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GLOBAL WEAK SOLUTIONS FOR A VISCOUS LIQUID-GAS MODEL WITH SINGULAR PRESSURE LAW

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Abstract. We study a viscous two-phase liquid-gas model relevant for well and pipe flow modelling. The gas is assumed to be polytropic whereas the liquid is treated as an incompressible fluid leading to a pressure law which becomes singular when transition to single-phase liquid flow occurs. In order to handle this difficulty we reformulate the model in terms of Lagrangian variables and study the model in a free-boundary setting where the gas and liquid mass are of compact support initially and discontinuous at the boundaries. Then, by applying an appropriate variable transformation, point-wise control on masses can be obtained which guarantees that no single-phase regions will occur when the initial state represents a true mixture of both phases. This paves the way for deriving a global existence result for a class of weak solutions. The result requires that the viscous coefficient depends on the volume fraction in an appropriate manner. By assuming more regularity of the initial fluid velocity a uniqueness result is obtained for an appropriate (smaller) class of weak solutions.

1. **Introduction.** The starting point for the investigations of this work is a one-dimensional two-phase model of the drift-flux type. This model is frequently used to simulate unsteady, compressible flow of liquid and gas in pipes [1, 3, 4, 15, 7, 26, 11, 21]. The model consists of two mass conservation equations corresponding to each of the two phases and one equation for the conservation of the momentum of the mixture and is given in the following form:

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$$\begin{aligned} \partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u_g] &= 0 \\ \partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u_l] &= 0 \\ \partial_t[\alpha_l\rho_l u_l + \alpha_g\rho_g u_g] + \partial_x[\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + p] &= -q + \partial_x[\varepsilon\partial_x u_{mix}], \end{aligned} \tag{1}$$

with $u_{mix} = \alpha_g u_g + \alpha_l u_l$ and $\varepsilon \geq 0$. The model is supposed under isothermal conditions. The unknowns are ρ_l, ρ_g the liquid and gas densities, α_l, α_g volume fractions of liquid and gas satisfying $\alpha_g + \alpha_l = 1$, u_l, u_g velocities of liquid and gas, p common pressure for liquid and gas, and q representing external forces like gravity and friction. Since the momentum is given only for the mixture, we need an additional closure law, a so-called hydrodynamical closure law, which connects the two phase velocities. More generally, this law should be able to take into account the different flow regimes. In addition, we need a thermodynamical equilibrium model which specifies the fluid properties. More details will be given in the next section. Otherwise, we refer to [1, 2, 5, 6, 7, 11, 12, 13, 18, 20, 21, 22, 26] for various numerical schemes which have been developed for the study of the drift-flux model. See also [8] for a study of the relation between the drift-flux model and the more general two-fluid model where two separate momentum equations are used instead of a mixture momentum equation [3, 15].

Few results concerning existence, uniqueness, and stability seem to exist for two-phase liquid-gas models of the form (1). The main purpose of this work is to initiate some work in this direction. Many new challenges, compared to single-phase Navier-Stokes flow type of models, occur. Thus, in this work we focus on a simplified model obtained by assuming that fluid velocities are equal $u_g = u_l = u$ and by neglecting the external forces, i.e., $q = 0$. In addition, we neglect certain gas effects by considering a simplified momentum equation where acceleration terms depend solely on the liquid phase. This is motivated by the fact that liquid phase density typically is much higher than gas phase density. Consequently, we consider a model in the form

$$\begin{aligned} \partial_t[\alpha_g\rho_g] + \partial_x[\alpha_g\rho_g u] &= 0 \\ \partial_t[\alpha_l\rho_l] + \partial_x[\alpha_l\rho_l u] &= 0 \\ \partial_t[\alpha_l\rho_l u] + \partial_x[\alpha_l\rho_l u^2] + \partial_x p &= \partial_x[\varepsilon\partial_x u], \qquad p, \varepsilon \geq 0. \end{aligned} \tag{2}$$

Assuming polytropic gas law relation $p = C\rho_g^\gamma$ with $\gamma > 1$ and incompressible liquid $\rho_l = \text{Const}$ we get a pressure law of the form (see Section 2 for more details)

$$p(n, m) = C\Big(\frac{n}{\rho_l - m}\Big)^\gamma,$$

where we use the notation $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$. In particular, we see that pressure becomes singular at transition to pure liquid phase $\alpha_l = 1$ which yields $m = \rho_l$. In order to treat this difficulty we first assume that we consider (2) in a free boundary problem setting where the masses m and n initially occupy only a finite interval $[a, b] \subset \mathbb{R}$. That is,

$$n(x, 0) = n_0(x) > 0, \qquad m(x, 0) = m_0(x) > 0, \qquad u(x, 0) = u_0(x), \qquad x \in [a, b],$$

and $n_0 = m_0 = 0$ outside $[a, b]$. The viscosity coefficient ε is in general assumed to be a functional of the masses m and n, i.e. $\varepsilon = \varepsilon(n, m)$, and a main purpose of the current study is to identify an appropriate form which can guarantee that pressure does not blow up, that is, transition to single-phase liquid flow is avoided.

Rewriting the model (2) in terms of Lagrangian variables, the free boundaries are converted into fixed and we get a model in the form

$$\begin{aligned} \partial_t n + (nm)\partial_x u &= 0 \\ \partial_t m + m^2 \partial_x u &= 0 \\ \partial_t u + \partial_x p(n,m) &= \partial_x(\varepsilon(n,m) m \partial_x u), \qquad x \in (0,1), \end{aligned} \tag{3}$$

with boundary conditions

$$p(n,m) = \varepsilon(n,m) m u_x, \qquad \text{at } x = 0,1, \quad t \geq 0,$$

and initial data

$$n(x,0) = n_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x \in [0,1].$$

We obtain an existence result (Theorem 3.1) for the model (3) for a class of weak solutions and for a flow regime where the viscosity coefficient is of the form

$$\varepsilon = \varepsilon(m) = \frac{m^\beta}{(\rho_l - m)^{\beta+1}}, \qquad \beta \in (0, 1/3).$$

This relation implies a certain balance between the pressure and viscous forces as m is approaching the critical limit ρ_l which is sufficient to guarantee that the liquid mass m can be controlled pointwise from below and from above. This pointwise control is then transferred to the gas mass n through the common fluid velocity u and the two mass conservation equations of (3). More precisely, by assuming initially that the gas and liquid mass n and m do not disappear or blow up on $[0,1]$, that is,

$$C^{-1} \leq n(x,0) \leq C, \qquad 0 < \mu \leq \alpha_l(x,0) \leq 1 - \mu < 1,$$

for a suitable constant $C > 0$ and $\mu > 0$, then the same will be true for the masses n and m for all $t \in [0,T]$ for any specified time $T > 0$. This allows us to obtain various estimates which ensure convergence to a class of weak solutions. By imposing more regularity on the fluid velocity we also derive a uniqueness result (Theorem 6.3) in a corresponding smaller class of weak solutions.

The main tool in this analysis is the introduction of a suitable variable transformation allowing for application of ideas and techniques similar to those used in [24, 17, 19, 27, 25, 16] in previous studies of the single-phase Navier-Stokes equations. We conclude this section by noting that the model (2) where both fluids (gas and liquid) were assumed to be compressible and with a constant viscosity coefficient ε was studied in [9]. A global existence result was obtained for a class of weak solutions for rather general initial data. In a recent work [10] we deal with the model (3) in a context where it is assumed that the initial masses m and n are connected to the boundary in a continuous manner.

The rest of this paper is organized as follows. In Section 2 we give a detailed description of the model (1) and present the motivation for studying the simplified model (2). In Section 3 we give more details relevant for the model (3) obtained from (2), and we state the main theorem. In Section 4 we describe a priori estimates for an auxiliary model obtained from (3) by using an appropriate variable transformation. In Section 5 we consider approximate solutions to (3) obtained by regularizing initial data. By means of the estimates of Section 4, we get a number of estimates which imply compactness. Convergence to a weak solution then follows by standard arguments. Finally, in Section 6 we present a uniqueness result for an appropriate (smaller) class of weak solutions.

2. **Motivation.** The purpose of this section is to give further details relevant for the drift-flux model (1). Ultimately this will lead us to consider the simpler model (2).

2.1. **Specification of the model (1).** To close the system, we need to include the following additional equations. The volume fractions are related by

$$\alpha_l + \alpha_g = 1. \tag{4}$$

Thermodynamical laws specify fluid properties such as densities ρ_l, ρ_g and viscosities μ_l, μ_g. In particular we will assume that the liquid density has the following form

$$\rho_l = \rho_{l,0} + \frac{p - p_{l,0}}{a_l^2}, \tag{5}$$

where $a_l = 1000$ [m/s] is the velocity of sound in the liquid phase and $\rho_{l,0}$ and $p_{l,0}$ are given constants. Here we will assume that $\rho_{l,0} = 1000$ [kg/m^3] and $p_{l,0} = 1$ [bar]. It is often assumed that the liquid is incompressible, i.e.

$$\rho_l = \rho_{l,0}.$$

We assume that we consider a polytropic, isentropic ideal gas characterized by

$$p(\rho_g) = a_g^2 \rho_g^\gamma, \qquad \gamma \geq 1. \tag{6}$$

In other words, we have

$$\rho_g = \left(\frac{p}{a_g^2}\right)^{1/\gamma}, \qquad \gamma \geq 1, \tag{7}$$

where $a_g = 316$ [m/s] is the velocity of sound in the gas phase. Furthermore, the viscosity for liquid and gas are assumed to be

$$\mu_l = 5 \cdot 10^{-2} \text{ [Pa s]}, \qquad \mu_g = 5 \cdot 10^{-6} \text{ [Pa s]}. \tag{8}$$

Since we only have one momentum equation for the mixture of the two phases, the model must be supplemented with an additional hydrodynamical closure law whose purpose is to determine the fluid velocities u_l, u_g through a so-called slip relation. We may assume that the slip relation can be expressed by a general relation

$$f(\alpha_g, u_l, u_g, \rho_g, \rho_l) = 0.$$

A commonly used slip relation is given by

$$f(\alpha_g, u_l, u_g, \rho_g, \rho_l) = u_g - c_0 u_{mix} - c_1 = 0, \tag{9}$$

where

$$u_{mix} = \alpha_l u_l + \alpha_g u_g,$$

and c_0, c_1 are flow dependent coefficients. We refer to [7] and references therein for more details. For the source term q we have two components

$$q = F_f + F_g,$$

where $F_g = g(\alpha_l \rho_l + \alpha_g \rho_g)\sin\theta$ represents the gravity where g is the gravitational constant and θ is the inclination. Moreover, F_f represents forces between the wall and the fluids. Typically [7], the following simple expression for F_f is assumed

$$F_f = \frac{32 u_{mix} \mu_{mix}}{d^2}, \tag{10}$$

where d is the inner diameter and the mixed viscosity μ_{mix} is given by

$$\mu_{mix} = \alpha_l \mu_l + \alpha_g \mu_g,$$

where the viscosity μ_l, μ_g are given by (8).

The construction of simple, but efficient numerical schemes for the model (1) equipped with the above additional closure relations (4)–(10) has been studied more recently in [5, 6, 7]. For other works on numerical methods for this model we refer to [20, 26, 11, 12] and references therein. It is convenient to express the above system in the form

$$\partial_t \begin{pmatrix} n \\ m \\ nu_g + mu_l \end{pmatrix} + \partial_x \begin{pmatrix} nu_g \\ mu_l \\ nu_g^2 + mu_l^2 + p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -q + \partial_x[\varepsilon(n,m)\partial_x u_{mix}] \end{pmatrix}, \tag{11}$$

where $n = \alpha_g\rho_g$ and $m = \alpha_l\rho_l$ and $\varepsilon \geq 0$ is a (n,m)-dependent viscous coefficient and $u_{mix} = \alpha_l u_l + \alpha_g u_g$. In order to see how pressure p is related to the masses m and n we observe that the relation (4) can be written as

$$\frac{n}{\rho_g(p)} + \frac{m}{\rho_l(p)} = 1. \tag{12}$$

Using this, we can express the pressure p as a function P of n and m, i.e.

$$p = P(n,m).$$

In particular, assuming that liquid is incompressible we get from (12) that

$$\rho_g = \rho_l \frac{n}{\rho_l - m},$$

which can be plugged into (6) yielding

$$p(\rho_g) = a_g^2\rho_l^\gamma\Big(\frac{n}{\rho_l - m}\Big)^\gamma = k_1\Big(\frac{n}{\rho_l - m}\Big)^\gamma =: P(n,m), \qquad k_1 = a_g^2\rho_l^\gamma. \tag{13}$$

We will use this pressure law for the model we analyse in the next section.

2.2. **A simplified viscous two-phase model.** As a first step, instead of working directly with the full two-phase model (11) we suggest to replace it by a simpler one. We introduce a simplification by replacing the mixture momentum equation by the momentum equation of the liquid phase only. This is motivated by the fact that the liquid phase density is much higher than for the gas phase, typically to the order of $\rho_g/\rho_l \sim 0.001$, and therefore plays the dominating role in the mixture momentum conservation equation, as long as the amount of gas does not become too high. We justify this simplification by performing two different numerical experiments demonstrating that the simplified model for many flow cases can give a good approximation to the original two-phase model. To sum up, we consider the model

$$\begin{aligned} \partial_t n + \partial_x[nu_g] &= 0 \\ \partial_t m + \partial_x[mu_l] &= 0 \\ \partial_t[mu_l] + \partial_x[mu_l^2 + P(n,m)] &= -F_{f,l} - F_{g,l} + \partial_x[\varepsilon(m)\partial_x u_l] \end{aligned} \tag{14}$$

together with the constitutive relations

$$\alpha_l + \alpha_g = 1, \quad f(\alpha_g, u_l, u_g, \rho_g, \rho_l) = 0, \quad \rho_l = \rho_l(p), \quad \rho_g = \rho_g(p), \tag{15}$$

and where

$$F_{f,l} = \frac{32(\alpha_l u_l)(\alpha_l\mu_l)}{d^2}, \quad F_{g,l} = g(\alpha_l\rho_l)\sin\theta.$$

Below we compute the solutions produced by the model (11) and (14) respectively, for two different flow cases. The purpose is to demonstrate the difference between the simplified model and the original two-phase model. For both cases we consider the inviscid case where $\varepsilon = 0$ and horizontal flow where gravity has no impact.

A shock tube example. We have assumed that the liquid and gas density models are given by (5) and (7) with $\gamma = 1$. Moreover, for this example we consider a slip relation of the form (9) where $c_1 = 1.07$ and $c_2 = 0.216$. This example was also considered in [5]. The purpose of this test is to compare the full drift-flux model with the simplified model for a Riemann test problem. We consider the initial data

$$(\alpha_{g,L}, \alpha_{g,R}) = (0.55, 0.55), \;\; (u_{l,L}, u_{l,R}) = (10.37, 0.561), \;\; (p_L, p_R) = (80450, 24282).$$

We have neglected frictional forces, and the pipe we consider is of length 100 m. Results are presented in Fig. 1 demonstrating that the two models produce results whose difference is almost indiscernible.

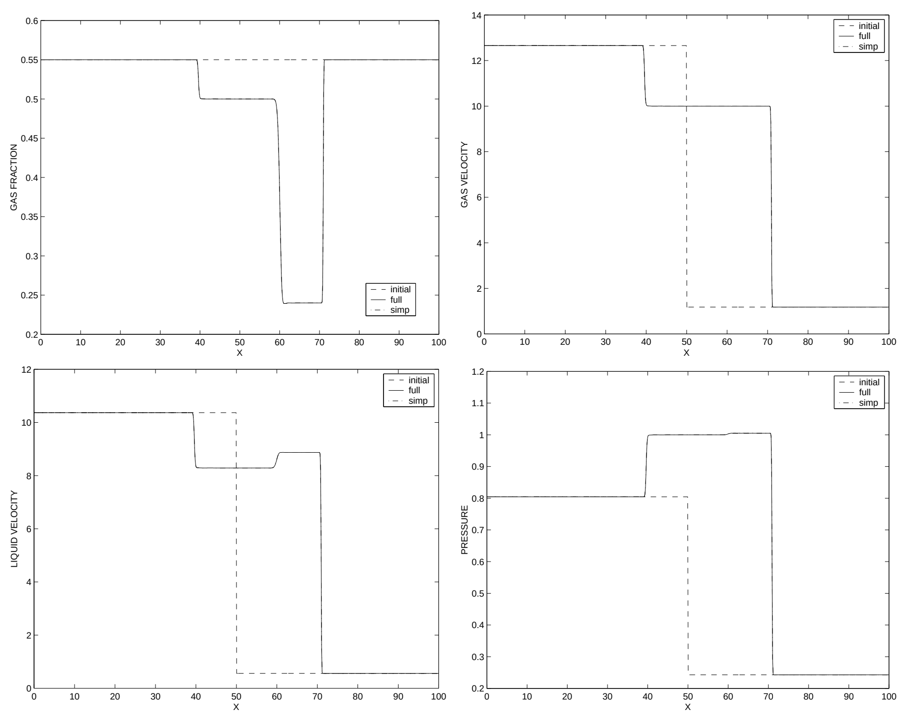

FIGURE 1. Snapshot of α_g, u_g (top) and u_l, p (bottom) at time $t = 1.0$. We have used 800 nodes.

A mass flow example. We compare the difference between the full model and the simplified model for a typical mass flow example taken from [7]. The slip relation for this example is given by (9) where

$$c_0 = 1, \qquad c_1 = 0.5(1 - \alpha_g)^{1/2}.$$

The results of this comparison is demonstrated in Fig. 2 and reflect that the difference is mild. Both examples serve as a justification of studying the model (14) as a reasonable approximation to the more complete model (11) for many flow scenarios.

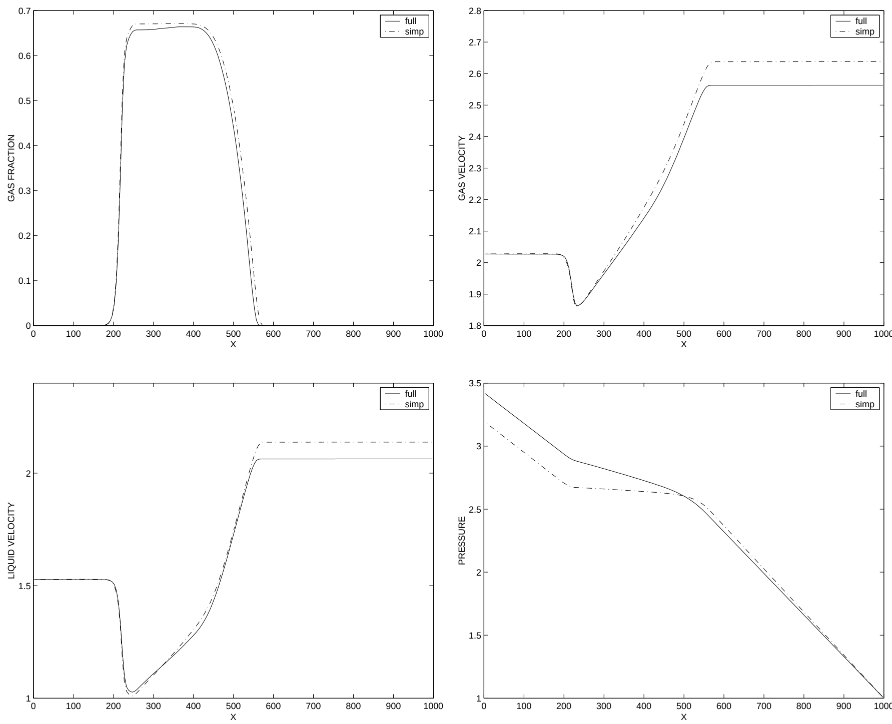

FIGURE 2. Snapshot of α_g, u_g (top) and u_l, p (bottom), at time $t = 175$ [s] with 200 nodes.

3. **A global existence result for a simplified viscous two-phase model.** In the following we shall work with one specific version of the model (14) and (15) where we assume that fluid velocities are equal, $u_l = u_g = u$, and where external force terms (friction and gravity) are neglected. In particular, we shall focus on the case where the liquid is assumed to be incompressible which implies that we use the pressure law (13). More precisely, we focus on the compressible gas-incompressible liquid two-phase model

$$
\begin{aligned}
\partial_t n + \partial_x [nu] &= 0 \\
\partial_t m + \partial_x [mu] &= 0 \\
\partial_t [mu] + \partial_x [mu^2] + \partial_x P(n,m) &= \partial_x [\varepsilon(m) \partial_x u],
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
P(n,m) &= k_1 \Big(\frac{n}{\rho_l - m} \Big)^{\gamma}, \qquad \gamma > 1, \\
\varepsilon(m) &= k_2 \frac{m^\beta}{(\rho_l - m)^{\beta+1}} = \frac{k_2}{\rho_l} \frac{\alpha_l^\beta}{(1-\alpha_l)^{\beta+1}}, \qquad \beta \in (0, 1/3),
\end{aligned}
\tag{17}
$$

where k_1 and k_2 are appropriate constants. One special feature of the above two-phase model (16)–(17) is that the pressure law becomes singular for pure liquid flow, that is, when $m = \rho_l \alpha_l = \rho_l$. To compensate for this, it is assumed that the viscosity coefficient $\varepsilon(m)$ reflects a similar behavior such that a proper balance between pressure and viscous forces takes place. Here it is in order to emphasize that as far as the viscous coefficient $\varepsilon(m)$ (17) is concerned, we currently do not directly motivate our choice from physical considerations (as is done for the single-phase Navier-Stokes equations [17]). Rather our choice is motivated by the desire for obtaining pointwise upper and lower control of the liquid mass m. In particular, other choices than the one given in (17) would also be of interest to consider.

3.1. **Main idea.** The idea of this paper is to study the model (16)–(17) in a setting where sufficient pointwise control on the masses m and n can be ensured. Motivated by previous studies of the single-phase Navier-Stokes model [24, 17, 19, 27, 25, 16], we propose to study (16) in a free-boundary setting where the gas and liquid masses m and n are of compact support initially and connect to the vacuum regions (where $n = m = 0$) discontinuously. Then, a main result is that by assuming that the gas and liquid mass n and m initially do not disappear or blow up on $[0, 1]$, that is,

$$C^{-1} \le n \le C, \qquad 0 < \mu \le \alpha_l \le 1 - \mu < 1,$$

for a suitable constant $C > 0$ and $\mu > 0$, then the same will be true for the masses n and m for all $t \in [0, T]$ for any time $T > 0$. This allows us to obtain various estimates which provide an existence result for a class of weak solutions.

We now give some more details relevant for the model (16) we shall deal with in the rest of this paper. We study the Cauchy problem (16) with initial data

$$(n, m, mu)(x, 0) = \begin{cases} (n_0, m_0, m_0 u_0) & x \in [a, b], \\ (0, 0, 0) & \text{otherwise}, \end{cases}$$

where $\min_{x\in[a,b]} n_0 > 0$, $\min_{x\in[a,b]} m_0 > 0$, and $n_0(x), m_0(x)$ are in H^1. In other words, we study the two-phase model in a setting where an initial true two-phase mixture region (a, b) is surrounded by vacuums states $n = m = 0$ on both sides. For a moment let us focus on the discontinuities of n_0, m_0 at the boundary points $x = a, b$. By Rankine-Hugoniot condition it follows that

$$\begin{aligned} S[n] &= [nu] \\ S[m] &= [mu] \\ S[mu] &= [mu^2 + P(n, m) - \varepsilon(m)u_x], \end{aligned} \tag{18}$$

where $[\cdot]$ represents the jump across a discontinuity line $x(t)$ where $S = x'(t)$. Thus, across any discontinuity at which u is continuous, i.e. $[u] = 0$, (18) is reduced to

$$[P(n, m)] = [\varepsilon(m)u_x], \qquad [u] = 0, \qquad S = u. \tag{19}$$

Letting $a(t)$ and $b(t)$ denote the particle paths initiating from $(a, 0)$ and $(b, 0)$, respectively, in the x-t coordinate system, these paths represent free boundaries, i.e., the interface of the gas-liquid mixture and the vacuum. In view of (19), using that $m = n = 0$ to the left of $a(t)$ and to the right of $b(t)$, they are determined by the equations

$$\begin{gathered} \frac{d}{dt}a(t) = u(a(t), t), \quad \frac{d}{dt}b(t) = u(b(t), t), \\ \left(-P(n, m) + \varepsilon(m)u_x\right)(a(t)^+, t) = 0, \quad \left(-P(n, m) + \varepsilon(m)u_x\right)(b(t)^-, t) = 0. \end{gathered} \tag{20}$$

Following along the line of previous studies for the single-phase Navier-Stokes equations [24, 17], it is convenient to replace the free boundaries $a(t)$ and $b(t)$ (which are unknown in Eulerian coordinates) by fixed boundaries by using Lagrangian coordinates. First, we introduce a new set of variables (ξ, τ) by using the coordinate transformation

$$\xi = \int_{a(t)}^{x} m(y,t)\,dy, \qquad \tau = t. \tag{21}$$

Thus, ξ represents a convenient rescaling of x. In particular, the free boundaries $x = a(t)$ and $x = b(t)$, in terms of the new variables ξ and τ, take the form

$$\tilde{a}(\tau) = 0, \qquad \tilde{b}(\tau) = \int_{a(t)}^{b(t)} m(y,t)\,dy = \int_{a}^{b} m_0(y)\,dy = \text{const},$$

where $\int_a^b m_0(y)\,dy$ is the total liquid mass initially, which we normalize to 1. In other words, the interval $[a, b]$ in the x-t system appears as the interval $[0, 1]$ in the ξ-τ system.

Next, we rewrite the model itself (16) in the new variables (ξ, τ). First, in view of the particle paths $X_\tau(x)$ given by

$$\frac{dX_\tau(x)}{d\tau} = u(X_\tau(x), \tau), \qquad X_0(x) = x,$$

the system (16) now takes the form

$$\begin{aligned} \frac{dn}{d\tau} + nu_x &= 0 \\ \frac{dm}{d\tau} + mu_x &= 0 \\ m\frac{du}{d\tau} + P(n,m)_x &= (\varepsilon(m)u_x)_x. \end{aligned}$$

Applying (21) to shift from (x, t) to (ξ, τ) we get

$$\begin{aligned} n_\tau + (nm)u_\xi &= 0 \\ m_\tau + (m^2)u_\xi &= 0 \\ u_\tau + P(n,m)_\xi &= (\varepsilon(m)mu_\xi)_\xi, \qquad \xi \in I := (0,1), \quad \tau \geq 0, \end{aligned}$$

with boundary conditions, in view of (20), given by

$$P(n,m) = \varepsilon(m)mu_\xi, \qquad \text{at } \xi = 0, 1, \quad \tau \geq 0.$$

In addition, we have the initial data

$$n(\xi,0) = n_0(\xi), \quad m(\xi,0) = m_0(\xi), \quad u(\xi,0) = u_0(\xi), \qquad \xi \in \bar{I} := [0,1].$$

In the following we replace the coordinates (ξ, τ) by (x, t) such that the model now takes the form

$$\begin{aligned} \partial_t n + (nm)\partial_x u &= 0 \\ \partial_t m + m^2 \partial_x u &= 0 \\ \partial_t u + \partial_x P(n,m) &= \partial_x(E(m)\partial_x u), \qquad x \in (0,1), \end{aligned} \tag{22}$$

with

$$P(n,m) = k_1 \Big(\frac{n}{\rho_l - m}\Big)^{\gamma}, \qquad \gamma > 1, \tag{23}$$

and

$$E(m) = k_2 \Big(\frac{m}{\rho_l - m}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3. \tag{24}$$

Moreover, boundary conditions are given by

$$P(n,m) = E(m)u_x, \qquad \text{at } x = 0,1, \quad t \geq 0, \tag{25}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x = \bar{I}. \tag{26}$$

3.2. **Main result.** Before we state the main result for the model (22)–(26), we describe the notation we apply throughout the paper. $W^{1,2}(I) = H^1(I)$ represents the usual Sobolev space defined over $I = (0,1)$ with norm $\|\cdot\|_{W^{1,2}}$. Moreover, $L^p(K,B)$ with norm $\|\cdot\|_{L^p(K,B)}$ denotes the space of all strongly measurable, pth-power integrable functions from K to B where K typically is subset of $\mathbb{R}$ and B is a Banach space. In addition, let $C^\alpha[0,1]$ for $\alpha \in (0,1)$ denotes the Banach space of functions on $[0,1]$ which are uniformly Hölder continuous with exponent α. Similarly, let $C^{\alpha,\alpha/2}(D_T)$ represent the Banach space of functions on $D_T = [0,1]\times[0,T]$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t.

Theorem 3.1 (Main Result). *Assume that $\gamma > 1$ and $\beta \in (0,1/3)$ respectively in (23) and (24), and that the initial data (n_0, m_0, u_0) satisfy*

(i) $\inf_{[0,1]} n_0 > 0$, $\sup_{[0,1]} n_0 < \infty$, $\inf_{[0,1]} m_0 > 0$, *and* $\sup_{[0,1]} m_0 < \rho_l$;

(ii) $n_0, m_0 \in W^{1,2}(I)$;

(iii) $u_0 \in L^{2k}(I)$, *for* $k \in \mathbb{N}$.

Then the initial-boundary problem (22)–(26) possesses a global weak solution (n,m,u) in the sense that for any $T > 0$,

(A) we have the following estimates:

$$\begin{aligned}
&n, m \in L^\infty([0,T], W^{1,2}(I)), \qquad n_t, m_t \in L^2([0,T], L^2(I)),\\
&u \in L^\infty([0,T], L^{2k}(I)) \cap L^2([0,T], H^1(I)),\\
&\mu \inf_{[0,1]}(c_0) \leq n(x,t) \leq (\rho_l - \mu)\sup_{[0,1]}(c_0), \qquad c_0 := \frac{n_0}{m_0},\\
&0 < \mu \leq m(x,t) \leq \rho_l - \mu < \rho_l, \qquad \forall (x,t) \in [0,1]\times[0,T],
\end{aligned}$$

for $\mu = \mu(\|c_0\|_{W^{1,2}(I)}, \|Q_0^\beta\|_{W^{1,2}(I)}, \|u_0\|_{L^{2k}(I)}, \sup_{[0,1]} c_0, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, T) > 0$ *where* $Q_0 = m_0/(\rho_l - m_0)$.

(B) Moreover, the following equations hold,

$$\begin{aligned}
&n_t + nmu_x = 0, \quad m_t + m^2 u_x = 0,\\
&(n,m)(x,0) = (n_0(x), m_0(x)), \ \text{ for a.e. } x \in (0,1) \text{ and any } t \geq 0,\\
&\int_0^\infty \int_0^1 \Big[u\phi_t + (P(n,m) - E(m)u_x)\phi_x\Big]\,dx\,dt + \int_0^1 u_0(x)\phi(x,0)\,dx = 0
\end{aligned} \tag{27}$$

for any test function $\phi(x,t) \in C_0^\infty(D)$, with $D := \{(x,t)\,|\,0 \leq x \leq 1,\ t \geq 0\}$.

The proof of Theorem 3.1 is based on a priori estimates for the approximate solutions of (22)–(26) and a corresponding limit procedure. As a part of this process it will be crucial to obtain pointwise upper and lower limits for m in order to control the quantities $\int_0^1 (m_x)^2\,dx$ and $\int_0^1 (n_x)^2\,dx$, see Corollary 2. The main idea in the following analysis is to focus, not on the mass m but instead the related quantity $Q(m) = m/(\rho_l - m)$ which connects pressure $P(n,m)$ and viscosity coefficient $E(m)$.

It turns out that we naturally can reformulate the model (22) in terms of the variables (c, Q, u) instead of (n, m, u) where $c = n/m$. Together with higher order regularity of u and $(Q^\beta)_x$, and energy-conservation obtained by adopting techniques used in [24, 17, 19, 27, 25, 16] for single-phase Navier-Stokes equations, pointwise upper and lower limits for $Q(m)$ can be derived. This, in turn, gives the required boundedness on m from below and above together with the L^2 estimate of m_x and n_x. From these estimates, which are derived in the coming section, we can rely on standard compactness arguments to prove Theorem 3.1. This is done in Section 5.

4. **Basic estimates.** Below we derive a priori estimates for (n, m, u) which are assumed to be a smooth solution of (22)–(26). We then construct the approximate solutions of (22) in Section 5 by mollifying the initial data n_0, m_0, u_0 and obtain global existence by taking the limit.

More precisely, similar to [16] we first assume that (n, m, u) is a solution of (22)–(26) on $[0, T]$ satisfying

$$\begin{aligned} &n, n_t, n_x, n_{tx}, m, m_x, m_t, m_{tx}, u, u_x, u_t, u_{xx} \in C^{\alpha,\alpha/2}(D_T) \text{ for some } \alpha \in (0,1), \\ &n(x,t) > 0, \qquad m(x,t) > 0, \qquad m(x,t) < \rho_l \quad \text{on } D_T = [0,1]\times[0,T]. \end{aligned} \tag{28}$$

In the following we will frequently take advantage of the fact that the model (22) can be rewritten in a form more amenable for deriving various useful estimates. We first describe this reformulation, and then present a number of a priori estimates.

4.1. **A reformulation of the model (22).** We introduce the variable

$$c = \frac{n}{m}, \tag{29}$$

and see from the first two equations of (22) that

$$\partial_t c = \frac{1}{m} n_t - \frac{n}{m^2} m_t = -\frac{nm}{m} u_x + \frac{nm^2}{m^2} u_x = 0.$$

Consequently, the model (22)–(26) then can be written in terms of the variables (c, m, u) in the form

$$\begin{aligned} \partial_t c &= 0 \\ \partial_t m + m^2 \partial_x u &= 0 \\ \partial_t u + \partial_x P(c,m) &= \partial_x (E(m)\partial_x u), \end{aligned} \tag{30}$$

with

$$P(c,m) = k_1 \Big(\frac{mc}{\rho_l - m}\Big)^\gamma, \qquad \gamma > 1,$$

and

$$E(m) = k_2 \Big(\frac{m}{\rho_l - m}\Big)^{\beta+1}, \qquad 0 < \beta < 1/3.$$

Moreover, boundary conditions are given by

$$P(c,m) = E(m) u_x, \qquad \text{at } x = 0, 1, \quad t \ge 0,$$

whereas initial data are

$$c(x,0) = c_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x = \bar{I}.$$

Furthermore, we introduce the variable

$$Q(m) = \frac{m}{\rho_l - m} = \frac{\alpha_l}{1 - \alpha_l} > 0, \tag{31}$$

since $m > 0$ and $m < \rho_l$, and observe that

$$Q(m)_t = \Big(\frac{m}{\rho_l - m}\Big)_t = \Big(\frac{1}{\rho_l - m} + \frac{m}{(\rho_l - m)^2}\Big)m_t$$
$$= \frac{\rho_l}{(\rho_l - m)^2} m_t = -\rho_l \frac{m^2}{(\rho_l - m)^2} u_x = -\rho_l Q(m)^2 u_x,$$

in view of the second equation of (30). Consequently, we rewrite the model (30) in the form

$$\begin{aligned} \partial_t c &= 0 \\ \partial_t Q(m) + \rho_l Q(m)^2 u_x &= 0 \\ \partial_t u + \partial_x P(c,m) &= \partial_x (E(m)\partial_x u), \end{aligned} \tag{32}$$

with

$$P(c,m) = k_1 c^\gamma Q(m)^\gamma, \qquad \gamma > 1,$$

and

$$E(m) = k_2 Q(m)^{\beta+1}, \qquad 0 < \beta < 1/3.$$

This model is then subject to the boundary conditions

$$P(c,m) = E(m)u_x, \qquad \text{at } x = 0,1, \quad t \geq 0. \tag{33}$$

In addition, we have the initial data

$$c(x,0) = c_0(x), \quad m(x,0) = m_0(x), \quad u(x,0) = u_0(x), \qquad x = [0,1]. \tag{34}$$

In particular, the first equation of (32) gives that

$$c(x,t) = c_0(x) = \frac{n_0}{m_0}(x) > 0, \qquad t > 0, \tag{35}$$

for initial data as prescribed in Theorem 3.1.

4.2. **A priori estimates.** Now we derive a priori estimates for (n,m,u) by making use of the reformulated model (32)–(34).

Lemma 4.1 (Energy estimate)**.** *We have the basic energy estimate*

$$\begin{aligned} \int_0^1 \Big(\frac{1}{2}u^2 + \frac{k_1 c_0^\gamma}{\rho_l(\gamma-1)} Q(m)^{\gamma-1}\Big)(t,x)\,dx + k_2 \int_0^t \int_0^1 Q(m)^{\beta+1}(u_x)^2\,dx\,ds \\ = \int_0^1 \Big(\frac{1}{2}u_0^2 + \frac{k_1 c_0^\gamma}{\rho_l(\gamma-1)} Q(m_0)^{\gamma-1}\Big)\,dx, \qquad \forall t \in [0,T]. \end{aligned} \tag{36}$$

Moreover,

$$Q(m)(x,t) \leq C_1, \qquad \forall (x,t) \in [0,1]\times[0,T], \tag{37}$$

where $C_1 = C_1(\sup_{[0,1]} Q_0, \|u_0\|_{L^2(I)}, \|c_0\|_{L^\gamma(I)})$*. Note that* $|Q(m_0)|$ *is pointwise bounded since* $\sup_{[0,1]} m_0 < \rho_l$ *and* $\inf_{[0,1]} m_0 > 0$*. Moreover, for any positive integer* k*,*

$$\int_0^1 u^{2k}(x,t)\,dx + k(2k-1)k_2 \int_0^t \int_0^1 u^{2k-2} Q(m)^{1+\beta}(u_x)^2\,dx\,dt \leq C_2, \tag{38}$$

where $C_2 = C_2(\sup_{[0,1]} c_0, \|u_0\|_{L^{2k}(I)}, T, k, C_1)$*.*

Proof. We multiply the third equation of (32) by u and integrate over $[0,1]$ in space. Applying the boundary condition (33) and the equation

$$\frac{k_1 c^\gamma}{\rho_l(\gamma-1)}(Q^{\gamma-1})_t + k_1 c^\gamma Q^\gamma u_x = 0,$$

obtained from the second equation of (32) by multiplying with $k_1 c^\gamma Q^{\gamma-2}$, we get

$$\frac{d}{dt}\int_0^1 \Big(\frac{1}{2}u^2 + \frac{k_1 c^\gamma}{\rho_l(\gamma-1)}Q^{\gamma-1}\Big)(t,x)\,dx + \int_0^1 E(m)(u_x)^2\,dx = 0.$$

From this, (36) follows.

Next, we focus on (37). From the second equation of (32) we deduce the equation

$$\frac{1}{\rho_l}(Q^\beta)_t + \beta Q^{\beta+1}u_x = 0. \tag{39}$$

Integrating over $[0,t]$, we get

$$Q^\beta(x,t) = Q^\beta(x,0) - \beta\rho_l \int_0^t Q^{\beta+1}u_x\,ds. \tag{40}$$

Then, we integrate the third equation of (32) over $[0,x]$ and get

$$\begin{aligned}\int_0^x u_t(y,t)\,dy + P(c,m) - P(c(0,t),m(0,t)) + (E(m)u_x)(0,t) &= E(m)u_x\\ &= k_2 Q(m)^{\beta+1}u_x.\end{aligned}$$

Using the boundary condition (33) and inserting the above relation into the right hand side of (40), we get

$$\begin{aligned}Q^\beta(x,t) &= Q^\beta(x,0) - \frac{\beta\rho_l}{k_2}\int_0^t \Big(\int_0^x u_t(y,t)\,dy + P(c,m)\Big)\,ds\\ &= Q^\beta(x,0) - \frac{\beta\rho_l}{k_2}\int_0^x (u(y,t)-u_0(y))\,dy - \frac{\beta\rho_l}{k_2}\int_0^t P(c,m)\,ds\end{aligned}$$

Consequently, since $P(c,m)\geq 0$

$$Q^\beta(x,t) \leq Q^\beta(x,0) + \frac{\beta\rho_l}{k_2}\int_0^1 |u(y,t)|\,dy + \frac{\beta\rho_l}{k_2}\int_0^1 |u_0(y)|\,dy.$$

Applying Hölder's inequality and (36) we can bound $\int_0^1 |u|\,dy$, hence the upper bound (37) follows.

Finally, we focus on estimate (38). Multiplying the third equation of (32) by $2ku^{2k-1}$, integrating over $[0,1]\times[0,t]$ and integration by parts together with application of the boundary conditions (33), we get

$$\begin{aligned}&\int_0^1 u^{2k}\,dx + 2k(2k-1)k_2\int_0^t\int_0^1 Q(m)^{\beta+1}(u_x)^2u^{2k-2}\,dx\,ds\\ &\qquad = \int_0^1 u_0^{2k}\,dx + 2k(2k-1)k_1\int_0^t\int_0^1 c_0^\gamma Q(m)^\gamma u^{2k-2}u_x\,dx\,ds.\end{aligned} \tag{41}$$

For the last term we apply Cauchy's inequality with ε, $ab \le (1/4\varepsilon)a^2 + \varepsilon b^2$, and get

$$\begin{aligned}
&\int_0^t \int_0^1 c_0^\gamma Q(m)^\gamma u^{2k-2} u_x \, dx \, ds \\
&\le \frac{1}{4\varepsilon} \int_0^t \int_0^1 c_0^{2\gamma} Q(m)^{2\gamma-\beta-1} u^{(2k-2)} \, dx \, ds + \varepsilon \int_0^t \int_0^1 Q(m)^{\beta+1} u^{(2k-2)} (u_x)^2 \, dx \, ds \\
&\le \frac{1}{4\varepsilon} \sup_{[0,1]} (c_0^{2\gamma}) \int_0^t \int_0^1 Q(m)^{2\gamma-\beta-1} u^{(2k-2)} \, dx \, ds \\
&\qquad\qquad + \varepsilon \int_0^t \int_0^1 Q(m)^{\beta+1} u^{(2k-2)} (u_x)^2 \, dx \, ds.
\end{aligned}$$

The last term clearly can be absorbed in the second term of the left-hand side of (41) by the choice $\varepsilon = k_2/2k_1$. Finally, let us see how we can bound the term $\int_0^t \int_0^1 u^{(2k-2)} Q(m)^{2\gamma-1-\beta} \, dx \, ds$. In view of Young's inequality $ab \le (1/p)a^p + (1/q)b^q$ where $1/p + 1/q = 1$, we get for the choice $p = k$ and $q = k/(k-1)$

$$\begin{aligned}
&\int_0^t \int_0^1 u^{(2k-2)} Q(m)^{2\gamma-1-\beta} \, dx \, ds \\
&\qquad \le \frac{1}{k} \int_0^t \int_0^1 Q(m)^{(2\gamma-1-\beta)k} \, dx \, ds + \frac{k-1}{k} \int_0^t \int_0^1 u^{2k} \, dx \, ds \\
&\qquad \le \frac{C_1^{(2\gamma-1-\beta)k}}{k} t + \frac{k-1}{k} \int_0^t \int_0^1 u^{2k} \, dx \, ds,
\end{aligned}$$

by using (37). To sum up, we get

$$\begin{aligned}
&\int_0^1 u^{2k} \, dx + k(2k-1)k_2 \int_0^t \int_0^1 Q(m)^{\beta+1} (u_x)^2 u^{2k-2} \, dx \, ds \\
&\le \int_0^1 u_0^{2k} \, dx + 2k(2k-1)k_1 \frac{1}{4\varepsilon} \sup_{[0,1]} (c_0^{2\gamma}) \Big[\frac{C_1^{2\gamma-1-\beta}}{k} t + \frac{k-1}{k} \int_0^t \int_0^1 u^{2k} \, dx \, ds \Big] \\
&= \int_0^1 u_0^{2k} \, dx + (2k-1) \frac{k_1^2}{k_2} \sup_{[0,1]} (c_0^{2\gamma}) \Big[C_1^{2\gamma-1-\beta} t + (k-1) \int_0^t \int_0^1 u^{2k} \, dx \, ds \Big].
\end{aligned} \tag{42}$$

In view of (42), an application of Gronwall's inequality then gives the estimate (38). ☐

Lemma 4.2 (Additional regularity)**.** *We have the estimate*

$$\int_0^1 (\partial_x Q^\beta(m))^2(x,t) \, dx \le C_3, \tag{43}$$

for a constant $C_3 = C_3(\|Q_0^\beta\|_{W^{1,2}(I)}, \|c_0\|_{W^{1,2}(I)}, \|u_0\|_{L^2(I)}, C_1, C_2, \sup_{[0,1]} c_0, T)$.

Proof. Using (39) in the third equation of (32) and integrating in time over $[0,t]$ we arrive at

$$u(x,t) - u_0(x) + \int_0^t \partial_x P(c,m)(x,s) \, ds = -\frac{k_2}{\beta \rho_l} (\partial_x Q^\beta(x,t) - \partial_x Q^\beta(x,0)). \tag{44}$$

Multiplying (44) by $\frac{\beta\rho_l}{k_2}(\partial_x Q^\beta)$ and integrating over $[0,1]$ in x, we get

$$\begin{aligned}
&\int_0^1 (\partial_x Q^\beta)^2\,dx\\
&= \int_0^1 (\partial_x Q^\beta)\partial_x Q_0^\beta\,dx - \frac{\beta\rho_l}{k_2}\int_0^1 (\partial_x Q^\beta)\Big[(u-u_0)+\int_0^t \partial_x P(c,m)\,ds\Big]\,dx\\
&\le \Big(\int_0^1 (\partial_x Q^\beta)^2\,dx\Big)^{1/2}\Big(\|\partial_x Q_0^\beta\|_{L^2(I)} + \frac{\beta\rho_l}{k_2}\|u-u_0\|_{L^2(I)} + \frac{\beta\rho_l}{k_2}\Big\|\int_0^t \partial_x P\,ds\Big\|_{L^2(I)}\Big)\\
&:= ab,
\end{aligned}$$

where we have used Hölder's inequality. Cauchy's inequality $ab \le a^2/2 + b^2/2$ then gives

$$\begin{aligned}
&\int_0^1 (\partial_x Q^\beta)^2\,dx\\
&\le \frac{1}{2}\int_0^1 (\partial_x Q^\beta)^2\,dx + \frac{1}{2}\Big(\|\partial_x Q_0^\beta\|_{L^2(I)} + \frac{\beta\rho_l}{k_2}\|u-u_0\|_{L^2(I)} + \frac{\beta\rho_l}{k_2}\Big\|\int_0^t \partial_x P\,ds\Big\|_{L^2(I)}\Big)^2\\
&\le \frac{1}{2}\int_0^1 (\partial_x Q^\beta)^2\,dx + C(\|Q_0^\beta\|_{W^{1,2}(I)}, \|u_0\|_{L^2(I)}, C_2) + \frac{\beta\rho_l T}{k_2}\int_0^t\int_0^1 (\partial_x P)^2\,dx\,ds,
\end{aligned}\tag{45}$$

by using Jensen's inequality and (38) with $k=1$. Moreover,

$$\begin{aligned}
\int_0^t\int_0^1 (\partial_x P)^2\,dx\,ds &= k_1^2\int_0^t\int_0^1 \Big(Q^\gamma (c^\gamma)_x + c^\gamma (Q^\gamma)_x\Big)^2\,dx\,ds\\
&\le 2k_1^2\Big(\int_0^t\int_0^1 Q^{2\gamma}(c^\gamma)_x^2\,dx\,ds + \int_0^t\int_0^1 c^{2\gamma}(Q^\gamma)_x^2\,dx\,ds\Big)\\
&\le 2k_1^2(\sup_{[0,1]} Q)^{2\gamma}\int_0^t\int_0^1 (c^\gamma)_x^2\,dx\,ds + 2k_1^2(\sup_{[0,1]} c)^{2\gamma}\int_0^t\int_0^1 (Q^\gamma)_x^2\,dx\,ds\\
&\le 2k_1^2 C_1^{2\gamma}\int_0^t\int_0^1 (c^\gamma)_x^2\,dx\,ds + 2k_1^2(\sup_{[0,1]} c_0)^{2\gamma}\int_0^t\int_0^1 (Q^\gamma)_x^2\,dx\,ds,
\end{aligned}\tag{46}$$

in view of estimate (37) and the initial pointwise bound on initial data c_0. Moreover,

$$\begin{aligned}
\int_0^t\int_0^1 (Q^\gamma)_x^2\,dx\,ds &= \Big(\frac{\gamma}{\beta}\Big)^2\int_0^t\int_0^1 Q^{2(\gamma-\beta)}(Q^\beta)_x^2\,dx\,ds\\
&\le \Big(\frac{\gamma}{\beta}\Big)^2 C_1^{2(\gamma-\beta)}\int_0^t\int_0^1 (Q^\beta)_x^2\,dx\,ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t\int_0^1 (c^\gamma)_x^2\,dx\,ds &= \gamma^2\int_0^t\int_0^1 c^{2(\gamma-1)}(c_x)^2\,dx\,ds\\
&\le \gamma^2(\sup_{[0,1]} c)^{2(\gamma-1)}\int_0^t\int_0^1 (c_x)^2\,dx\,ds \le \gamma^2(\sup_{[0,1]} c_0)^{2(\gamma-1)} t\|c_0\|_{W^{1,2}(I)}.
\end{aligned}\tag{47}$$

Consequently, we see that we must require that $c = c_0 \in W^{1,2}(I)$ in order to bound the right hand side of (47). In light of (46)–(47), we conclude from (45) that

$$\int_0^1 (\partial_x Q^\beta)^2\,dx \le C + C\int_0^t\int_0^1 (\partial_x Q^\beta)^2\,dx\,ds.$$

Thus, application of Gronwall's inequality gives the estimate (43). □

Lemma 4.3 (Pointwise lower limit)**.** *Let $0 < \beta < 1/3$. Then we have a pointwise lower limit on $Q(m)$ of the form*

$$Q(m)(x,t) \geq C_4, \qquad \forall (x,t) \in [0,1]\times[0,T], \tag{48}$$

where the constant $C_4 = C_4(C_2, C_3, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, T, \|u_0\|_{L^2(I)}, \|c_0\|_{L^\gamma(I)})$.

Proof. We first define

$$v(x,t) = \frac{1}{Q(x,t)}, \qquad V(t) = \max_{[0,1]\times[0,t]} v(x,s).$$

We calculate as follows:

$$\begin{aligned} v(x,t) - v(0,t) &= \int_0^x \partial_x v\,dx \leq \int_0^1 |\partial_x Q| v^2\,dx \\ &= \frac{1}{\beta}\int_0^1 v^{\beta+1}|\partial_x Q^\beta|\,dx \\ &\leq \frac{1}{\beta}\Big(\int_0^1 |\partial_x Q^\beta|^2\,dx\Big)^{1/2}\Big(\int_0^1 v^{2(\beta+1)}\,dx\Big)^{1/2} \\ &\leq \frac{C_3^{1/2}}{\beta}\Big(\int_0^1 v\,dx\Big)^{1/2}\Big((\max_{[0,1]} v(\cdot,t))^{2\beta+1}\Big)^{1/2} \\ &\leq \frac{C_3^{1/2}}{\beta}\Big(\int_0^1 v\,dx\Big)^{1/2}\Big(\max_{[0,1]} v(\cdot,t)\Big)^{\beta+1/2}, \end{aligned} \tag{49}$$

where we have used (43). Next, we focus on how to estimate $\int_0^1 v\,dx$. The starting point is the observation that the second equation of (32) can be written as

$$v_t - \rho_l u_x = 0.$$

Integrating over $[0,1]\times[0,t]$ we get

$$\begin{aligned} \int_0^1 v(x,t)\,dx &= \int_0^1 v(x,0)\,dx + \rho_l\int_0^t [u(1,s) - u(0,s)]\,ds \\ &\leq (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l \int_0^t \max_{[0,1]} |u(\cdot,s)|\,ds \\ &\leq (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l\sqrt{t}\Big(\int_0^t \max_{[0,1]} |u(\cdot,s)|^2\,ds\Big)^{1/2} \\ &\leq (\inf_{[0,1]} Q_0)^{-1} + 2\rho_l\sqrt{t}\Big(\int_0^t \|u^2(s)\|_{L^\infty(I)}\,ds\Big)^{1/2}, \end{aligned} \tag{50}$$

where we have used Hölder's inequality. In light of Sobolev's inequality $\|f\|_{L^\infty(I)} \leq C\|f\|_{W^{1,1}(I)}$ it follows that the last term of (50) can be estimated as follows:

$$
\begin{aligned}
\int_0^t \|u^2(s)\|_{L^\infty(I)}\,ds &\leq C\int_0^t \|u^2(s)\|_{W^{1,1}(I)}\,ds \\
&= C\Big(\int_0^t\int_0^1 u^2\,dx\,ds + \int_0^t\int_0^1 |(u^2)_x|\,dx\,ds\Big) \\
&\leq C\int_0^t\Big(\int_0^1 u^4\,dx\Big)^{1/2} ds + 2C\int_0^t\int_0^1 Q^{\frac{1+\beta}{2}}|u||u_x|v^{\frac{1+\beta}{2}}\,ds\,ds \\
&\leq CtC_2^{1/2} + 2C\Big(\int_0^t\int_0^1 Q^{1+\beta}u_x^2u^2\,dx\,ds\Big)^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2} \\
&\leq CtC_2^{1/2} + 2CC_2^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2},
\end{aligned}
\tag{51}
$$

where we have used (38) with $k = 2$ and Hölder's inequality. Combining (50) and (51) we get

$$
\begin{aligned}
&\int_0^1 v(x,t)\,dx \\
&\quad\leq C(\inf_{[0,1]} Q_0) + 2\rho_l\sqrt{t}\Big[CtC_2^{1/2} + 2CC_2^{1/2}\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/2}\Big]^{1/2} \\
&\quad\leq C(\inf_{[0,1]} Q_0, C_2, T) + C(\inf_{[0,1]} Q_0, C_2, T)\Big(\int_0^t\int_0^1 v^{1+\beta}\,dx\,ds\Big)^{1/4} \\
&\quad= C(\inf_{[0,1]} Q_0, C_2, T) + C(\inf_{[0,1]} Q_0, C_2, T)\Big(\int_0^t\int_0^1 v^{2\beta}v^{1-\beta}\,dx\,ds\Big)^{1/4} \\
&\quad\leq C(\inf_{[0,1]} Q_0, C_2, T) + C(\inf_{[0,1]} Q_0, C_2, T)V(t)^{2\beta/4}\Big(\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds\Big)^{1/4}.
\end{aligned}
\tag{52}
$$

Now we focus on estimating $\int_0^t\int_0^1 v^{1-\beta}\,dx\,ds$. For that purpose, we note that the second equation of (32), by multiplying with $Q^{\frac{\beta-1}{2}-1}$, can be written as

$$
(Q^{\frac{\beta-1}{2}})_t = \rho_l\frac{1-\beta}{2}Q^{\frac{\beta+1}{2}}u_x.
$$

Integrating this equation over $[0,t]$ we get

$$
Q^{\frac{\beta-1}{2}}(x,t) = Q^{\frac{\beta-1}{2}}(x,0) + \rho_l\frac{1-\beta}{2}\int_0^t Q^{\frac{\beta+1}{2}}u_x\,ds.
$$

Consequently, using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we get

$$
\begin{aligned}
Q^{\beta-1}(x,t) &\leq 2Q^{\beta-1}(x,0) + 2\rho_l^2\Big(\frac{1-\beta}{2}\Big)^2\Big(\int_0^t Q^{\frac{\beta+1}{2}}u_x\,ds\Big)^2 \\
&\leq 2Q^{\beta-1}(x,0) + 2\rho_l^2 t\Big(\frac{1-\beta}{2}\Big)^2\int_0^t Q^{\beta+1}u_x^2\,ds,
\end{aligned}
$$

by Jensen's inequality. Integrating over $[0,1]$ in space yields

$$\begin{aligned}\int_0^1 v^{1-\beta}(x,t)\,dx &\le 2\int_0^1 v^{1-\beta}(x,0)\,dx + 2\rho_l^2 t\Big(\frac{1-\beta}{2}\Big)^2\int_0^1\int_0^t Q^{\beta+1}u_x^2\,ds\,dx\\ &\le C(\inf_{[0,1]} Q_0) + 2\rho_l^2 t\Big(\frac{1-\beta}{2}\Big)^2\int_0^1\int_0^t Q^{\beta+1}u_x^2\,ds\,dx \\ &\le C(\inf_{[0,1]} Q_0, \|u_0\|_{L^2(I)}, \sup_{[0,1]} Q_0, \|c_0\|_{L^\gamma(I)}, T),\end{aligned} \tag{53}$$

by using (36). Thus, (52) and (53) imply that

$$\begin{aligned}\int_0^1 v\,dx &\le C(\inf_{[0,1]} Q_0, C_2, T) + C(\inf_{[0,1]} Q_0, C_2, T, \|u_0\|_{L^2(I)}, \sup_{[0,1]} Q_0, \|c_0\|_{L^\gamma(I)})V(t)^{\beta/2}\\ &\le D_3[1+V(t)^{\beta/2}],\end{aligned}$$

for an appropriate coefficient $D_3 = D_3(\inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, C_2, T, \|u_0\|_{L^2(I)}, \|c_0\|_{L^\gamma(I)})$. Substituting this into (49) we get

$$\begin{aligned}v(x,t) - v(0,t) &\le \frac{C_3^{1/2}}{\beta}\Big(\int_0^1 v\,dx\Big)^{1/2}\Big(\max_{[0,1]} v(\cdot,t)\Big)^{\beta+1/2}\\ &\le \frac{(C_3D_3)^{1/2}}{\beta}[1+V(t)^{\beta/2}]^{1/2}V(t)^{\beta+1/2}\\ &\le E_3\max(V(t)^{(5/4)\beta+1/2}, 2^{1/2}),\end{aligned} \tag{54}$$

for $E_3 = E_3(C_3, D_3)$. Here we have used the inequality $(1+x^{\beta/4})x^{\beta+1/2} \le Cx^{(5/4)\beta+1/2}$ which holds for $x \ge 1$ and an appropriate constant C. This follows by observing that

$$f(x) = Cx^{(5/4)\beta+1/2} - x^{\beta+1/2}(1+x^{\beta/4}) = x^{\beta+1/2}((C-1)x^{\beta/4}-1) \ge 0,$$

for $x \ge 1$ and $C \ge 2$.

We must check that $v(0,t)$ remains bounded in $[0,T]$. From the boundary condition (33) we have

$$k_1 c_0^\gamma Q^\gamma - k_2 Q^{\beta+1}u_x\Big|_{x=0} = 0.$$

Using that $u_x Q^2 \rho_l = -Q_t$ for $x = 0$ we get

$$y' = -Ky^{\gamma-\beta+1},$$

where

$$K = \frac{\rho_l k_1}{k_2}c_0(0)^\gamma, \qquad y(t) = Q(0,t), \qquad y_0 = Q(0,0).$$

Hence,

$$\frac{1}{\beta-\gamma}(y^{\beta-\gamma} - y_0^{\beta-\gamma}) = -Kt, \qquad \text{or} \quad y^{\beta-\gamma} = -K(\beta-\gamma)t + y_0^{\beta-\gamma}.$$

Equivalently,

$$(y^{-1})^{\gamma-\beta} = (y_0^{-1})^{\gamma-\beta}\Big(K(\gamma-\beta)y_0^{\gamma-\beta}t + 1\Big).$$

Consequently, for $t \in [0,T]$

$$v(0,t) = v(0,0)\Big(K(\gamma-\beta)Q(0,0)^{\gamma-\beta}t+1\Big)^{1/(\gamma-\beta)} \le C(\sup_{[0,1]} c_0, \inf_{[0,1]} Q_0, T).$$

In conclusion, from (54) we have

$$V(T) \le C(\sup_{[0,1]} c_0, \inf_{[0,1]} Q_0, T) + E_3 V(T)^{(5/4)\beta+1/2}.$$

Since $\beta < 2/6 < 2/5$ we see that $(5/4)\beta + 1/2 < 1$. Therefore, it is clear from the general inequality $x \le C(1 + x^{\xi})$ with $0 < \xi < 1$, that $x \le C$ for some constant C. Consequently, $V(T) \le C_4$ where (in view of the above estimates)

$$C_4 = C_4(C_2, C_3, \inf_{[0,1]} Q_0, \sup_{[0,1]} Q_0, T, \|u_0\|_{L^2(I)}, \|c_0\|_{L^{\gamma}(I)}).$$

Thus, the result (48) follows. □

We have the following estimate which ensures that no transition to single-phase flow occurs.

Corollary 1. *There is a constant $\mu = \mu(C_1, C_4) > 0$ such that*

$$\begin{aligned} \mu \le & \alpha_l \le 1 - \mu \qquad\qquad (\textit{equivalently}, \quad \mu\rho_l \le m \le \rho_l - \mu\rho_l), \\ \rho_l \mu \inf_{[0,1]}(c_0) \le & n \le \rho_l(1-\mu) \sup_{[0,1]}(c_0), \end{aligned} \tag{55}$$

for $c_0 = n_0/m_0$.

Proof. In view of (31) and the bounds (37) and (48) it is clear that the first estimate of (55) follows. The second follows from the first and the fact that $n = c_0 m$ which is a consequence of (29) and (35). □

Corollary 2. *We have the estimates*

$$\int_0^1 (\partial_x m)^2\, dx \le C_5, \qquad \int_0^1 (\partial_x n)^2\, dx \le C_6, \tag{56}$$

for a constant $C_5 = C_5(C_3, C_4)$ and $C_6 = C_6(\|c_0\|_{W^{1,2}(I)}, \sup_{[0,1]} c_0, C_5)$.

Proof. It follows that

$$\partial_x Q(m)^{\beta} = \beta Q(m)^{\beta-1} Q'(m)\partial_x m = \beta\rho_l Q(m)^{\beta-1}\frac{Q(m)^2}{m^2}\partial_x m = \beta\rho_l \frac{Q(m)^{\beta+1}}{m^2}\partial_x m,$$

since $Q'(m) = (\rho_l/m^2)Q(m)^2$. In view of this calculation and the pointwise upper and lower limits for $Q(m)$, as well as m, given by (37), (48), and (55), it follows by application of Lemma 4.2 that the first estimate of (56) holds. The second follows directly from the relation

$$\partial_x n = m\partial_x c_0 + c_0 \partial_x m, \qquad \text{since } n = c_0 m,$$

and the estimate

$$\int_0^1 (\partial_x n)^2\, dx \le 2\rho_l^2 \int_0^1 (\partial_x c_0)^2 + 2(\sup_{[0,1]} c_0)^2 \int_0^1 (\partial_x m)^2\, dx \le C_6,$$

by the first estimate of (56) and the assumptions on the initial data n_0 and m_0. □

Remark 1. Note that the estimate of Lemma 4.2 can be generalized such that $\partial_x Q^{\beta}(\cdot, t)$ lies in $L^{2k}(I)$ for any integer k. As a consequence, the estimate of Lemma 4.3 can be shown to hold under the weaker assumption $\beta \in (0,1)$. This follows by a slight modification of the above calculations according to [16]. Consequently, the result of Theorem 3.1 can also be shown to hold for the more general case where $\beta \in (0,1)$.

5. **Proof of existence result.** Now focus is on the model (22). All arguments in this section closely follow along the line of [16], however, for completeness we include the main steps. First, we introduce the Friedrichs mollifier $j_\delta(x)$. Let $\psi(x) \in C_0^\infty(\mathbb{R})$ satisfy $\psi(x) = 1$ when $|x| \le 1/2$ and $\psi(x) = 0$ when $|x| \ge 1$, and define $\psi_\delta := \psi(x/\delta)$.

Mollifying. We extend n_0, m_0, u_0 to $\mathbb{R}$ by using

$$n_0(x) := \begin{cases} n_0(1), & x \in (1,\infty), \\ n_0(x), & x \in [0,1], \\ n_0(0), & x \in (-\infty, 0), \end{cases} \qquad m_0(x) := \begin{cases} m_0(1), & x \in (1,\infty), \\ m_0(x), & x \in [0,1], \\ m_0(0), & x \in (-\infty, 0), \end{cases}$$

whereas we extend $u_0(x)$ to $\mathbb{R}$ by defining it to be zero outside the interval $[0,1]$. Approximate initial data $(n_0^\delta, m_0^\delta, u_0^\delta)$ to (n_0, m_0, u_0) are now defined as follows:

$$\begin{aligned} &n_0^\delta(x) = (n_0 * j_\delta)(x), \qquad m_0^\delta(x) = (m_0 * j_\delta)(x), \\ &u_0^\delta = (u_0 * j_\delta)(x)[1 - \psi_\delta(x) - \psi_\delta(1-x)] + (u_0 * j_\delta)(0)\psi_\delta(x) + (u_0 * j_\delta)(1)\psi_\delta(1-x) \\ &+ \frac{k_1}{k_2}(c_0^\delta)^\gamma Q(m_0^\delta)^{\gamma-\beta-1}(0)\int_0^x \psi_\delta(y)\,dy - \frac{k_1}{k_2}(c_0^\delta)^\gamma Q(m_0^\delta)^{\gamma-\beta-1}(1)\int_x^1 \psi_\delta(1-y)\,dy. \end{aligned}$$

Then it follows that $n_0^\delta, m_0^\delta \in C^{1+s}[0,1]$, $u_0^\delta \in C^{2+s}[0,1]$ for any $0 < s < 1$, and n_0^δ, m_0^δ and u_0^δ are compatible with the boundary conditions (25). Moreover, it follows that

$$\begin{aligned} |(u_0 * j_\delta)(0)|^{2k} \int_0^1 \psi_\delta^{2k}\,dx &\le C\delta \Big(\int_0^\delta u_0(x) j_\delta(x)\,dx \Big)^{2k} \\ &\le C\delta \int_0^\delta u_0^{2k}\,dx \Big(\int_0^\delta j_\delta^{2k/(2k-1)}(x)\,dx \Big)^{2k-1} \\ &\le C \int_0^\delta u_0^{2k}(x)\,dx \to 0 \quad \text{as } \delta \to 0. \end{aligned}$$

Similarly, it follows that $|(u_0 * j_\delta(1)|^{2k} \int_0^1 \psi_\delta^{2k}(1-x)\,dx \to 0$. Therefore, recalling the definition of $u_0^\delta(x)$ we see that as $\delta \to 0$,

$$u_0^\delta \to u_0 \;\text{ in } L^{2k}(I).$$

In addition,

$$n_0^\delta \to n_0, \quad m_0^\delta \to m_0 \qquad \text{uniformly in } [0,1],$$

as $\delta \to 0$.

Now, we consider the initial boundary value problem (22)–(26) with the initial data (n_0, m_0, u_0) replaced by $(n_0^\delta, m_0^\delta, u_0^\delta)$. For this problem standard arguments can be used (the energy estimates and the contraction mapping theorem) to obtain the existence of a unique *local* solution $(n^\delta, m^\delta, u^\delta)$ with n^δ, n_t^δ, n_x^δ, n_{tx}^δ, m^δ, m_x^δ, m_t^δ, m_{tx}^δ, u^δ, u_x^δ, u_t^δ, $u_{xx}^\delta \in C^{\alpha,\alpha/2}([0,1]\times[0,T^*])$ for some $T^* > 0$.

In view of the estimates of Section 4.2, it follows that n^δ and m^δ are pointwise bounded from above and below, $(u^\delta)^k$, n_x^δ, and m_x^δ are bounded in $L^\infty([0,T], L^2(I))$ and u_x^δ is bounded in $L^2((0,T), L^2(I))$ for any $T > 0$. Furthermore, we can differentiate the equations in (22) and apply the energy method to derive bounds of high-order derivatives of $(n^\delta, m^\delta, u^\delta)$. Then the Schauder theory for linear parabolic equations can be applied to conclude that the $C^{\alpha,\alpha/2}(D_T)$-norm of n^δ, n_t^δ, n_x^δ, n_{tx}^δ, m^δ, m_x^δ, m_t^δ, m_{tx}^δ, u^δ, u_x^δ, u_t^δ, u_{xx}^δ is a priori bounded. Therefore, we can continue the local solution globally in time and obtain that there exists a unique global solution $(n^\delta, m^\delta, u^\delta)$ of (22)–(26) with initial data $(n_0^\delta, m_0^\delta, u_0^\delta)$ such that for any $T > 0$, the regularity of (28) holds.

Estimates and Compactness. Clearly, in view of the estimates of Section 4.2 and the model itself (22), we have for $t \in [0,T],\ k \in \mathbb{N}$

$$\begin{aligned} &\int_0^1 (u^\delta)^{2k}(x,t)\,dx + \int_0^1 (n_x^\delta)^2(x,t)\,dx + \int_0^1 (m_x^\delta)^2(x,t)\,dx \le C,\\ &\qquad \mu \le m^\delta(x,t) \le \rho_l - \mu,\\ &\qquad \mu \inf_{[0,1]} c_0(x) \le n^\delta(x,t) \le (\rho_l - \mu)\sup_{[0,1]} c_0(x), \qquad \text{for } (x,t) \in [0,1]\times[0,T],\\ &\int_0^T \int_0^1 \Big[(u_x^\delta)^2 + (n_t^\delta)^2 + (m_t^\delta)^2\Big](x,s)\,dx\,ds \le C, \end{aligned} \tag{57}$$

where the constants $C, \mu > 0$ do not depend on δ. Note that the boundedness of m_t^δ (n_t^δ) in $L^2([0,T], L^2(I))$ follows in view of the equation $m_t^\delta + (m^\delta)^2 u_x^\delta = 0$ $(n_t^\delta + n^\delta m^\delta u_x^\delta = 0)$, the estimates of Corollary 1, and the energy estimate (36) of Lemma 4.1. Hence, we can extract a subsequence of $(n^\delta, m^\delta, u^\delta)$, still denoted by $(n^\delta, m^\delta, u^\delta)$, such that as $\delta \to 0$,

$$\begin{aligned} &u^\delta \rightharpoonup u \text{ weak-* in } L^\infty([0,T], L^{2k}(I)),\\ &n^\delta \rightharpoonup n \text{ weak-* in } L^\infty([0,T], W^{1,2}(I)),\\ &m^\delta \rightharpoonup m \text{ weak-* in } L^\infty([0,T], W^{1,2}(I)),\\ &(n_t^\delta, m_t^\delta, u_x^\delta) \rightharpoonup (n_t, m_t, u_x) \text{ weakly in } L^2([0,T], L^2(I)). \end{aligned} \tag{58}$$

Next, we show that (n, m, u) obtained in (58) in fact is a weak solution of (22)–(26). The classical Sobolev imbedding (Morrey's inequality) $W^{1,2k}(0,1) \hookrightarrow C^{1-1/(2k)}[0,1]$ applied with $k = 1$ gives that for any $x_1, x_2 \in (0,1)$ and $t \in [0,T]$

$$|m^\delta(x_1,t) - m^\delta(x_2,t)| \le C|x_1 - x_2|^{1/2}. \tag{59}$$

To control continuity in time, in view of the sequence of imbeddings $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1) \hookrightarrow L^2(0,1)$, we can apply Lions-Aubin lemma (see for example [23], Section 1.3.12) for a constant $\nu > 0$ (arbitrary small) to find a constant C_ν such that

$$\begin{aligned} &\|m^\delta(t_1) - m^\delta(t_2)\|_{L^\infty(I)}\\ &\le \nu\|m^\delta(t_1) - m^\delta(t_2)\|_{W^{1,2}(I)} + C_\nu\|m^\delta(t_1) - m^\delta(t_2)\|_{L^2(I)}\\ &\le 2\nu\|m^\delta(t)\|_{W^{1,2}(I)} + C_\nu|t_1 - t_2|^{1/2}\|m_t^\delta\|_{L^2([0,T],L^2(I))}\\ &\le C\nu + C_\nu C|t_1 - t_2|^{1/2}, \end{aligned} \tag{60}$$

where we have used (57) to derive the last two inequalities. Consequently, (59) and (60) together with the triangle inequality show that $\{m^\delta\}$ is equi-continuous on $D_T = [0,1]\times[0,T]$. Hence, by Arzela-Ascoli's theorem and a diagonal process for t, we can extract a subsequence of $\{m^\delta\}$, such that

$$m^\delta(x,t) \to m(x,t) \text{ strongly in } C^0(D_T).$$

The same arguments apply to n yielding

$$n^\delta(x,t) \to n(x,t) \text{ strongly in } C^0(D_T).$$

Clearly, m_t is also bounded in $L^2([0,T], L^2(I))$ and from the estimate

$$\begin{aligned}\|m(t_1)-m(t_2)\|^2_{L^2(I)} &= \int_0^1 |m(t_1)-m(t_2)|^2\,dx = \int_0^1 \Big|\int_{t_1}^{t_2} m_t\,ds\Big|^2\,dx \\ &\le \int_0^1 \Big(\int_{t_1}^{t_2} |m_t|\,ds\Big)^2\,dx \le |t_1-t_2| \int_0^T\int_0^1 m_t^2\,dx\,ds,\end{aligned}$$

where we have used Hölder's inequality, we may also conclude that

$$m \in C^{1/2}([0,T], L^2(I)).$$

Similarly, the same arguments apply to n. Thus, we conclude that the limit functions (n,m,u) from (58) satisfy the first two equations $n_t + nmu_x = 0$ and $m_t + m^2u_x = 0$ of (27) for a.e. $x \in (0,1)$ and any $t \ge 0$. To show that the last integral equality holds, we multiply the third equation of (22) by $\phi \in C_0^\infty(D)$ with $D = [0,1]\times[0,\infty)$ and integrate over $(0,T)\times(0,1)$, followed by integration by parts with respect to x and t. Taking the limit as $\delta \to 0$, we see that (n,m,u) also must satisfy weakly the third equation of (27).

6. **A uniqueness result.** In this section we present a uniqueness result for the two-phase model (22) similar to the one presented in [16] for the single-phase Navier-Stokes equations. For that purpose we need more regularity of the fluid velocity u. More precisely, for initial data $u_0 \in H^1(I)$ we have the following result.

Lemma 6.1. *Let (n,m,u) be a weak solution of (22)–(26) in the sense of Theorem 3.1. If $u_0 \in H^1(I)$, then*

$$u \in L^\infty([0,T], H^1(I)) \cap L^2([0,T], H^2(I)), \qquad u_t \in L^2([0,T], L^2(I)). \tag{61}$$

More precisely, the following estimate holds:

$$\|u_t\|_{L^2(D_T)} + \|u_{xx}\|_{L^2(D_T)} + \|u_x\|_{L^\infty([0,T],L^2(I))} \le C, \tag{62}$$

where the constant C depends on the quantities involved in the estimates of Lemma 4.1–4.3.

Proof. We consider the global smooth solutions $(n^\delta, m^\delta, u^\delta)$ described in the previous section with initial data $(n_0^\delta, m_0^\delta, u_0^\delta)$ which possess smoothness properties as described by (28). It follows that (see Section 3 in [16] for more details)

$$\partial_x u_0^\delta \to \partial_x u_0 \qquad \text{in } L^2(I).$$

For the coming calculation the superscript δ is neglected. We multiply the third equation of (32) by u_t and integrate over $[0,1]\times[0,T]$. Applying integration by parts together with the boundary condition (33) we get

$$\begin{aligned}\int_0^t\int_0^1 u_t^2\,dx\,ds &- \int_0^1 [P(c,m)u_x - E(m)u_x^2]\,dx \\ &+ \int_0^1 [P(c_0,m_0)u_{0,x} - E(m_0)u_{0,x}^2]\,dx \\ &+ \int_0^t\int_0^1 [P(c,m) - E(m)u_x]_t u_x\,dx\,ds = 0.\end{aligned} \tag{63}$$

For the last term we have

$$[P(c,m) - E(m)u_x]_t u_x = -k_1\rho_l\gamma c^\gamma Q^{\gamma+1}(u_x)^2 + k_2(\beta+1)\rho_l Q^{\beta+2}(u_x)^3 - k_2 Q^{\beta+1}(\frac{1}{2}u_x^2)_t,$$

where we have used the second equation of (32). Observing that

$$k_2 Q^{\beta+1}(\frac{1}{2}u_x^2)_t = (\frac{k_2}{2}Q^{\beta+1}u_x^2)_t - \frac{k_2}{2}(\beta+1)Q^\beta Q_t u_x^2 = (\frac{1}{2}E(m)u_x^2)_t + \frac{k_2}{2}(\beta+1)\rho_l Q^{\beta+2}u_x^3,$$

it follows that

$$\int_0^t\int_0^1 [P(c,m) - E(m)u_x]_t u_x\,dx\,ds = -k_1\rho_l\gamma\int_0^t\int_0^1 c^\gamma Q^{\gamma+1}(u_x)^2\,dx\,ds + \frac{1}{2}k_2(\beta+1)\rho_l\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds - \frac{1}{2}\int_0^1 E(m)u_x^2\,dx + \frac{1}{2}\int_0^1 E(m_0)u_{0,x}^2\,dx. \quad (64)$$

From (63) and (64) it follows that

$$\int_0^t\int_0^1 u_t^2\,dx\,ds + \frac{1}{2}\int_0^1 E(m)u_x^2\,dx = \frac{1}{2}\int_0^1 E(m_0)u_{0,x}^2\,dx + \int_0^1 P(c,m)u_x\,dx - \int_0^1 P(c_0,m_0)u_{0,x}\,dx + k_1\rho_l\gamma\int_0^t\int_0^1 c^\gamma Q^{\gamma+1}(u_x)^2\,dx\,ds - \frac{1}{2}k_2(\beta+1)\rho_l\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds. \quad (65)$$

The second term on the right hand side of (65) can be absorbed in the second term on the left hand side by using the Cauchy inequality with ε

$$2ab \le \varepsilon a^2 + \varepsilon^{-1}b^2, \qquad a,b>0, \qquad \varepsilon>0. \quad (66)$$

Together with application of the estimates of (57) and regularity of initial data, we get an estimate of the form

$$\int_0^t\int_0^1 u_t^2\,dx\,ds + \int_0^1 u_x^2\,dx \le C + C\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds. \quad (67)$$

The last term of (67), in view of (57), can be estimated as follows.

$$\begin{aligned}\int_0^t\int_0^1 Q^{\beta+2}(u_x)^3\,dx\,ds &\le C\int_0^t \max_{[0,1]}(Q^{1+\beta}u_x)(\cdot,s)\Big(\int_0^1 u_x^2\,dx\Big)\,ds\\ &\le C\int_0^t \max_{[0,1]}|(E(m)u_x - P(c,m))(\cdot,s)|\Big(\int_0^1 u_x^2\,dx\Big)\,ds + C\int_0^t\int_0^1 u_x^2\,dx\,ds\\ &\le C\int_0^t\Big(\int_0^1 |(E(m)u_x - P(c,m))_x|\,dx\Big)\Big(\int_0^1 u_x^2\,dx\Big)\,ds + C\\ &= C\int_0^t\Big(\int_0^1 |u_t|\,dx\Big)\Big(\int_0^1 u_x^2\,dx\Big)\,ds + C\\ &\le \frac{1}{2}\int_0^t\int_0^1 u_t^2\,dx\,ds + C\int_0^t\Big(\int_0^1 u_x^2\,dx\Big)^2\,ds + C,\end{aligned}$$

where we again have used (66) to obtain the last inequality. Inserting this in (67) gives

$$\int_0^t \int_0^1 u_t^2 \, dx \, ds + \int_0^1 u_x^2 \, dx \leq C + \int_0^t \|u_x(s)\|_{L^2(I)}^2 \int_0^1 u_x^2 \, dx \, ds, \ \ \forall t \in [0,T]. \quad (68)$$

Since $\int_0^T \|u_x(s)\|_{L^2(I)}^2 \, ds < \infty$ (see (57)), application of Gronwall's inequality to (68) gives the estimate

$$\int_0^t \int_0^1 u_t^2 \, dx \, ds + \int_0^1 u_x^2 \, dx \leq C. \quad (69)$$

The last equation of (22), the estimates of (57) and the estimate (69) imply that

$$\int_0^T \int_0^1 u_{xx}^2 \, dx \, ds \leq C.$$

Thus, (61) and (62) have been shown. □

Taking advantage of the additional regularity of Lemma 6.1 we now derive a stability result.

Lemma 6.2. *Let (n_1, m_1, u_1) be an arbitrary weak solution of (22)–(26), in the sense of Theorem 3.1, which also satisfies (61). Let (n_2, m_2, u_2) be another weak solution subject to the same initial data. Then we have the stability estimate*

$$\begin{aligned} &\|u_1 - u_2\|_{L^2(I)}^2 + \|1/Q(m_1) - 1/Q(m_2)\|_{L^2(I)}^2 \\ &\qquad \leq \int_0^t C(s) \|1/Q(m_1) - 1/Q(m_2)\|_{L^2(I)}^2 \, ds, \end{aligned} \quad (70)$$

where the non-negative constant $C(s)$ satisfies $\int_0^T C(s) \, ds < \infty$.

Proof. We consider the reformulated model as expressed by (32)–(35). In particular, $c_1 = c_2 := c_0$. In the following it will be useful to work with the quantity $v_i = 1/Q(m_i)$, $i = 1, 2$. We then get

$$(Q_i^\beta)_t + \rho_l \beta Q_i^{\beta+1} u_{ix} = 0, \qquad (v_i)_t = \rho_l u_{ix}, \qquad i = 1, 2. \quad (71)$$

The last equation of (32) yields

$$\begin{aligned} &(u_1 - u_2)_t + k_1([c_0(x) Q(m_1)]^\gamma - [c_0(x) Q(m_2)]^\gamma)_x \\ &\qquad = k_2 (Q(m_1)^{\beta+1} u_{1x} - Q(m_2)^{\beta+1} u_{2x})_x. \end{aligned}$$

Multiplying by $(u_1 - u_2)$, integrating over $[0,1]$ together with integration by parts and application of boundary conditions (33) give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (u_1-u_2)^2\,dx \\
&= k_1\int_0^1 ([c_0(x)Q(m_1)]^\gamma - [c_0(x)Q(m_2)]^\gamma)(u_1-u_2)_x\,dx \\
&\qquad - k_2\int_0^1 (Q(m_1)^{\beta+1}u_{1x} - Q(m_2)^{\beta+1}u_{2x})(u_1-u_2)_x\,dx \\
&= \frac{k_1}{\rho_l}\int_0^1 c_0(x)^\gamma (v_1^{-\gamma}-v_2^{-\gamma})(v_1-v_2)_t\,dx \\
&\qquad - k_2\int_0^1 (v_1^{-(\beta+1)} - v_2^{-(\beta+1)})(u_{1x}-u_{2x})u_{2x}\,dx \\
&\qquad\qquad - k_2\int_0^1 Q(m_1)^{\beta+1}(u_{1x}-u_{2x})^2\,dx \\
&\le -\frac{k_1}{2\rho_l}\frac{d}{dt}\int_0^1 c_0(x)^\gamma a(x,t)(v_1-v_2)^2\,dx \\
&\qquad + \frac{k_1}{2\rho_l}\int_0^1 c_0(x)^\gamma a_t(x,t)(v_1-v_2)^2\,dx \\
&\qquad\qquad + \frac{C_0}{2}\int_0^1 (u_{1x}-u_{2x})^2\,dx + C_1\int_0^1 (v_1-v_2)^2(u_{2x})^2\,dx \\
&\qquad\qquad\qquad - C_0\int_0^1 (u_{1x}-u_{2x})^2\,dx,
\end{aligned}
\tag{72}
$$

where we have used that

$$
\begin{aligned}
a(x,t) &= \frac{f(v_1)-f(v_2)}{v_1-v_2} \\
&= \int_0^1 f'(\tau(v_1-v_2)+v_2)\,d\tau = \gamma\int_0^1 \frac{1}{(\tau(v_1-v_2)+v_2)^{(\gamma+1)}}\,d\tau,
\end{aligned}
\tag{73}
$$

with $f(v) = -v^{-\gamma}$, i.e. $f'(v) = \gamma v^{-(\gamma+1)}$ so that

$$
\begin{aligned}
&\int_0^1 c_0(x)^\gamma (v_1^{-\gamma}-v_2^{-\gamma})(v_1-v_2)_t\,dx \\
&\quad = -\int_0^1 c_0(x)^\gamma a(x,t)(v_1-v_2)(v_1-v_2)_t\,dx \\
&\quad = -\frac{1}{2}\int_0^1 c_0(x)^\gamma a(x,t)((v_1-v_2)^2)_t\,dx \\
&\quad = -\frac{1}{2}\frac{d}{dt}\int_0^1 c_0(x)^\gamma a(x,t)(v_1-v_2)^2\,dx + \frac{1}{2}\int_0^1 c_0(x)^\gamma a_t(x,t)(v_1-v_2)^2\,dx.
\end{aligned}
$$

In addition, we have used that $|g(y_1) - g(y_1)| \le \max|g'(y)||y_1 - y_2|$ for $g(y) = y^{-(\beta+1)}$ together with the upper and lower limits for v_i, $i = 1,2$ given by (57), as well as the inequality (66). These estimates also imply that $a(x,t)$ given by (73) has a positive lower limit on $D_T = [0,1] \times [0,T]$. Moreover,

$$a_t(x,t) = \int_0^1 f''(\tau(v_1 - v_2) + v_2)(\tau(v_{1t} - v_{2t}) + v_{2t})\, d\tau,$$

so that

$$|a_t(x,t)| \le \int_0^1 |f''(\tau(v_1 - v_2) + v_2)|(\tau|v_{1t} - v_{2t}| + |v_{2t}|)\, d\tau \le C(|v_{1t} - v_{2t}| + |v_{2t}|),$$

where C depends on lower and upper limits of v_1 and v_2. Consequently,

$$\begin{aligned}
\frac{k_1}{2\rho_l} \int_0^1 c_0(x)^\gamma a_t(x,t)(v_1 - v_2)^2\, dx &\le C \int_0^1 (|v_{1t} - v_{2t}| + |v_{2t}|)(v_1 - v_2)^2\, dx \\
&= C \int_0^1 |v_{1t} - v_{2t}|(v_1 - v_2)^2\, dx + C \int_0^1 |v_{2t}|(v_1 - v_2)^2\, dx \\
&\le C\varepsilon \int_0^1 (v_{1t} - v_{2t})^2 (v_1 - v_2)^2\, dx \\
&\quad + C\varepsilon^{-1} \int_0^1 (v_1 - v_2)^2\, dx + C \int_0^1 |v_{2t}|(v_1 - v_2)^2\, dx \\
&\le \frac{C_0}{4\rho_l^2} \int_0^1 (v_{1t} - v_{2t})^2\, dx + C \int_0^1 (1 + |v_{2t}|)(v_1 - v_2)^2\, dx \\
&= \frac{C_0}{4} \int_0^1 (u_{1x} - u_{2x})^2\, dx + C \int_0^1 (1 + |v_{2t}|)(v_1 - v_2)^2\, dx,
\end{aligned}$$

where we have used (66) with an appropriate choice of $\varepsilon > 0$, the upper and lower limits of v_1 and v_2, and (71). Inserting this in (72) we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_0^1 (u_1 - u_2)^2\, dx + \frac{k_1}{2\rho_l}\frac{d}{dt} \int_0^1 c_0(x)^\gamma a(x,t)(v_1 - v_2)^2\, dx + \frac{C_0}{4} \int_0^1 (u_{1x} - u_{2x})^2\, dx \\
\le C \int_0^1 (1 + |u_{2x}|)(v_1 - v_2)^2\, dx + C_1 \int_0^1 (v_1 - v_2)^2 (u_{2x})^2\, dx \\
\le C \int_0^1 (1 + |u_{2x}|)^2 (v_1 - v_2)^2\, dx,
\end{aligned}$$

for a suitable choice of the constant C. Integrating over $[0,t]$ we get the inequality

$$\begin{aligned}
\int_0^1 (u_1 - u_2)^2\, dx + \int_0^1 c_0(x)^\gamma a(x,t)(v_1 - v_2)^2\, dx + \int_0^t \int_0^1 (u_{1x} - u_{2x})^2\, dx\, ds \\
\le C \int_0^t \int_0^1 (1 + |u_{2x}|)^2 (v_1 - v_2)^2\, dx\, ds.
\end{aligned}$$

Using that $\inf a(x,t) > 0$ and $\inf c_0(x) > 0$ we get

$$\begin{aligned}
\int_0^1 (u_1 - u_2)^2(x,t)\, dx + \int_0^1 (v_1 - v_2)^2(x,t)\, dx \\
\le C \int_0^t \int_0^1 (1 + |u_{2x}|)^2 (v_1 - v_2)^2(x,s)\, dx\, ds \\
\le C \int_0^t \|(1 + |u_{2x}|)^2\|_{L^\infty(I)} \int_0^1 (v_1 - v_2)^2(x,s)\, dx\, ds.
\end{aligned}$$

This shows the estimate (70). Finally, it follows by Sobolev's imbedding theorem $\|f\|_{L^\infty(I)} \leq C\|f\|_{W^{1,1}(I)}$ that

$$
\begin{aligned}
\int_0^t \|(1+u_{2x})^2\|_{L^\infty(I)}\,ds &\leq C\int_0^t \|(1+u_{2x})^2\|_{W^{1,1}(I)}\,ds \\
&= C\int_0^t\int_0^1 (1+u_{2x})^2\,dxds + C\int_0^t\int_0^1 |((1+u_{2x})^2)_x|\,dx\,ds \\
&\leq C + C\int_0^t\int_0^1 |(1+u_{2x})u_{2xx}|\,dx\,ds \\
&\leq C + C\Big(\int_0^t\int_0^1 (1+u_{2x})^2\,dx\,ds\Big)^{1/2}\Big(\int_0^t\int_0^1 u_{2xx}^2\,dx\,ds\Big)^{1/2} \leq C,
\end{aligned}
$$

since $u_2 \in L^\infty([0,T],H^1(I)) \cap L^2([0,T],H^2(I))$ (see Lemma 6.1). □

Now, we can conclude that the following uniqueness result holds.

Theorem 6.3 (Uniqueness)**.** *Under the assumptions of Theorem 3.1 and the additional regularity assumption $u_0 \in H^1(I)$, the weak solutions are unique.*

Proof. Clearly, the results of Lemma 6.1 and Lemma 6.2 hold which lead to the inequality (70). Thus, application of Gronwall's inequality to (70) yields immediately that

$$Q(m_1(x,t)) = Q(m_2(x,t)), \quad u_1(x,t) = u_2(x,t) \quad \text{a.e. } (x,t) \in D_T = [0,1]\times[0,T].$$

The fact that $Q(m)$ is monotone implies the desired result. □

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