{\mathbb{R}} \eta_\delta' \widetilde{G}_{xx}^r * (u - \bar{u})dxd\tau. \end{aligned} \quad (31)$$

Taking the limit $\delta \to 0$, we get

$$\int_{\mathbb{R}} [u(x,t) - \bar{u}(x,t)]^+ \, dx \leq \int_{\mathbb{R}} [u_0(x) - \bar{u}_0(x)]^+ \, dx + R, \quad (32)$$

where $R = \varepsilon \int_0^t \int_{\mathbb{R}} \operatorname{sgn}(u - \bar{u})^+ \widetilde{G}_{xx}^r * (u - \bar{u}) \, dxd\tau$. We must estimate R.

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \operatorname{sgn}(u - \bar{u})^+ [G_{xx}^r \chi_{|x-x'|>r/\sqrt{2}}] * (u - \bar{u}) \, dxd\tau \\ &\leq \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_{xx}^r \chi_{|x-x'|>r/\sqrt{2}} \cdot [u - \bar{u}]^+(x',t) \, dx' \, dxd\tau \\ &= \int_0^t \int_{\mathbb{R}} [u - \bar{u}]^+(x',t) \int_{\mathbb{R}} G_{yy}^r \chi_{|y|>r/\sqrt{2}} \, dy \, dx' d\tau \\ &= 2 \int_0^t \int_{\mathbb{R}} [u - \bar{u}]^+(x',t) \int_{-\infty}^{-r/\sqrt{2}} G_{yy}^r \, dy \, dx' d\tau = 2c \int_0^t \int_{\mathbb{R}} [u - \bar{u}]^+(x',t) \, dx' d\tau, \end{aligned}$$

where we have used the transformation $y = x - x'$ for a fixed x'. Consequently,

$$\int_0^t \int_{\mathbb{R}} \operatorname{sgn}(u-\bar{u})^+ \Big([G^r_{xx}\chi_{|x-x'|>r/\sqrt{2}}] * (u-\bar{u}) - 2c(u-\bar{u}) \Big)\, dx d\tau \le 0,$$

and it follows from (32) that $\int_{\mathbb{R}} (u-\bar{u})^+\, dx \le \int_{\mathbb{R}} (u_0 - \bar{u}_0)^+\, dx$. From this, (28) and (29) follow. To show that (30) holds, we multiply (24) by a regularization of $p|u|^{p-1}\operatorname{sgn}(u)$ and observe that

$$p|u|^{p-1}\operatorname{sgn}(u)u_t = \partial_t(|u|^p), \quad p|u|^{p-1}\operatorname{sgn}(u)(u^2)_x = \partial_x(\frac{2p}{p+1}\operatorname{sgn}(u)|u|^{p+1}),$$
$$p|u|^{p-1}\operatorname{sgn}(u)\mu\partial^2_{xx}u = \mu(u_x p|u|^{p-1}\operatorname{sgn}(u))_x - \mu p(p-1)|u|^{p-2}(u_x)^2.$$

Consequently, $\int_{\mathbb{R}} |u|^p\, dx \le \int_{\mathbb{R}} |u_0|^p\, dx + \varepsilon \int_{\mathbb{R}} p|u|^{p-1}\operatorname{sgn}(u)\widetilde{G}^r_{xx} * u\, dx$. Moreover,

$$\int_{\mathbb{R}} |u|^{p-1}\operatorname{sgn}(u)[G^r_{xx}\chi_{|x-x'|>r/\sqrt{2}}] * u\, dx \le \|u\|_p^{p-1} \|G^r_{xx}\chi_{|x-x'|>r/\sqrt{2}} * u\|_p$$
$$\le \|u\|_p^{p-1}\|G^r_{xx}\chi_{|x-x'|>r/\sqrt{2}}\|_1 \|u\|_p \le 2c\|u\|_p^p.$$

Therefore, we conclude that $\int_{\mathbb{R}} p|u|^{p-1}\operatorname{sgn}(u)\widetilde{G}^r_{xx} * u\, dx \le 0$, which implies that $\|u\|_p \le \|u_0\|_p$ for all $p \ge 1$. Thus, (30) follows.

Lemma 2. *Let u^μ be the solution to* (26) *with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as initial data with $\int_{\mathbb{R}} |u_0(x+h) - u_0(x)|\, dx \le \omega(|h|)$, for any $h \in \mathbb{R}$, for some nondecreasing function ω on $\mathbb{R}^+$ with $\omega(r) \downarrow 0$ as $r \downarrow 0$. Then there exists a constant C, depending only on $\|u_0\|_\infty$ such that, for any $t > 0$,*

$$\int_{\mathbb{R}} |u^\mu(x+h,t) - u^\mu(x,t)|\, dx \le \omega(|h|), \quad \text{for any } h \in \mathbb{R}, \tag{33}$$
$$\int_{\mathbb{R}} |u^\mu(x,t+k) - u^\mu(x,t)|\, dx \le C(k + k^{2/3} + \mu k^{1/3})\|u_0\|_1 + 4\omega(k^{1/3}), \tag{34}$$

for any $k > 0$.

This can be proved along the line of, for example [LM03]. In view of Lemma 2, it follows by classical arguments that the sequence u^μ is compact in L^1_{loc}. More precisely, the following theorem holds.

Theorem 1. *Let u^μ be the solution to* (26) *with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as initial datum. Then, as $\mu \downarrow 0$, for any $T > 0$, $u^\mu \to u$ strongly in $L^p_{loc}(\mathbb{R} \times [0,T])$ for $p < \infty$, and $u \in L^1(\mathbb{R} \times [0,T]) \cap L^\infty(\mathbb{R} \times [0,T])$ is an entropy solution to* (24) *in the sense of Definition 1.*

Finally, we also mention that L^1-stability of entropy weak solutions (thus, uniqueness) can be proved along the line of [LM03]. It is also of interest to study the difference between the solutions of (4) and (24). We have the following result.

Theorem 2. *Let u and $\widetilde{u}$ denote entropy weak solutions, respectively of* (4) *and* (24) *with initial data $u_0 \in BV(\mathbb{R})$. Then, for any $t > 0$,*

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^1(\mathbb{R})} \leq \frac{2\sqrt{2}}{3\sqrt{3}} \varepsilon r t e^{2\varepsilon t} \|u_0\|_{BV(\mathbb{R})}. \tag{35}$$

Proof. Let $e = u - \widetilde{u}$, where u and $\widetilde{u}$ are viscous approximations to (4) and (24). Then $e_t + (u^2 - \widetilde{u}^2)_x = \varepsilon \widetilde{G}_x^r * e_x + \varepsilon [G_x^r - c(1-2H)] \chi_{|x-x'|<r/\sqrt{2}} * u_x + \mu e_{xx}$. Along the line of Lemma 3.1 we get

$$\begin{aligned}\int_{\mathbb{R}} e(x,t)^+ \, dx \leq \int_{\mathbb{R}} e_0(x)^+ \, dx + \varepsilon \int_0^t \int_{\mathbb{R}} \operatorname{sgn}(e)^+ \widetilde{G}_{xx}^r * e \, dx d\tau \\ + \varepsilon \int_0^t \int_{\mathbb{R}} \operatorname{sgn}(u - \bar{u})^+ [G_x^r - c(1-2H)] \chi_{|x-x'|<r/\sqrt{2}} * u_x \, dx d\tau.\end{aligned} \tag{36}$$

Here $\int_0^t \int_{\mathbb{R}} \operatorname{sgn}(e)^+ \widetilde{G}_{xx}^r * e \, dx d\tau \leq 0$ whereas the last term is estimated as follows

$$\begin{aligned}&\Big| \int_0^t \int_{\mathbb{R}} \operatorname{sgn}(u - \bar{u})^+ [G_x^r - c(1-2H)] \chi_{|x-x'|<r/\sqrt{2}} * u_x \, dx d\tau \Big| \\ &\leq \Big\| [G_x^r - c(1-2H)] \chi_{|x-x'|<r/\sqrt{2}} \Big\|_{L^1(\mathbb{R})} \int_0^t \|u_x\|_{L^1(\mathbb{R})} d\tau \leq \sqrt{2} c r t e^{2\varepsilon t} \|u_0\|_{BV(\mathbb{R})},\end{aligned}$$

where we have used that $\|[G_x^r - c(1-2H)] \chi_{|x-x'|<r/\sqrt{2}}\|_{L^1(\mathbb{R})} \leq 2G_x^r(r/\sqrt{2}) r/\sqrt{2} = \sqrt{2} c r$ and $\|u(\cdot,t)\|_{BV(\mathbb{R})} \leq e^{2\varepsilon t} \|u_0\|_{BV(\mathbb{R})}$, taken from Lemma 4.2 in [EK06].

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