ASYMPTOTIC STABILITY OF THE COMPRESSIBLE GAS-LIQUID MODEL WITH WELL-FORMATION INTERACTION AND GRAVITY

STEINAR EVJE, QINGQING LIU, AND CHANGJIANG ZHU*

ABSTRACT. In this paper we are concerned with the initial-boundary value problem of the compressible gas-liquid model with well-formation interaction and gravity. The asymptotic behavior of solutions to steady states is established. Also the time-decay rates of perturbed solutions in the sense of L^∞ norm are obtained under some suitable assumptions on the initial date, if $\gamma > 1$ (associated with pressure law of gas) and $\beta \in (0, \frac{\gamma}{2}] \cap (0, \gamma - \alpha\gamma) \cap (0, \frac{\gamma+\alpha\gamma}{3}]$ where β characterizes the viscosity coefficient and α describes the mass decay rate at the boundary. A main purpose of this work is to clarify the role played by the well-reservoir interaction term. The analysis demonstrates that it is essential to take into account information about sign as well as size of the interaction term in order to obtain time-independent estimates when it operates in combination with gravity.

Contents

1. Introduction

The compressible drift-flux gas-liquid model is often used in chemical engineering to describe the dynamics of two-phase flow, see [1, 2]. This model is different from the two-fluid model in the sense that it has one mixture momentum equation instead of two separate momentum equations. We refer to [10] for more on the relationship between the two-fluid and drift-flux models. In this paper, we are concerned with a gas-liquid model where gas is allowed to flow between a wellbore and surrounding formation governed by a given function $A(x,t)$. From an application point of view, $A(x,t) > 0$ means that there is inflow of gas along the well and $A(x,t) < 0$ means that there is outflow of gas along the well, see [4]. More precisely, the corresponding model can be written in Eulerian coordinates as

$$\begin{cases} \partial_t n + \partial_x[nu] = nA(x,t), \\ \partial_t m + \partial_x[mu] = 0, \\ \partial_t[(m+n)u] + \partial_x[(m+n)u^2] + \partial_x P = gm + \partial_x[\varepsilon \partial_x u], \quad a(t) < x < b, \end{cases} \tag{1.1}$$

Date: August 22, 2013.
2000 *Mathematics Subject Classification.* 76T10, 35L65, 35B40.
Key words and phrases. Two-phase flow, Stationary solutions, Convergence rates.
*Corresponding author. Email: cjzhu@mail.ccnu.edu.cn (C.J. Zhu).

where the free boundary function $a(t)$ satisfies

$$\begin{cases} \dfrac{da(t)}{dt} = u(a(t),t), \quad t > 0, \\ a(0) = a. \end{cases}$$

Here, $n = n(x,t) \geq 0$, $m = m(x,t) \geq 0$ respectively are masses of the gas and liquid. $u = u(x,t)$ denotes velocity of the phases. Initial data is given as

$$m(x,0) = m_0(x), \quad n(x,0) = n_0(x), \quad u(x,0) = u_0(x), \tag{1.2}$$

and the boundary conditions

$$n(a(t),t) = 0, \quad m(a(t),t) = 0, \quad u(b,t) = 0, \quad t > 0. \tag{1.3}$$

The pressure function $P(\cdot,\cdot)$ and viscosity function $\varepsilon(\cdot,\cdot)$ depending on the masses satisfy

$$P(n,m) = K_1\left(\frac{n}{\rho_l - m}\right)^{\gamma}, \quad \varepsilon(n,m) = \frac{K_2 n^{\beta}}{(\rho_l - m)^{\beta+1}}, \quad K_1, \ K_2 > 0. \tag{1.4}$$

Throughout this paper, we take $K_1 = K_2 = 1$, for the sake of simplicity only. Note that we use a slightly simplified gravity term which only depends on the liquid mass. For this case, as we shall show later, the corresponding steady equation is similar to the steady equation of 1D compressible Navier-Stokes equations with gravity, which has been studied in [3, 28, 30]. However, the gas mass is not conserved and the variable $c = c(x,t) = \frac{n}{n+m}$ which is used to reformulate the two-phase model (see Section 2) becomes time-dependent.

It was Evje who first studied the model (1.1) without gravity in [4]. He proved the global existence of the weak solutions to the initial boundary problem when the viscosity was taken as $\varepsilon(n,m) = \frac{(m+n)^{\beta}}{(\rho_l - m)^{\beta+1}}$, $\beta \in (0, \frac{1}{3})$. However, it is interesting to clarify the long-time behavior of the model (1.1). In particular, it is desirable to get a more detailed understanding of how the stability properties of the model are possibly influenced by the interaction term $A(x,t)$. We give a partial answer to this by exploring the model in the framework used in [3, 28, 30] for studies of single-phase Navier-Stokes equations. It turns out that both size and sign on $A(x,t)$ play a key role in the analysis to show convergence to the steady state. Moreover, a decay rate associated with $A(x,t)$ is required to estimate convergence rates. The case with more general choices of $A(x,t)$ is left for future study. However, we refer to [9] for a result with a more general interaction term $A(x,t)$ but where the effect of gravity is not included.

Before we proceed we will briefly describe some previous works on the gas-liquid model. There have been extensive investigations into the simplified gas-liquid model. In the 1D case, it takes the following form,

$$\begin{cases} \partial_t n + \partial_x[n u_g] = 0, \\ \partial_t m + \partial_x[m u_l] = 0, \\ \partial_t[m u_l] + \partial_x[m u_l^2] + \partial_x P = -f m^3 u_l |u_l| + g m + \partial_x[\varepsilon \partial_x u_{mix}], \quad u_{mix} = \alpha_g u_g + \alpha_l u_l, \end{cases} \tag{1.5}$$

where $\alpha_g + \alpha_l = 1$. For the simplified model obtained by assuming $u_l = u_g = u$ and neglecting frictional force $-f m^3 u_l |u_l|$ and gravity gm in (1.5), there has been a number of research results where the viscosity function $\varepsilon(n,m)$ is taken different forms. First of all, Evje and Karlsen [7] studied the existence of the global weak solutions when the initial masses were discontinuously connected to a vacuum for the viscosity function $\varepsilon(n,m)$ taking the following form

$$\varepsilon(n,m) = \varepsilon(m) = \frac{m^{\beta}}{(\rho_l - m)^{\beta+1}}, \quad \beta \in \left(0, \frac{1}{3}\right). \tag{1.6}$$

This result was later extended to the case $\beta \in (0,1]$ by Yao and Zhu [21]. When the initial masses were continuously connected to a vacuum, Evje, Flatten and Friis [11] also obtained the global existence and uniqueness of weak solutions for the viscosity $\varepsilon(n,m)$ taking the following form

$$\varepsilon(n,m) = \frac{n^\beta}{(\rho_l - m)^{\beta+1}}, \quad \beta \in \left(0, \frac{1}{3}\right). \tag{1.7}$$

Taking viscosity as a constant, Yao and Zhu in [23] obtained the global existence of weak solutions where the initial masses were continuously connected to a vacuum. With regard to asymptotic behavior, Liu and Zhu in [18] showed that the masses n and m tend to zero as time goes to infinity, moreover, they obtained a stabilization rate estimates of the mass functions for any $\beta > 0$. Later, by assuming $u_l = u_g$ in (1.5), Friis and Evje in [8] proved the existence of global weak solutions to the initial boundary value problem of (1.5) with initial data (1.2) and boundary conditions $[p(n,m) - \varepsilon(n,m)mu_x](0,t) = 0,\ u(1,t) = 0$ when the viscosity $\varepsilon(n,m)$ was taken the form (1.6). Recently, Fan, Liu and Zhu in [12] considered the long time behavior of the weak solutions to initial boundary value problem of (1.5) by overcoming the difficulties which came from the frictional force $-fm^3u|u|$. For the case $u_g \neq u_l$, Evje in [5] obtained the local existence by neglecting external forces like friction and gravity. Furthermore, the model has also been studied in Eulerian coordinates with the simplified momentum equation and constant viscosity coefficient [6]. For a similar result where the model is studied in a 2D setting we refer to [22]. See also [24] for a result on blow-up phenomena of the 2D gas-liquid model. We remark that the main tool in obtaining the above results is the introduction of a suitable variable transformation by which one can use ideas and techniques similar to those used in [25, 26, 27] on Navier-Stokes equations.

In this paper we rewrite our problem into (2.5)-(2.7) by using the transformations (2.4) and investigate it in the framework of [3, 12, 28, 30]. Main observations we would like to highlight from the analysis are:

- Size and sign of $A(x,t)$ appear to be crucial to obtain the energy estimate Lemma 3.1 which allows us to control the estimate of the fluid velocity. Similarly, both size and sign of $A(x,t)$ are exploited in the proof of the time-independent upper and lower bounds of $\frac{c}{1-c}Q$, see Lemma 3.2, which plays a very important role in studying the long-time behavior of $\frac{c}{1-c}Q$ and u, as expressed by Theorem 2.1.
- In order to prove an estimate of the rate at which the mass-related variable $\frac{c}{1-c}Q$ tends to $\frac{c_\infty}{1-c_\infty}Q_\infty$ and velocity u tends to zero, as expressed by Theorem 2.2, it is necessary to give more information about the rate at which the interaction term $A(x,t)$ will tend to zero in an averaged sense, see condition (2.21) for the precise statement.

Many studies have been made for the asymptotic behavior of Navier-Stokes equations with density-dependent viscosity and vacuum. For the case without external force, Guo and Zhu in [13, 14] gave the asymptotic behavior and decay rate of the density function $\rho(x,t)$ when the initial density was continuously connected to a vacuum. Zhu in [29] investigated the asymptotic behavior and decay rate estimates on the density function $\rho(x,t)$ by overcoming some new difficulties which came from the appearance of boundary layers when the initial density was discontinuously connected to a vacuum. In [13, 14, 29], the auxiliary function $w(x,t)$ introduced by Nagasawa in [16] was used to investigate the decay rate of $\rho(x,t)$. For the other case with gravity, under some assumptions on the initial data, Zhang and Fang in [28] proved that the solution converges to the stationary states as time goes to infinity provided $\theta \in (0, \gamma-1) \cap (0, \frac{\gamma}{2}]$ and $\gamma > 1$. The stabilization rates were also estimated in several norms. Duan in [3] generalized part result in [28], and showed that the solution converges to the stationary state in the sense of integral when $\gamma = 2$, $\theta = 1$. Recently, Zhu and Zi in [30] improved the results in [3, 28] in the sense that $\theta \in (0, \gamma - 1] \cap (0, \frac{\gamma}{2}]$.

We conclude this section by stating the arrangement of the rest of this paper. In Section 2, the system is transformed into a more simple one, then we give the definition of the weak solution and state the main results. In Section 3, we derive some crucial uniform estimates for studying the asymptotic behavior and the decay rate estimates. In Section 4, the asymptotic behavior of weak solution will be given. In Section 5, we will establish stabilization rate estimates of the solution as time tends to infinity.

2. Formulation of problem and main results

To solve the free boundary problems (1.1)-(1.3), it is convenient to convert the free boundaries to the fixed boundaries by using Lagrangian coordinates. To do this, let

$$\xi = \int_{a(t)}^{x} m(y,t)dy, \quad \tau = t.$$

Then the free boundaries $x = a(t)$ and $x = b$ become $\xi = 0$ and $\xi = \int_{a(t)}^{b} m(y,t)dy = \int_a^b m_0(y)dy$ by the conservation of mass, where $\int_a^b m_0(y)dy$ is the total liquid mass initially. We normalize $\int_a^b m_0(y)dy$ to 1. Hence in the Lagrangian coordinates, the free boundary problem (1.1)-(1.3) can be transformed into the following fixed boundary form,

$$\begin{cases} n_\tau + (nm)u_\xi = nA, \\ m_\tau + m^2 u_\xi = 0, \\ (m+n)u_\tau + m(P(n,m))_\xi = -unA + gm + m(\varepsilon(n,m)mu_\xi)_\xi. \end{cases} \tag{2.1}$$

Initial data is given as,

$$n(\xi,0) = n_0(\xi),\ m(\xi,0) = m_0(\xi),\ u(\xi,0) = u_0(\xi),\ \xi \in [0,1], \tag{2.2}$$

and boundary conditions,

$$n(0,\tau) = 0, \quad m(0,\tau) = 0, \quad u(1,\tau) = 0, \quad \tau \geq 0. \tag{2.3}$$

In the following, we replace the coordinates (ξ, τ) by (x,t). Introduce the variables,

$$c = \frac{n}{n+m}, \qquad Q(m) = \frac{m}{\rho_l - m} \geq 0. \tag{2.4}$$

From the first two equations of (2.1) we get

$$c_t = \frac{n_t}{n+m} - \frac{n}{(n+m)^2}(m_t + n_t) = c(1-c)A,$$

and

$$\begin{aligned} Q(m)_t = \left(\frac{m}{\rho_l - m}\right)_t &= \left(\frac{1}{\rho_l - m} + \frac{m}{(\rho_l - m)^2}\right) m_t \\ &= \frac{\rho_l}{(\rho_l - m)^2} m_t = -\frac{\rho_l m^2}{(\rho_l - m)^2} u_x = -\rho_l Q(m)^2 u_x. \end{aligned}$$

Then we obtain

$$\begin{cases} c_t = c(1-c)A, \\ Q_t + \rho_l Q^2 u_x = 0, \\ \left(\dfrac{1}{1-c}\right) u_t + P(c,Q)_x = -u\left(\dfrac{c}{1-c}\right)A + \partial_x(E(c,Q)\partial_x u) + g, \end{cases} \tag{2.5}$$

with

$$P(c,Q) = \left(\frac{c}{1-c}\right)^\gamma Q^\gamma, \qquad \gamma > 1,$$

and

$$E(c,Q)=\left(\frac{c}{1-c}\right)^{\beta}Q^{\beta+1}.$$

The boundary conditions are given as follows,

$$c(0,t)=0,\quad Q(0,t)=0,\quad u(1,t)=0, \tag{2.6}$$

and the initial data

$$c(x,0)=c_0(x)\geq 0,\ \ Q(x,0)=Q_0(x)\geq 0,\ \ u(x,0)=u_0(x),\quad x\in[0,1]. \tag{2.7}$$

Here we have used $\dfrac{n}{m}=\dfrac{c}{1-c}$, and c_0, Q_0 are given from $[n_0,m_0]$ according the transformations (2.4).

Throughout this paper, our assumptions on the flow rate function $A(x,t)$ and the initial data are as follows:

(A_1) $\displaystyle\int_0^t\|A(\cdot,s)\|_{L^\infty([0,1])}ds\leq C\epsilon_0,\ A(x,t)\leq 0,\ \ |A(x,t)|\leq C,\ \forall\ (x,t)\in[0,1]\times[0,+\infty),$
$\displaystyle\int_0^t(\|A_x(\cdot,s)\|^2_{L^\infty([0,1])}+\|A_t(\cdot,s)\|^2_{L^\infty([0,1])})ds\leq C;$

(A_2) There are positive constants $K_1,\ K_2,\ K_3$ and K_4 such that
$K_1x^\alpha\leq m_0(x)\leq K_2x^\alpha<\rho_l$ and $K_3x^{\frac{1}{\gamma}}\leq n_0(x)\leq K_4x^{\frac{1}{\gamma}}$, where $\alpha<\frac{1}{\gamma}$.
In particular, this implies that there exist positive constants C_1, C_2, such that $\frac{c_0(x)}{1-c_0(x)}=\frac{n_0}{m_0}\sim x^{\frac{1}{\gamma}-\alpha}$, $Q_0=\frac{m_0}{\rho_l-m_0}\sim x^\alpha$, $C_1x^{\frac{1}{\gamma}}\leq\frac{c_0(x)}{1-c_0(x)}Q_0(x)\leq C_2x^{\frac{1}{\gamma}}$, and $0\leq\inf\limits_{x\in[0,1]}c_0(x),\ \sup\limits_{x\in[0,1]}c_0(x)<1;$

(A_3) $u_0\in H^1([0,1];$

(A_4) $\left(\left(\frac{c_0}{1-c_0}\right)^\beta Q_0^{1+\beta}u_{0x}\right)_x\in L^2([0,1]),\ x^{\frac{3\gamma-\beta-\alpha\gamma}{4\gamma}}\left(\left(\frac{c_0}{1-c_0}Q_0\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)\in L^2([0,1]).$

Under assumptions (A_1)-(A_4), we will study the asymptotic behavior of $\frac{c}{1-c}Q$ and u provided that the global weak solution to the initial boundary value problem (2.5)-(2.7) exists. In the Lagrangian coordinates, the definition of the weak solution to (2.5)-(2.7) can be stated as follows:

Definition 2.1. **(Weak solution)** *The function $(c(x,t),Q(x,t),u(x,t))$ is called a weak solution to the initial boundary problem* (2.5)-(2.7), *if*

$$c,\ Q\in L^\infty([0,1]\times[0,\infty))\cap C^1([0,\infty);L^2(0,1)), \tag{2.8}$$

$$u\in L^\infty([0,1]\times[0,\infty))\cap C^{\frac{1}{2}}([0,\infty);L^2(0,1)), \tag{2.9}$$

$$\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x\in L^\infty([0,1]\times[0,\infty))\cap L^2([0,\infty);H^1(0,1)). \tag{2.10}$$

Furthermore, the following equations hold:

$$c_t=c(1-c)A,\quad a.e.,$$
$$Q_t+\rho_lQ^2u_x=0,\quad a.e.,$$

and

$$\int_0^\infty\int_0^1\left\{\left(\frac{u}{1-c}\right)\phi_t+\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x\right)\phi_x+g\phi\right\}dxdt$$
$$+\int_0^1\left(\frac{u_0}{1-c_0}\right)(x)\phi(x,0)dx=0,$$

for any test functions $\phi(x,t)\in C_0^\infty(\Omega)$ with $\Omega=\{(x,t):0\leq x\leq 1,\ t\geq 0\}$.

Remark 2.1. (Existence of the global weak solution). *To our knowledge, by using the standard line method (see* [4, 17, 28] *for example), it is easy to obtain the global existence of the weak solutions to* (2.5)-(2.7)*. The details are omitted.*

As in [4], we can solve function $c(x,t)$ from the first equation of (2.5), which will be used frequently later. Based on the assumptions (A_1) and (A_2), we can also list some properties of $c(x,t)$.

Proposition 2.1. *Under the conditions of Theorem 2.1, it holds that for* $0<x<1,\ t>0$,

$$\frac{c(x,t)}{1-c(x,t)}=\frac{c_0(x)}{1-c_0(x)}\exp\left(\int_0^t A(x,s)ds\right). \tag{2.11}$$

Based on the assumptions on $A(x,t)$*, we have the following properties*

$$0\le \inf_{x\in[0,1]} c(x,t),\ \ \sup_{x\in[0,1]} c(x,t)<1. \tag{2.12}$$

We denote $c_\infty(x)$ to be the steady state of $c(x,t)$. According to (2.11), $c_\infty(x)$ should be defined as following,

$$\frac{c_\infty(x)}{1-c_\infty(x)}=\frac{c_0(x)}{1-c_0(x)}\exp\left(\int_0^{+\infty} A(x,s)ds\right), \tag{2.13}$$

and $A(x,t)\le 0$ implies that

$$c_0(x)\ge c(x,t)\ge c_\infty(x). \tag{2.14}$$

Let $\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(x),0\right)$ be the stationary solution of equations (2.5), and $\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(x)$ is given by the following equations:

$$\begin{cases}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma_x=g,\\ \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(0)=0.\end{cases} \tag{2.15}$$

Then

$$\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(x)=(gx)^{\frac{1}{\gamma}}. \tag{2.16}$$

We refer to (2.13) for a characterization of $c_\infty(x)$.

Theorem 2.1. *Under the assumptions* (A_1)*-*(A_4)*,* $\gamma>1$ *and* $\beta\in(0,\frac{\gamma}{2}]\cap(0,\frac{\gamma+\alpha\gamma}{3}]$*, let* $\left(\left(\frac{c}{1-c}Q\right)(x,t),u(x,t)\right)$ *be the weak solution to the initial boundary value problem (2.5)-(2.7). There exists a constant* $0<\epsilon_0<1$*, such that if*

$$\|u_0\|^2_{L^2}\le\epsilon_0,\qquad \int_0^1 x^{1-3\alpha}\left(Q_0-Q_\infty\right)^2dx\le\epsilon_0, \tag{2.17}$$

then we have the following asymptotic behavior

$$\lim_{t\to\infty}\left(\frac{c}{1-c}Q\right)(x,t)=\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(x), \tag{2.18}$$

uniformly in $x\in[0,1]$ *and*

$$\lim_{t\to\infty}\sup_{x\in[\delta,1]}|u(x,t)|=0, \tag{2.19}$$

for any $0<\delta<1$*. Furthermore, if we assume further* $\beta<\gamma-\alpha\gamma$*, then* (2.19) *can be improved as*

$$\lim_{t\to\infty}\sup_{x\in[0,1]}|u(x,t)|=0. \tag{2.20}$$

Theorem 2.2. *Under the conditions of Theorem 2.1, assuming that $\beta < \gamma - \alpha\gamma$, and under the additional conditions on $A(x,t)$:*

$$\begin{cases} \displaystyle\int_0^t (1+s)\|A(\cdot,s)\|_{L^\infty([0,1])} ds \le C, \\ \displaystyle\int_0^t (1+s)\|A_x(\cdot,s)\|^2_{L^\infty([0,1])} ds \le C, \\ \displaystyle\int_0^t \left[1 - \exp\left(-\gamma \int_s^{+\infty} \|A(\cdot,\tau)\|_{L^\infty([0,1])} d\tau\right)\right] ds \le C, \end{cases} \tag{2.21}$$

the following estimates hold:

$$\left\| \left(\left(\frac{c}{1-c} Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty} Q_\infty\right)^\beta \right)(\cdot,t) \right\|_{L^\infty([0,1])} \le C(1+t)^{-\frac{\beta}{2\gamma-\alpha\gamma}}, \tag{2.22}$$

and

$$\|u(\cdot,t)\|_{L^\infty([0,1])} \le C(1+t)^{-\frac{1}{2}}, \tag{2.23}$$

for all $t \ge 0$.

Remark 2.2. *Various assumptions on $A(x,t)$ have been imposed as expressed by assumption (A_1) and* (2.21)*. As an example of $A(x,t)$ which satisfies all the assumptions we may consider $A(x,t)$ in the following form*

$$A(x,t) = \frac{\epsilon_0 \phi(x)}{(1+t)^{2+\delta}},$$

where ϵ_0 and δ are small positive constants, $\phi(x) \in W^{1,\infty}([0,1])$ and $\phi(x) \le 0$.

3. Uniform *a priori* estimates

In this section, we devote ourselves to some uniform-in-time *a priori* estimates for the solutions to (2.5)-(2.7) by classical energy method. The basic energy estimate is carried out by making use of size and sign of $A(x,t)$.

Lemma 3.1. ***(Basic energy estimate).*** *Under the conditions in Theorem 2.1, the following energy estimate holds:*

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left(\frac{1}{1-c}\right) u^2 dx + \int_0^1 \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)} \left[\left(\frac{c}{1-c}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right] dx \\ &+ \int_0^1 \frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \int_{Q_\infty}^Q \frac{h^\gamma - Q_\infty^\gamma}{h^2} dh dx + \int_0^t \int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_x^2 dx ds \\ \le& C\epsilon_0. \end{aligned} \tag{3.1}$$

Proof. Multiplying $(2.5)_2$ and $(2.5)_3$ by $\left(\frac{c}{1-c}\right)^\gamma Q(m)^{\gamma-2}$ and u, respectively, summing the resulting equations and integrating it over $[0,1]$ with respect to x, using (2.15) and the boundary conditions (2.6), integrating by parts, one gets

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \left[\left(\frac{1}{1-c}\right)\frac{u^2}{2} + \frac{1}{\rho_l(\gamma-1)}\left(\frac{c}{1-c}\right)^\gamma Q^{\gamma-1} + \frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma \frac{1}{Q}\right] dx \\ &+ \int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_x^2 dx \\ =& -\frac{1}{2}\int_0^1 \left(\frac{1}{1-c}\right) u^2 [cA] dx + \frac{\gamma}{\rho_l(\gamma-1)} \int_0^1 \left(\frac{c}{1-c}\right)^\gamma Q^{\gamma-1} A dx. \end{aligned} \tag{3.2}$$

Here we have also used the first equation of (2.5). Note that

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\left\{\frac{1}{\rho_l(\gamma-1)}\left(\frac{c}{1-c}\right)^\gamma Q^{\gamma-1}+\frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\frac{1}{Q}\right\}dx\\
=&\frac{1}{\rho_l(\gamma-1)}\frac{d}{dt}\int_0^1\left\{\left(\frac{c}{1-c}\right)^\gamma Q^{\gamma-1}-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma Q^{\gamma-1}\right\}dx\\
&+\frac{1}{\rho_l(\gamma-1)}\frac{d}{dt}\int_0^1\left\{\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma Q^{\gamma-1}-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma Q_\infty^{\gamma-1}\right\}dx\\
&+\frac{1}{\rho_l}\frac{d}{dt}\int_0^1\left\{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\frac{1}{Q}-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma Q_\infty^{\gamma-1}\right\}dx\\
=&\frac{d}{dt}\int_0^1\frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]dx+\frac{d}{dt}\int_0^1\frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\int_{Q_\infty}^Q\frac{h^\gamma-Q_\infty^\gamma}{h^2}dhdx,
\end{aligned}
$$

and

$$
\frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\geq 0,
$$

in view of $A(x,t)\leq 0$, and

$$
\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\int_{Q_\infty}^Q\frac{h^\gamma-Q_\infty^\gamma}{h^2}dh\geq 0.
$$

Consequently,

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_0^1\left\{\left(\frac{1}{1-c}\right)\frac{u^2}{2}+\frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\right\}dx\right)\\
&+\frac{d}{dt}\frac{1}{\rho_l}\int_0^1\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\int_{Q_\infty}^Q\frac{h^\gamma-Q_\infty^\gamma}{h^2}dhdx+\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dx\\
=&-\frac{1}{2}\int_0^1\left(\frac{1}{1-c}\right)u^2[cA]dx+\frac{\gamma}{\rho_l(\gamma-1)}\int_0^1\left(\frac{c}{1-c}\right)^\gamma Q^{\gamma-1}Adx\\
\leq&-\frac{1}{2}\int_0^1\left(\frac{1}{1-c}\right)u^2[cA]dx+\frac{\gamma}{\rho_l(\gamma-1)}\int_0^1\left[\left(\frac{c}{1-c}\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]Q^{\gamma-1}Adx. \qquad (3.3)
\end{aligned}
$$

In deriving the last inequality in (3.3), we have also used the sign of $A(x,t)$. Integrating the above inequality with respect to t over $[0,t]$, one gets,

$$
\begin{aligned}
&\int_0^1\left\{\left(\frac{1}{1-c}\right)\frac{u^2}{2}+\frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\right\}dx\\
&+\int_0^1\frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\int_{Q_\infty}^Q\frac{h^\gamma-Q_\infty^\gamma}{h^2}dhdx+\int_0^t\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dxds\\
\leq&C\int_0^t\|A(s)\|_{L^\infty([0,1])}\int_0^1\left\{\frac{1}{1-c}u^2+\frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\right\}dx\\
&+\int_0^1\left(\frac{1}{1-c_0}\right)\frac{u_0^2}{2}dx+\int_0^1\frac{Q_0^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c_0}{1-c_0}\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]dx\\
&+\int_0^1\frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\int_{Q_\infty}^{Q_0}\frac{h^\gamma-Q_\infty^\gamma}{h^2}dhdx. \qquad (3.4)
\end{aligned}
$$

It follows from (A_2), (2.16), (2.17), and (2.13) that

$$\begin{aligned}
&\int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \int_{Q_\infty}^{Q_0} \frac{h^\gamma - Q_\infty^\gamma}{h^2} dhdx \\
\leq & C\int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma Q_\infty^{-2}|Q_0-Q_\infty||Q_0^\gamma - Q_\infty^\gamma|dx \\
\leq & C\int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma Q_\infty^{-2}\gamma\xi^{\gamma-1}\left|Q_0-Q_\infty\right|^2 dx \\
\leq & C\int_0^1 x^{1-3\alpha}\left(Q_0-Q_\infty\right)^2 dx \leq C\epsilon_0,
\end{aligned}$$

where ξ is between Q_0 and Q_∞ and we have used that $Q_\infty \sim x^\alpha$. And it follows from (2.13) and (A_1) that

$$\begin{aligned}
&\int_0^1 \frac{Q_0^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c_0}{1-c_0}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right] dx \\
= & \int_0^1 \frac{Q_0^{\gamma-1}}{\rho_l(\gamma-1)}\left(\frac{c_0}{1-c_0}\right)^\gamma \left[1-\exp\int_0^{+\infty}\gamma A(x,s)ds\right] dx \\
\leq & C\int_0^1 \left(\frac{c_0}{1-c_0}\right)\left(\frac{c_0}{1-c_0}Q_0\right)^{\gamma-1}\left|1-e^{-\gamma C\epsilon_0}\right| dx \leq C\epsilon_0.
\end{aligned}$$

Using the above estimates, (3.4) and Gronwall inequality, we can get (3.1). This completes the proof. ☐

Lemma 3.2. *Under the conditions in Theorem 2.1, we have,*

$$C_3\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(x) \leq \left(\frac{c}{1-c}Q\right)(x,t) \leq C_4\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(x), \tag{3.5}$$

where $(x,t) \in \Omega = \{(x,t) : 0 \leq x \leq 1,\ t \geq 0\}$, C_3 and C_4 are two positive constants, independent of t.

Proof. Let $Y(x,t) = \left(\frac{c}{1-c}Q\right)^\beta (x,t)\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\beta}(x)$, then due to the first two equations of (2.5), we have

$$\begin{aligned}
Y_t = & \beta\left(\frac{c}{1-c}Q\right)^{\beta-1}\left[\left(\frac{c}{1-c}\right)_t Q + \left(\frac{c}{1-c}\right)Q_t\right]\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\beta} \\
= & \beta\left(\frac{c}{1-c}Q\right)^{\beta-1}\left[\left(\frac{c}{1-c}\right)AQ - \rho_l\left(\frac{c}{1-c}\right)Q^2u_x\right]\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\beta} \\
= & \beta Y(x,t)A - \rho_l\beta\left(\frac{c}{1-c}\right)^\beta Q^{\beta+1}u_x\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\beta}.
\end{aligned} \tag{3.6}$$

It follows from $(2.5)_3$ that

$$\left(\frac{c}{1-c}\right)^\beta Q^{\beta+1}u_x = \int_0^x \left(\frac{1}{1-c}u\right)_t dx + \left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma. \tag{3.7}$$

Then substituting (3.7) into (3.6), we have

$$\begin{aligned}
Y_t = & \beta Y(x,t)A - \rho_l\beta\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\beta}\int_0^x\left(\frac{1}{1-c}u\right)dx\right)_t \\
& + \beta\rho_l\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta}[1-Y^{\frac{\gamma}{\beta}}].
\end{aligned} \tag{3.8}$$

Claim: $Y(t) \leq \max\{1, Y_0\} + C(C\epsilon_0)^{\frac{1}{2}} := C_3$.

It suffices to verify the above claim for points $t = t_2$ such that $Y(t_2) > \max\{1, Y_0\}$. By virtue of the continuity of Y on $\mathbb{R}^+$ and the initial condition $Y(0) \le \max\{1, Y_0\}$ for each such points (if any), there exists a point $t_1 \in [0, t_2)$ such that

$$Y(t) > \max\{1, Y_0\} \quad \text{for} \quad t_1 < t \le t_2,$$

and $Y(t_1) = \max\{1, Y_0\}$. Firstly, we notice that for any $0 \le t_1 < t_2$,

$$\begin{aligned} &\left| \int_{t_1}^{t_2} \rho_l \beta \left(\left(\frac{c_\infty}{1 - c_\infty} Q_\infty \right)^{-\beta} \int_0^x \left(\frac{1}{1-c} u \right) dx \right)_t d\tau \right| \\ \le & C \left(\frac{c_\infty}{1 - c_\infty} Q_\infty \right)^{-\beta} x^{\frac{1}{2}} \sup_{t \ge 0} \left(\int_0^1 \left(\frac{1}{1-c} u \right)^2 dx \right)^{\frac{1}{2}} \\ \le & C (C\epsilon_0)^{1/2}. \end{aligned} \tag{3.9}$$

Here we have used (2.12) and (3.1) and the following fact,

$$\left(\frac{c_\infty}{1 - c_\infty} Q_\infty \right)^{-\beta} x^{\frac{1}{2}} \le C,$$

which follows from (2.16) and the restriction $\gamma \ge 2\beta$ directly. Integrating the differential equation (3.8) over (t_1, t_2), and taking into account the choice of the points t_1 and t_2,

$$\begin{aligned} Y(t_2) = & Y(t_1) + \beta \int_{t_1}^{t_2} Y(x,s) A(x,s) ds - \rho_l \beta \int_{t_1}^{t_2} \left(\left(\frac{c_\infty}{1 - c_\infty} Q_\infty \right)^{-\beta} \int_0^x \left(\frac{1}{1-c} u \right) dx \right)_t ds \\ & + \beta \rho_l \int_{t_1}^{t_2} \left(\frac{c_\infty}{1 - c_\infty} Q_\infty \right)^{\gamma - \beta} [1 - Y^{\frac{\gamma}{\beta}}] ds \\ \le & \max\{1, Y_0\} + \beta \max\{1, Y_0\} \int_{t_1}^{t_2} A(x,s) ds + C(C\epsilon_0)^{1/2}. \end{aligned}$$

Since $A \le 0$, the claim holds true.

Next, we estimate the lower bound of $Y(x,t)$. Similar to the estimate of the upper bound of $Y(x,t)$, one has

$$Y(t) \ge \min\{1, Y_0\} - 2C(C\epsilon_0)^{\frac{1}{2}} := C_4.$$

It suffices to verify the above inequality for points $t = t_2$ such that $Y(t_2) < \min\{1, Y_0\}$, for each such points (if any), there exists a point $t_1 \in [0, t_2)$ such that

$$Y(t) < \min\{1, Y_0\} \quad \text{for} \quad t_1 < t \le t_2,$$

and $Y(t_1) = \min\{1, Y_0\}$. Therefore, we have

$$\begin{aligned} Y(t_2) = & Y(t_1) + \beta \int_{t_1}^{t_2} Y(x,s) A(x,s) ds - \rho_l \beta \int_{t_1}^{t_2} \left(\left(\frac{c_\infty}{1 - c_\infty} Q_\infty \right)^{-\beta} \int_0^x \left(\frac{1}{1-c} u \right) dx \right)_t ds \\ & + \beta \rho_l \int_{t_1}^{t_2} \left(\frac{c_\infty}{1 - c_\infty} Q_\infty \right)^{\gamma - \beta} [1 - Y^{\frac{\gamma}{\beta}}] ds \\ \ge & \min\{1, Y_0\} - \beta \min\{1, Y_0\} \int_{t_1}^{t_2} \|A(\cdot, s)\|_{L^\infty} ds - C(C\epsilon_0)^{1/2} \\ \ge & \min\{1, Y_0\} - 2C(C\epsilon_0)^{1/2}. \end{aligned}$$

Here we have used that for any $t_1 \le t \le t_2$,

$$\int_{t_1}^{t_2} Y A ds \ge \int_{t_1}^{t_2} \min\{1, Y_0\} A ds \ge - \min\{1, Y_0\} \int_{t_1}^{t_2} \|A(\cdot, s)\|_{L^\infty([0,1])} ds \ge -C(C\epsilon_0)^{\frac{1}{2}},$$

since $A \le 0$ and $\int_0^t \|A(\cdot, s)\|_{L^\infty([0,1])} ds \le C\epsilon_0$. The proof of Lemma 3.2 is complete. □

It is interesting to note that both the size $\int_0^t \|A(\cdot,s)\|_{L^\infty([0,1])}ds \le C\epsilon_0$ and the sign of $A(x,t)$ are essential in the proof of the upper bound and lower bound of $\left(\frac{c}{1-c}Q\right)$.

Lemma 3.3. *Under the conditions in Theorem 2.1, it holds that*

$$\int_0^t \int_0^1 u^2 dxds \le C, \tag{3.10}$$

where C is a positive constant, independent of t.

Proof. Since $u(1,t)=0$, we have

$$\begin{aligned} |u(x,t)| &\le \int_x^1 |u_y(y,t)|dy \\ &= \int_x^1 |u_y(y,t)| \left(\frac{c}{1-c}\right)^{\frac{\beta}{2}} Q^{\frac{1+\beta}{2}} \left(\frac{c}{1-c}\right)^{-\frac{\beta}{2}} Q^{-\frac{1+\beta}{2}} dy \\ &\le \left(\int_x^1 \left(\frac{c}{1-c}\right)^{\beta} Q^{1+\beta} u_y^2 dy\right)^{\frac{1}{2}} \left(\int_x^1 \left(\frac{c}{1-c}\right)^{-\beta} Q^{-(1+\beta)} dy\right)^{\frac{1}{2}} \\ &\le C\left(\int_x^1 \left(\frac{c}{1-c}\right)^{\beta} Q^{1+\beta} u_y^2 dy\right)^{\frac{1}{2}} \left(\int_x^1 \left(\frac{c}{1-c}\right) \left(\frac{c}{1-c}Q\right)^{-(1+\beta)} dy\right)^{\frac{1}{2}}. \end{aligned}$$

It is easy to see from (2.16), (3.5) and Proposition 2.1 that

$$\int_0^1 \int_x^1 \left(\frac{c}{1-c}\right)\left(\frac{c}{1-c}Q\right)^{-(1+\beta)} dydx \le C\int_0^1 \int_x^1 y^{-\frac{\alpha\gamma+\beta}{\gamma}} dydx \le C,$$

provided $\beta < 2\gamma - \alpha\gamma$, and

$$\int_0^1 u^2 dx \le C\int_0^1 \left(\frac{c}{1-c}\right)^{\beta} Q^{1+\beta} u_x^2 dx. \tag{3.11}$$

Using Lemma 3.1, we get (3.10) immediately. This completes the proof of Lemma 3.3. □

For the next result we must make use of some regularity on $A(\cdot,t)$ relatively time variable. As stated in assumption (A_1) we shall assume that $\int_0^t \|A_t(\cdot,s)\|^2_{L^\infty([0,1])}ds \le C$.

Lemma 3.4. *Under the conditions in Theorem 2.1, it holds that*

$$\int_0^1 u_t^2 dx + \int_0^t \int_0^1 \left(\frac{c}{1-c}\right)^{\beta} Q^{1+\beta} u_{xt}^2 dx \le C, \tag{3.12}$$

where C is a positive constant, independent of t.

Proof. Differentiating $(2.5)_3$ with respect to t, multiplying the resulting equation by u_t and integrating over $[0,1]$ with respect to x, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{1}{1-c}\right) u_t^2 dx + \int_0^1 \left(\left(\frac{c}{1-c}Q\right)^{\gamma}\right)_{xt} u_t dx \\ =& -\frac{3}{2}\int_0^1 \left(\frac{c}{1-c}\right) A u_t^2 dx - \int_0^1 u\left(\frac{c}{1-c}\right) A^2 u_t dx - \int_0^1 u\left(\frac{c}{1-c}\right) A_t u_t dx \\ &+ \int_0^1 \left(\left(\frac{c}{1-c}\right)^{\beta} Q^{1+\beta} u_x\right)_{xt} u_t dx. \end{aligned} \tag{3.13}$$

Using the boundary conditions (2.6) and integrating by parts, we have

$$\begin{aligned}\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma\right)_{xt} u_t dx &= -\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma\right)_t u_{xt} dx \\ &= \gamma\rho_l \int_0^1 \left(\frac{c}{1-c}\right)^\gamma Q^{\gamma+1} u_x u_{xt} dx - \gamma \int_0^1 \left(\frac{c}{1-c}Q\right)^\gamma A u_{xt} dx, \quad (3.14)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \left(\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_x\right)_{xt} u_t dx &= -\int_0^1 \left(\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_x\right)_t u_{xt} dx \\ = (1+\beta)\rho_l \int_0^1 \left(\frac{c}{1-c}\right)^\beta & Q^{2+\beta} u_x^2 u_{xt} dx - \int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_{xt}^2 dx \\ &- \beta \int_0^t \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} A u_x u_{xt} dx. \qquad (3.15)\end{aligned}$$

Substituting (3.14) and (3.15) into (3.13), and using Cauchy inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{1}{1-c}\right) u_t^2 dx + \int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_{xt}^2 dx \\ =& -\frac{3}{2}\int_0^1 \left(\frac{c}{1-c}\right) A u_t^2 dx - \int_0^1 u \left(\frac{c}{1-c}\right) A^2 u_t dx - \int_0^1 u \left(\frac{c}{1-c}\right) A_t u_t dx \\ &+ (1+\beta)\rho_l \int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{2+\beta} u_x^2 u_{xt} dx - \beta \int_0^t \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} A u_x u_{xt} dx \\ &- \gamma\rho_l \int_0^1 \left(\frac{c}{1-c}\right)^\gamma Q^{\gamma+1} u_x u_{xt} dx + \gamma \int_0^1 \left(\frac{c}{1-c}Q\right)^\gamma A u_{xt} dx \\ \leq& C\int_0^1 (|A| + |A_t|^2)\left(\frac{1}{1-c}\right) u_t^2 dx + C\int_0^1 |A|^3 \left(\frac{c}{1-c}\right) u^2 dx + C\int_0^1 \left(\frac{c}{1-c}\right) u^2 dx \\ &+ \frac{1}{2}\int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_{xt}^2 dx + C\int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{\beta+3} u_x^4 dx + C\int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{\beta+1} u_x^2 A^2 dx \\ &+ C\int_0^1 \left(\frac{c}{1-c}\right)^{2\gamma-\beta} Q^{2\gamma-\beta+1} u_x^2 dx + C\int_0^1 \left(\frac{c}{1-c}\right)^{2\gamma-\beta} Q^{2\gamma-\beta-1} A^2 dx, \qquad (3.16)\end{aligned}$$

which implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{1}{1-c}\right)u_t^2dx+\frac{1}{2}\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{1+\beta}u_{xt}^2dx\\
\le& C\int_0^1(|A|+|A_t|^2)\left(\frac{1}{1-c}\right)u_t^2dx+C\int_0^1|A|^3\left(\frac{c}{1-c}\right)u^2dx+C\int_0^1\left(\frac{c}{1-c}\right)u^2dx\\
&+C\int_0^1\left(\frac{c}{1-c}\right)^{2\gamma-2\beta}Q^{2\gamma-2\beta}\left(\frac{c}{1-c}\right)^{\beta}Q^{1+\beta}u_x^2dx+C\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{1+\beta}u_x^2(Qu_x)^2dx\\
&+C\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{\beta+1}u_x^2dx+C\|A\|^2_{L^\infty[0,1]}\int_0^1\left(\frac{c}{1-c}\right)\left(\frac{c}{1-c}Q\right)^{2\gamma-\beta-1}dx\\
\le& C\int_0^1(|A|+|A_t|^2)\left(\frac{1}{1-c}\right)u_t^2dx+C\int_0^1u^2dx+C\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{1+\beta}u_x^2(Qu_x)^2dx\\
&+C\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{\beta+1}u_x^2dx+C\|A\|^2_{L^\infty([0,1])},
\end{aligned}\tag{3.17}
$$

where we have used (3.5). By (2.16) and Lemma 3.1, we get

$$
\begin{aligned}
(Qu_x)^2&=\left(\left(\frac{c}{1-c}\right)^{\beta}Q^{1+\beta}u_x\right)^2\left(\frac{c}{1-c}Q\right)^{-2\beta}\\
&=\left(\frac{c}{1-c}Q\right)^{-2\beta}\left\{\int_0^x\left(\frac{1}{1-c}u\right)_t dy+\left(\frac{c}{1-c}Q\right)^{\gamma}-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma}\right\}^2\\
&\le C\left(\frac{c}{1-c}Q\right)^{-2\beta}\left\{\left(\int_0^x\left(\frac{1}{1-c}u\right)_t dy\right)^2+\left(\frac{c}{1-c}Q\right)^{2\gamma}+\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{2\gamma}\right\}\\
&\le C\left(\frac{c}{1-c}Q\right)^{-2\beta}x\int_0^1\left[\frac{1}{1-c}u_t+\left(\frac{1}{1-c}\right)_t u\right]^2dx+C\\
&\le C\int_0^1\left(\frac{1}{1-c}\right)^2u_t^2dx+C.
\end{aligned}\tag{3.18}
$$

Here we have used the fact

$$
\left(\frac{c}{1-c}Q\right)^{-2\beta}x\le C,\quad \int_0^1\left(\frac{c}{1-c}\right)^2A^2u^2dx\le C,
$$

which follow from $\beta\le\frac{\gamma}{2}$ and Lemma 3.1 respectively. Then it follows from (3.17), (3.18) that

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\left(\frac{1}{1-c}\right)u_t^2dx+\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{1+\beta}u_{xt}^2dx\\
\le& C\left(\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{\beta+1}u_x^2dx+\|A\|_{L^\infty([0,1])}+\|A_t\|^2_{L^\infty([0,1])}\right)\int_0^1\left(\frac{1}{1-c}\right)u_t^2dx\\
&+C\int_0^1\left(\frac{c}{1-c}\right)^{\beta}Q^{\beta+1}u_x^2dx+C\|A\|^2_{L^\infty([0,1])}+C\int_0^1u^2dx.
\end{aligned}\tag{3.19}
$$

According to assumption (A_4), $\beta \le \frac{\gamma+\alpha\gamma}{3}$, and taking care of $(2.5)_3$, using the following fact

$$
\begin{aligned}
\left(\frac{c_0}{1-c_0}Q_0\right)^{\gamma}_x &= \frac{\gamma}{\beta}\left(\frac{c_0}{1-c_0}Q_0\right)^{\gamma-\beta}\left(\frac{c_0}{1-c_0}Q_0\right)^{\beta}_x \\
&= \frac{\gamma}{\beta}\left(\frac{c_0}{1-c_0}Q_0\right)^{\gamma-\beta}\left[\left(\frac{c_0}{1-c_0}Q_0\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x\right] \\
&\quad + \frac{\gamma}{\beta}\left(\frac{c_0}{1-c_0}Q_0\right)^{\gamma-\beta}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x \\
&\le Cx^{\frac{\gamma-\beta}{\gamma}}\left|\left(\frac{c_0}{1-c_0}Q_0\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x\right| + C \\
&\le Cx^{\frac{3\gamma-\beta-\alpha\gamma}{4\gamma}+\frac{\gamma-3\beta+\alpha\gamma}{4\gamma}}\left|\left(\frac{c_0}{1-c_0}Q_0\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x\right| + C \\
&\le Cx^{\frac{3\gamma-\beta-\alpha\gamma}{4\gamma}}\left|\left(\frac{c_0}{1-c_0}Q_0\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x\right| + C,
\end{aligned}
$$

we obtain $\int_0^1 u_t^2(x,0)dx \le C$. Then (3.19), Gronwall inequality, Lemma 3.1 and Lemma 3.3 imply (3.12). This proves Lemma 3.4. □

Corollary 3.1. *It follows from (3.12) and (3.18) that*

$$\|Qu_x\|_{L^\infty([0,1]\times[0,\infty))} \le C, \tag{3.20}$$

where C is a positive constant, independent of t.

Lemma 3.5. *Assume the conditions in Theorem 2.1 hold, and let θ be a fixed constant.*
(i) if $\theta > 0$, then

$$\int_0^t\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^{\gamma} - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma}\right)\left(\left(\frac{c}{1-c}Q\right)^{\theta} - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\theta}\right) dxds \le C,$$

(ii) if $\theta \in (0, \frac{\gamma+\alpha\gamma-\beta}{2}]$, then

$$\int_0^t\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^{\gamma} - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma}\right)\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta} - \left(\frac{c}{1-c}Q\right)^{-\theta}\right) dxds \le C,$$

where C is a positive constant, independent of t.

Proof. We only prove (ii) here, since the proof of (i) is similar to the proof of (ii). Due to $(2.5)_3$, using the second equation of (2.5) and (2.16), we have

$$
\begin{aligned}
\left(\frac{c}{1-c}Q\right)^{\gamma} - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma} &= -\int_0^x \left(\frac{1}{1-c}u\right)_t dy + \left(\frac{c}{1-c}\right)^{\beta} Q^{1+\beta}u_x \\
&= -\int_0^x \left(\frac{1}{1-c}u\right)_t dy - \frac{1}{\rho_l\beta}\left(\frac{c}{1-c}Q\right)^{\beta}_t \\
&\quad + \frac{1}{\rho_l}\left(\frac{c}{1-c}Q\right)^{\beta} A.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^t\int_0^1\left(\left(\frac{c}{1-c}Q\right)^{\gamma}-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma}\right)\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta}-\left(\frac{c}{1-c}Q\right)^{-\theta}\right)dxds\\
&=-\frac{1}{\beta\rho_l}\int_0^t\int_0^1\left(\frac{c}{1-c}Q\right)^{\beta}_t\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta}-\left(\frac{c}{1-c}Q\right)^{-\theta}\right)dxds\\
&\quad-\int_0^t\int_0^1\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta}-\left(\frac{c}{1-c}Q\right)^{-\theta}\right)\int_0^x\left(\frac{1}{1-c}u\right)_t dydxds\\
&\quad+\frac{1}{\rho_l}\int_0^t\int_0^1\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta}-\left(\frac{c}{1-c}Q\right)^{-\theta}\right)\left(\frac{c}{1-c}Q\right)^{\beta}Adxds\\
&=I_1+I_2+I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=-\frac{1}{\beta\rho_l}\int_0^t\int_0^1\left(\frac{c}{1-c}Q\right)^{\beta}_t\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta}-\left(\frac{c}{1-c}Q\right)^{-\theta}\right)dxds\\
&=\frac{1}{\rho_l\beta}\int_0^1\left(\left(\frac{c_0}{1-c_0}Q_0\right)^{\beta}-\left(\frac{c}{1-c}Q\right)^{\beta}\right)\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta}dx\\
&\quad+\frac{1}{\rho_l}\int_0^t\int_0^1\left(\frac{c}{1-c}Q\right)^{\beta-\theta-1}\left(\frac{c}{1-c}Q\right)_t dsdx\\
&=I_1^{(1)}+I_1^{(2)}.
\end{aligned}
$$

By using (A_2), (2.16) and (3.5), one easily gets

$$\left|I_1^{(1)}\right|\le C\int_0^1 x^{\frac{\beta-\theta}{\gamma}}dx\le C, \tag{3.21}$$

provided $\theta<\beta+\gamma$.

Next we estimate $I_1^{(2)}$ by two cases.

Case 1: $\beta\neq\theta$.

In this case, if $\theta<\beta+\gamma$, similarly to (3.21), we have

$$\left|I_1^{(2)}\right|=\left|\frac{1}{\rho_l(\beta-\theta)}\int_0^1\left(\left(\frac{c}{1-c}Q\right)^{\beta-\theta}-\left(\frac{c_0}{1-c_0}Q_0\right)^{\beta-\theta}\right)dx\right|\le C.$$

Case 2: $\beta=\theta$.

In this case, also by (A_2), (3.5) and (2.16), we have

$$\left|I_1^{(2)}\right|=\left|\frac{1}{\rho_l}\int_0^1\left(\ln\left(\frac{c}{1-c}Q\right)-\ln\left(\frac{c_0}{1-c_0}Q_0\right)\right)dx\right|\le C\int_0^1|\ln C|dx\le C.$$

Hence, if $\theta<\beta+\gamma$, we have

$$|I_1|\le C. \tag{3.22}$$

Now we estimate I_2 as follows:

$$\begin{aligned} I_2 = & -\int_0^1 \left(\left(\frac{c_\infty}{1-c_\infty} Q_\infty \right)^{-\theta} - \left(\frac{c}{1-c} Q \right)^{-\theta} \right) \int_0^x \frac{1}{1-c} u dy dx \\ & + \int_0^1 \left(\left(\frac{c_\infty}{1-c_\infty} Q_\infty \right)^{-\theta} - \left(\frac{c_0}{1-c_0} Q_0 \right)^{-\theta} \right) \int_0^x \frac{1}{1-c_0} u_0 dy dx \\ & + \int_0^t \int_0^1 \left(\left(\frac{c_\infty}{1-c_\infty} Q_\infty \right)^{-\theta} - \left(\frac{c}{1-c} Q \right)^{-\theta} \right)_t \left(\int_0^x \frac{1}{1-c} u dy \right) dx ds \\ = & I_2^{(1)} + I_2^{(2)}, \end{aligned}$$

where

$$\begin{aligned} \left| I_2^{(1)} \right| \leq & \int_0^1 \left| \left(\left(\frac{c_\infty}{1-c_\infty} Q_\infty \right)^{-\theta} - \left(\frac{c}{1-c} Q \right)^{-\theta} \right) \left(\int_0^x \frac{1}{1-c} u dy \right) \right| dx \\ & + \int_0^1 \left| \left(\left(\frac{c_\infty}{1-c_\infty} Q_\infty \right)^{-\theta} - \left(\frac{c_0}{1-c_0} Q_0 \right)^{-\theta} \right) \int_0^x \frac{1}{1-c_0} u_0 dy \right| dx \\ \leq & C \int_0^1 x^{-\frac{\theta}{\gamma}} x^{\frac{1}{2}} \left\{ \left(\int_0^1 \left(\frac{1}{1-c} \right)^2 u^2 dx \right)^{\frac{1}{2}} + \left(\int_0^1 \left(\frac{1}{1-c_0} \right)^2 u_0^2 dx \right)^{\frac{1}{2}} \right\} dx \\ \leq & C \int_0^1 x^{\frac{1}{2} - \frac{\theta}{\gamma}} dx \leq C. \end{aligned} \tag{3.23}$$

Here, $\theta < \frac{3}{2}\gamma$, (A_2), (2.16), (3.1), (3.5) and Hölder inequality were used.

From $(2.5)_1$ and $(2.5)_2$, Hölder inequality, we have

$$\begin{aligned} & I_2^{(2)} = \int_0^t \int_0^1 \left(\left(\frac{c_\infty}{1-c_\infty} Q_\infty \right)^{-\theta} - \left(\frac{c}{1-c} Q \right)^{-\theta} \right)_t \left(\int_0^x \frac{1}{1-c} u dy \right) dx ds \\ & = -\rho_l \theta \int_0^t \int_0^1 \left(\frac{c}{1-c} \right)^{-\theta} Q^{1-\theta} u_x \left(\int_0^x \frac{1}{1-c} u dy \right) dx ds \\ & \quad + \theta \int_0^t \int_0^1 \left(\frac{c}{1-c} Q \right)^{-\theta} A \left(\int_0^x \frac{1}{1-c} u dy \right) dx ds \\ & \leq C \left(\int_0^t \int_0^1 x \left(\frac{c}{1-c} \right)^{-\beta-2\theta} Q^{1-\beta-2\theta} \left(\frac{c}{1-c} \right)^{\beta} Q^{1+\beta} u_x^2 dx ds \right)^{\frac{1}{2}} \left(\int_0^t \int_0^1 \left(\frac{1}{1-c} u \right)^2 dx ds \right)^{\frac{1}{2}} \\ & \quad + \int_0^t \int_0^1 x^{\frac{1}{2}} \left(\frac{c}{1-c} Q \right)^{-\theta} A \left(\int_0^x \left(\frac{1}{1-c} u \right)^2 dx \right)^{\frac{1}{2}} dx ds \\ & \leq C \sup_{[0,1]\times[0,\infty)} \left(x \frac{1-c}{c} \left(\frac{c}{1-c} Q \right)^{1-\beta-2\theta} \right)^{\frac{1}{2}} \left(\int_0^t \int_0^1 \left(\frac{c}{1-c} \right)^{\beta} Q^{1+\beta} u_x^2 dx ds \right)^{\frac{1}{2}} \\ & \quad + C \int_0^t \|A\|_{L^\infty([0,1])} \int_0^1 x^{\frac{1}{2}} \left(\frac{c}{1-c} Q \right)^{-\theta} dx ds \\ & \leq C, \end{aligned} \tag{3.24}$$

provided $\theta \le \frac{\gamma-\beta+\alpha\gamma}{2}$. Here we have use the assumption (A_1), (2.16), (3.1), (3.5) and (3.10). Next, we estimate I_3,

$$
\begin{aligned}
I_3 &= \frac{1}{\rho_l}\int_0^t\int_0^1 \left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta} - \left(\frac{c}{1-c}Q\right)^{-\theta}\right)\left(\frac{c}{1-c}Q\right)^{\beta} Adxds \\
&\le \int_0^t\int_0^1 x^{\frac{\beta-\theta}{\gamma}} Adxds \le \int_0^t \|A\|_{L^\infty([0,1])}\int_0^1 x^{\frac{\beta-\theta}{\gamma}} dxds \le C. \qquad (3.25)
\end{aligned}
$$

Note that $\gamma > 1$, $\alpha \le \frac{1}{\gamma}$ and $\beta \in (0, \frac{\gamma}{2}]$ imply that $(0, \beta+\gamma) \cap (0, \frac{\gamma+\alpha\gamma-\beta}{2}] \cap (0, \frac{3}{2}\gamma) = (0, \frac{\gamma+\alpha\gamma-\beta}{2}]$. It follows from (3.21)-(3.25) that (ii) holds. This completes the proof of Lemma 3.5. ☐

Lemma 3.6. *Assume the conditions in Theorem 2.1 hold, then*

$$
\begin{aligned}
&\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}} \left(\left(\frac{c}{1-c}Q\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x\right)^2 dx \\
&\qquad + \int_0^t\int_0^1 x^{\frac{5\gamma-3\beta-\alpha\gamma}{2\gamma}} \left(\left(\frac{c}{1-c}Q\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x\right)^2 dxdt \le C, \quad (3.26)
\end{aligned}
$$

where C is a positive constant, independent of t.

Proof. From (2.5) and (2.16), we get

$$
\left(\left(\frac{c}{1-c}Q\right)^{\beta}\right)_{tx} - \beta\left(\left(\frac{c}{1-c}Q\right)^{\beta} A\right)_x + \beta\rho_l(E(c,Q)u_x)_x = 0,
$$

and

$$
\gamma\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}\right)_x = \frac{\beta g}{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta}}.
$$

Hence, combining these with (2.5), we get

$$
\begin{aligned}
&\left(\left(\frac{c}{1-c}Q\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x + \beta\rho_l\frac{1}{1-c}u\right)_t \\
&\qquad + \gamma\rho_l\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)^{\beta}_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta}_x + \beta\rho_l\frac{1}{1-c}u\right) \\
&= \beta\rho_l^2\gamma\frac{1}{1-c}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}u + \beta\rho_l g\left(1 - \frac{\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}}{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta}}\right) + \beta\left(\left(\frac{c}{1-c}Q\right)^{\beta}A\right)_x. \qquad (3.27)
\end{aligned}
$$

Multiplying (3.27) by $x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)$, then integrating the resulting equation on $[0,1]\times[0,t]$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dx\\
&+\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dxds\\
=&\frac{1}{2}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c_0}{1-c_0}Q_0\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c_0}u_0\right)^2dx\\
&+\beta\rho_l^2\gamma\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)\frac{1}{1-c}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}udxds\\
&+\beta\rho_l g\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)\left(1-\frac{\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}}{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta}}\right)dxds\\
&+\beta\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)\left(\left(\frac{c}{1-c}Q\right)^\beta A\right)_x dxds\\
=&\sum_{i=1}^4 J_i,
\end{aligned}
\tag{3.28}
$$

Now we estimate J_1–J_4 as follows:

First, by the assumption (A_4), and Cauchy-Schwarz inequality, we have

$$
J_1\le C,
$$

$$
\begin{aligned}
J_2\le&\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dxds\\
&+C\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\frac{1}{1-c}\right)^2u^2dxds\\
\le&\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dxds\\
&+C\int_0^t\int_0^1 x^{\frac{5\gamma-3\beta-\alpha\gamma}{2\gamma}}\left(\frac{1}{1-c}\right)^2u^2dxds\\
\le&\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dxds+C,
\end{aligned}
$$

$$
\begin{aligned}
J_3 \le & \frac{1}{8}\gamma\rho_l \int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}} \left(\frac{c}{1-c}Q\right)^{\gamma-\beta} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \beta\rho_l \frac{1}{1-c}u\right)^2 dxds \\
& + C\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}} \left(\frac{c}{1-c}Q\right)^{\beta-\gamma} \left(1-\frac{\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}}{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta}}\right)^2 dxds \\
\le & \frac{1}{8}\gamma\rho_l \int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}} \left(\frac{c}{1-c}Q\right)^{\gamma-\beta} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \beta\rho_l \frac{1}{1-c}u\right)^2 dxds \\
& + C\int_0^t\int_0^1 W(x,t)\left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta} - \left(\frac{c}{1-c}Q\right)^{-\theta}\right) dxds,
\end{aligned}
$$

where

$$
W(x,t) = \frac{x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}} \left(\frac{c}{1-c}Q\right)^{\beta-\gamma}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{2\beta-2\gamma}\left\{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta} - \left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\right\}^2}{\left\{\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right\}\left\{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta} - \left(\frac{c}{1-c}Q\right)^{-\theta}\right\}}.
$$

It is easy to see that

$$
|W(x,t)| \le Cx^{\frac{\beta-\gamma+2\theta-\alpha\gamma}{2\gamma}} \le C,
$$

provided $\theta = \frac{\alpha\gamma+\gamma-\beta}{2}$. In the following, we devote ourselves to deal with J_4, which contains the function $A(x,t)$.

$$
\begin{aligned}
J_4 = \beta & \int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \beta\rho_l\frac{1}{1-c}u\right) \\
& \times \left(\left(\frac{c}{1-c}Q\right)^\beta_x A - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x A\right) dxds \\
& + \beta\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \beta\rho_l\frac{1}{1-c}u\right) \\
& \times \left(\frac{\beta}{\gamma}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta-\gamma} gA + \left(\frac{c}{1-c}Q\right)^\beta A_x\right) dxds = J_4^{(1)} + J_4^{(2)},
\end{aligned}
$$

$$
\begin{aligned}
J_4^{(1)} \leq & C\int_0^t \|A\|_{L^\infty([0,1])}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta\right)^2 dxds\\
&+C\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\frac{1}{1-c}u\left(\left(\frac{c}{1-c}Q\right)_x^\beta A-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta A\right)dxds\\
\leq & C\int_0^t \|A\|_{L^\infty([0,1])}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta\right)^2 dxds\\
&+\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta\right)^2 dxds\\
&+C\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\beta-\gamma}\left(\frac{1}{1-c}u\right)^2 A^2 dxds\\
\leq & C+C\int_0^t \|A\|_{L^\infty([0,1])}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta\right)^2 dxds\\
&+\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta\right)^2 dxds,
\end{aligned}
$$

$$
\begin{aligned}
J_4^{(2)} \leq & \int_0^t \|A\|_{L^\infty([0,1])}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c}u\right)^2 dxds\\
&+C\int_0^t \|A\|_{L^\infty[0,1]}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{2\beta-2\gamma} dxds\\
&+\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c}u\right)^2 dx\\
&+C\int_0^t \|A_x\|^2_{L^\infty[0,1]}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\beta-\gamma}\left(\frac{c}{1-c}Q\right)^{2\beta} dxds\\
\leq & C+C\int_0^t \|A\|_{L^\infty([0,1])}\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c}u\right)^2 dxds\\
&+\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1 x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c}u\right)^2 dx.
\end{aligned}
$$

Here we have used the assumption (A_1), (2.16) and (3.5). Taking $\theta=\frac{\alpha\gamma+\gamma-\beta}{2}$ in Lemma 3.5 (ii), substituting J_1-J_4 into (3.28), we obtain (3.26) immediately by using Gronwall inequality. This completes the proof of Lemma 3.6. □

4. Asymptotic behavior

To apply the uniform estimates obtained above to study the asymptotic behavior of $\left(\frac{c}{1-c}Q\right)(x,t)$ and $u(x,t)$, we introduce the following lemma (cf. [3, 28, 20]), and omit the details of the proof.

Lemma 4.1. *Suppose that* $y\in W^{1,1}_{loc}(\mathbb{R}^+)$ *satisfies*

$$y=y_1'+y_2,$$

and

$$|y_2| \leq \sum_{i=1}^{n} \alpha_i, \quad |y'| \leq \sum_{i=1}^{n} \beta_i, \quad on \ \ \mathbb{R}^+,$$

where $y_1 \in W^{1,1}_{loc}(\mathbb{R}^+)$, $\lim\limits_{s\to+\infty} y_1(s) = 0$ *and* α_i, $\beta_i \in L^{p_i}(\mathbb{R}^+)$ *for some* $p_i \in [1,\infty)$, $i = 1,\cdots,n$. *Then*

$$\lim_{s\to+\infty} y(s) = 0.$$

Proof of Theorem 2.1.

At first, we consider the convergence of $u(x,t)$. To this end, motivated by [19], we introduce a function of t as follows:

$$f(t) = \int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\beta Q_\infty^{1+\beta} u_x^2 dx.$$

Then by Lemma 2.1 and Lemma 3.1, we get

$$\begin{aligned}\int_0^\infty |f(t)|dt &= \int_0^\infty \int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\beta Q_\infty^{1+\beta} u_x^2 dxdt \\ &\leq C \int_0^\infty \int_0^1 \left(\frac{\frac{c}{1-c}}{\frac{c_\infty}{1-c_\infty}}\right) \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_x^2 dxdt \\ &\leq C.\end{aligned}$$

Furthermore, by Lemmas 2.1, 3.1, 3.2, 3.4 and Cauchy inequality, we have

$$\int_0^\infty \left|\frac{df(t)}{dt}\right| dt \leq \int_0^\infty \int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\beta Q_\infty^{1+\beta} u_x^2 dxdt + \int_0^\infty \int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\beta Q_\infty^{1+\beta} u_{xt}^2 dxdt \leq C.$$

It follows from Lemma 4.1 that

$$\lim_{t\to\infty} \int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\beta Q_\infty^{1+\beta} u_x^2 dx = 0.$$

Therefore, by Hölder inequality, we have for $x \in [\delta, 1]$ with $0 < \delta < 1$,

$$\begin{aligned}|u(x,t)| &= \left|\int_x^1 u_z(z,t)dz\right| = \left|\int_x^1 \left(\frac{c_\infty}{1-c_\infty}\right)^{\frac{\beta}{2}} Q_\infty^{\frac{1+\beta}{2}} u_z(z,t) \left(\frac{c_\infty}{1-c_\infty}\right)^{-\frac{\beta}{2}} Q_\infty^{-\frac{1+\beta}{2}} dz\right| \\ &\leq \left(\int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\beta Q_\infty^{\beta+1} u_x^2(x,t)dx\right)^{\frac{1}{2}} \left(\int_\delta^1 \left(\frac{c_\infty}{1-c_\infty}\right)^{-\beta} Q_\infty^{-(1+\beta)} dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^\beta Q_\infty^{\beta+1} u_x^2(x,t)dx\right)^{\frac{1}{2}} \to 0, \end{aligned} \tag{4.1}$$

as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \sup_{x\in[\delta,1]} u(x,t) = 0. \tag{4.2}$$

Furthermore, when $\delta = 0$,

$$\int_0^1 \left(\frac{c_\infty}{1-c_\infty}\right)^{-\beta} Q_\infty^{-(1+\beta)} dx \leq \int_0^1 x^{\frac{1}{\gamma}-\alpha} x^{-\frac{1+\beta}{\gamma}} dx \leq C,$$

provided $\beta < \gamma - \alpha\gamma$. Thus, when $\beta < \gamma - \alpha\gamma$, we have

$$\lim_{t\to\infty} \sup_{x\in[0,1]} u(x,t) = 0. \tag{4.3}$$

Next we consider the convergence of $\left(\frac{c}{1-c}Q\right)(x,t)$. Firstly, we shall show that $\left(\frac{c}{1-c}Q\right)(x,t)$ tends to the stationary state $\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(x)$ in the sense of integral as $t\to\infty$. The similar conclusion for Navier-Stokes equation was obtained in [3, 28, 30] before.

Taking $\theta=\gamma$ in Lemma 3.5 (i), we have

$$\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dx\in L^1(\mathbb{R}^+). \tag{4.4}$$

On the other hand, it is easy to see that

$$\begin{aligned}
&\left|\frac{d}{dt}\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dx\right|\\
\le&\, C\left|\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\frac{c}{1-c}\right)^\gamma Q^{\gamma+1}u_xdx\right|\\
&+C\left|\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\frac{c}{1-c}Q\right)^\gamma Adx\right|\\
\le&\, C\left(\int_0^1\left(\frac{c}{1-c}\right)^{2\gamma-\beta}Q^{2\gamma+1-\beta}dx\right)^{\frac12}\left(\int_0^1\left(\frac{c}{1-c}Q\right)^\beta Q^{1+\beta}u_x^2dx\right)^{\frac12}+C\|A(s)\|_{L^\infty([0,1])}\\
\le&\, C\left(\int_0^1\left(\frac{c}{1-c}Q\right)^\beta Q^{1+\beta}u_x^2dx\right)^{\frac12}+C\|A(s)\|_{L^\infty([0,1])}.
\end{aligned} \tag{4.5}$$

Let $y_1=0$ and y_2 be defined by (4.4) in Lemma 4.1. Clearly, the first term on the right side of (4.5) is in $L^2([0,+\infty))$ and the second term in $L^1([0,+\infty))$. Then Lemma 4.1 yields

$$\lim_{t\to\infty}\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dx=0. \tag{4.6}$$

Consequently,

$$\begin{aligned}
&\int_0^1\left(\left(\frac{c}{1-c}Q\right)-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)\right)^{2\gamma}dx\\
=&\int_0^1\frac{\left(\left(\frac{c}{1-c}Q\right)-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)\right)^{2\gamma}}{\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dx\\
\le&\, C\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dx\to0,\quad\text{as }\ t\to\infty.
\end{aligned} \tag{4.7}$$

For $q\in(0,2\gamma)$, we have from (4.7) and Hölder inequality that

$$\int_0^1\left|\frac{c}{1-c}Q-\frac{c_\infty}{1-c_\infty}Q_\infty\right|^qdx\le C\left(\int_0^1\left(\frac{c}{1-c}Q-\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{2\gamma}dx\right)^{\frac{q}{2\gamma}}\to0. \tag{4.8}$$

On the other hand, for $q \in (2\gamma, \infty)$, we have from (2.16) and (3.5) that

$$\begin{aligned}
&\int_0^1 \left|\frac{c}{1-c}Q - \frac{c_\infty}{1-c_\infty}Q_\infty\right|^q dx \\
= &\int_0^1 \left|\frac{c}{1-c}Q - \frac{c_\infty}{1-c_\infty}Q_\infty\right|^{q-2\gamma} \left(\frac{c}{1-c}Q - \frac{c_\infty}{1-c_\infty}Q_\infty\right)^{2\gamma} dx \\
\le &\, C\int_0^1 \left(\frac{c}{1-c}Q - \frac{c_\infty}{1-c_\infty}Q_\infty\right)^{2\gamma} dx \to 0.
\end{aligned} \tag{4.9}$$

Hence, by (4.8) and (4.9), we get

$$\left\|\left(\frac{c}{1-c}Q - \frac{c_\infty}{1-c_\infty}Q_\infty\right)(\cdot,t)\right\|_{L^q([0,1])} \to 0, \quad \text{as} \quad t\to\infty, \quad q\in(0,\infty). \tag{4.10}$$

We are now in a position to show the uniform convergence of $\left(\frac{c}{1-c}Q\right)(x,t)$. To this end, choosing a positive number k large enough, which is to be determined later, applying Hölder inequality, (2.16) and (3.5), we have by $\left(\frac{c}{1-c}Q\right)(0,t) = \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(0) = 0$,

$$\begin{aligned}
0 \le &\left|\left(\tfrac{c}{1-c}Q\right)^\beta (x,t) - \left(\tfrac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta (x)\right|^k \\
\le &\, k\int_0^x \left|\left(\frac{c}{1-c}Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right|^{k-1} \left|\left(\left(\frac{c}{1-c}Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right)_x\right| dy \\
\le &\, k\left(\int_0^1 \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\eta} \left|\left(\frac{c}{1-c}Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right|^{2k-2} dx\right)^{\frac12} \\
&\times \left(\int_0^1 \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\eta} \left(\left(\frac{c}{1-c}Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right)_x^2\right)^{\frac12} \\
\le &\, C\left(\int_0^1 \left|\left(\frac{c}{1-c}Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right|^{2k-2-\frac{\eta}{\beta}} dx\right)^{\frac12} \\
&\times \left(\int_0^1 \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\eta} \left(\left(\frac{c}{1-c}Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right)_x^2\right)^{\frac12},
\end{aligned} \tag{4.11}$$

where $\eta = \frac{3\gamma-\beta-\alpha\gamma}{2}$. Note that

$$\left|\left(\frac{c}{1-c}Q\right)^\beta - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right| \le C\left|\left(\frac{c}{1-c}Q\right) - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)\right|^{\min\{\beta,1\}}. \tag{4.12}$$

Now letting $2k-2-\frac{\eta}{\beta} > 0$, we deduce from (4.10), (4.11) and Lemma 3.6 that

$$\lim_{t\to\infty}\left|\left(\frac{c}{1-c}Q\right)^\beta (x,t) - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta (x)\right|^k = 0, \tag{4.13}$$

uniformly in $x\in[0,1]$, and (2.18) follows. This completes the proof of Theorem 2.1.

5. Stabilization rate estimates

In this section, using the method in [28], we will give the stabilization rate estimates of the weak solution $\left(\left(\frac{c}{1-c}Q\right)(x,t), u(x,t)\right)$ under the condition $\beta\in(0,\gamma-\alpha\gamma)\cap(0,\frac{\gamma}{2}]\cap(0,\frac{\gamma+\alpha\gamma}{3}]$.

Compared with the corresponding results in [28, 30], the only difference is that we must deal with the terms involving function $A(x,t)$.

Lemma 5.1. *Assume the conditions in Theorem 2.2 hold, we have*

$$\int_0^1 \left(u^2 + x^{1-3\alpha}(Q-Q_\infty)^2\right)dx \le \frac{C}{1+t}, \quad \forall\, t \ge 0, \tag{5.1}$$

and

$$\int_0^\infty \int_0^1 (1+t)\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta} u_x^2\, dx dt \le C. \tag{5.2}$$

Proof. Multiplying (3.3) by $(1+t)$ and taking integration over $[0,t]$, we get

$$\begin{aligned}
&(1+t)\int_0^1 \left\{\frac{1}{2}\left(\frac{1}{1-c}\right)u^2 + \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\right\}dx \\
&+(1+t)\int_0^1 \frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \int_{Q_\infty}^{Q} \frac{h^\gamma - Q_\infty^\gamma}{h^2}\, dh dx + \int_0^t \int_0^1 (1+s)\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2\, dx ds \\
\le& \int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1 \left\{\frac{1}{2}\left(\frac{1}{1-c}\right)u^2 + \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\right\}dx ds \\
&+\frac{1}{2}\int_0^1 \left(\frac{1}{1-c_0}\right)u_0^2 dx + \int_0^1 \frac{Q_0^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c_0}{1-c_0}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]dx \\
&+\int_0^1 \frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \int_{Q_\infty}^{Q_0} \frac{h^\gamma - Q_\infty^\gamma}{h^2}\, dh dx \\
&+\int_0^t \int_0^1 \left\{\frac{1}{2}\left(\frac{1}{1-c}\right)u^2 + \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\right\}dx ds \\
&+\int_0^t \int_0^1 \frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \int_{Q_\infty}^{Q} \frac{h^\gamma - Q_\infty^\gamma}{h^2}\, dh dx ds \\
\le& \int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1 \left\{\frac{1}{2}\left(\frac{1}{1-c}\right)u^2 + \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]\right\}dx ds \\
&+\int_0^t \int_0^1 \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]dx ds \\
&+\int_0^t \int_0^1 \frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \int_{Q_\infty}^{Q} \frac{h^\gamma - Q_\infty^\gamma}{h^2}\, dh dx ds + C,
\end{aligned}$$

where we have used Lemma 3.1 and Lemma 3.3. Furthermore, due to (2.11), (2.13) and the assumption (2.21) we can estimate as follows:

$$\begin{aligned}
&\int_0^t \int_0^1 \frac{Q^{\gamma-1}}{\rho_l(\gamma-1)}\left[\left(\frac{c}{1-c}\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right]dx ds \\
&= \frac{1}{\rho_l(\gamma-1)}\int_0^t \int_0^1 Q^{-1}\left(\frac{c}{1-c}Q\right)^\gamma \left[1 - \left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \left(\frac{c}{1-c}\right)^{-\gamma}\right]dx ds \\
&= \frac{1}{\rho_l(\gamma-1)}\int_0^t \int_0^1 Q^{-1}\left(\frac{c}{1-c}Q\right)^\gamma \left[1 - \exp\left(\gamma\int_s^{+\infty} A(x,\tau)d\tau\right)\right]dx ds \\
&\le C\int_0^t \int_0^1 x^{1-\alpha}\left[1 - \exp\left(-\gamma\int_s^{+\infty} \|A(\cdot,\tau)\|_{L^\infty([0,1])}d\tau\right)\right]dx ds \\
&\le C\int_0^t \left[1 - \exp\left(-\gamma\int_s^{+\infty} \|A(\cdot,\tau)\|_{L^\infty([0,1])}d\tau\right)\right]ds \le C,
\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^t\int_0^1 \frac{1}{\rho_l}\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma \int_{Q_\infty}^{Q} \frac{h^\gamma - Q_\infty^\gamma}{h^2} dhdx \\
&\le C\int_0^t\int_0^1 Q_\infty^{-2}|Q-Q_\infty|\left|\left(\frac{c_\infty}{1-c_\infty}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right| dxds \\
&\le C\int_0^t\int_0^1 Q_\infty^{-2}|Q-Q_\infty|\left|\left(\frac{c_\infty}{1-c_\infty}Q\right)^\gamma - \left(\frac{c}{1-c}Q\right)^\gamma\right| dxds \\
&\qquad\qquad + C\int_0^t\int_0^1 Q_\infty^{-2}|Q-Q_\infty|\left|\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right| dxds \\
&=: A + B,
\end{aligned}
$$

where

$$
\begin{aligned}
A =& C\int_0^t\int_0^1 Q_\infty^{-2}|Q-Q_\infty|\left|\left(\frac{c_\infty}{1-c_\infty}Q\right)^\gamma - \left(\frac{c}{1-c}Q\right)^\gamma\right| dxds \\
=& C\int_0^t\int_0^1 Q_\infty^{-2}|Q-Q_\infty|\left(\frac{c}{1-c}Q\right)^\gamma\left[1-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\left(\frac{c}{1-c}\right)^{-\gamma}\right] dxds \\
\le& C\int_0^t\int_0^1 x^{1-\alpha}\left[1-\exp\left(-\gamma\int_s^{+\infty}\|A(\cdot,\tau)\|_{L^\infty([0,1])}d\tau\right)\right] dxds \\
\le& C\int_0^t \left[1-\exp\left(-\gamma\int_s^{+\infty}\|A(\cdot,\tau)\|_{L^\infty([0,1])}d\tau\right)\right] ds \le C,
\end{aligned}
$$

and

$$
\begin{aligned}
B =& C\int_0^t\int_0^1 Q_\infty^{-2}|Q-Q_\infty|\left|\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right| dxds \\
=& C\int_0^t\int_0^1 \frac{Q_\infty^{-2}|Q-Q_\infty|}{\left|\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta} - \left(\frac{c}{1-c}Q\right)^{-\theta}\right|}\left|\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right| \\
&\qquad\qquad \times\left|\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta} - \left(\frac{c}{1-c}Q\right)^{-\theta}\right| dxds \\
\le& C\int_0^t\int_0^1 x^{\frac{\theta-\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{-\theta} - \left(\frac{c}{1-c}Q\right)^{-\theta}\right) dxds \\
\le& C.
\end{aligned}
$$

Choose $\theta = \alpha\gamma$, then (5.1) and (5.2) are immediately obtained from Gronwall inequality and Lemma 3.5. This completes the proof of Lemma 5.1. □

Corollary 5.1. *Assume the conditions in Theorem 2.2 hold, we have*

$$
\int_0^t\int_0^1 (1+s)u^2 dxds \le C. \tag{5.3}
$$

Proof. The result can be easily obtained by (3.11) and Lemma 5.1 and the details are omitted. □

Corollary 5.2. *Assume the conditions in Theorem 2.2 hold, we have*

$$
\int_0^t\int_0^1 (1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2 dxds \le C. \tag{5.4}
$$

Proof. Due to $(2.5)_3$, we have

$$
\begin{aligned}
&\int_0^t\int_0^1(1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right)^2dxds\\
=&\int_0^t\int_0^1(1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right)\left(-\int_0^x\left(\frac{1}{1-c}u\right)_tdy\right)dxds\\
&+\int_0^t\int_0^1(1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}\right)^\gamma\right)\left(\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x\right)dxds\\
:=&J_1+J_2.
\end{aligned}
$$

Next, we estimate terms on the right hand side of the above equality. Integrating by parts with respect to t,

$$
\begin{aligned}
J_1=&\int_0^1x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c_0}{1-c_0}Q_0\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\int_0^x\frac{1}{1-c_0}u_0dy\right)dx\\
&-(1+t)\int_0^1x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\int_0^x\frac{1}{1-c}udy\right)dx\\
&+\int_0^t\int_0^1x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\int_0^x\frac{1}{1-c}udy\right)dxds\\
&+\int_0^t\int_0^1(1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\gamma\left(\frac{c}{1-c}Q\right)^\gamma A-\gamma\rho_l\left(\frac{c}{1-c}\right)^\gamma Q^{\gamma+1}u_x\right)\left(\int_0^x\frac{1}{1-c}udy\right)dxds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
J_1\le&\int_0^1x^{\frac12-\frac{\beta+\alpha\gamma}{\gamma}+1}\left(\int_0^1\left(\frac{1}{1-c_0}u_0\right)^2dy\right)^{\frac12}dx\\
&+(1+t)\int_0^1x^{\frac12-\frac{\beta+\alpha\gamma}{\gamma}+1-2\alpha}(Q-Q_\infty)^2\left(\int_0^1\left(\frac{1}{1-c}u\right)^2dy\right)^{\frac12}dx\\
&+C\int_0^t\int_0^1\left(\int_0^x\left(\frac{1}{1-c}u\right)^2dy\right)dxds+\int_0^t\int_0^1x^{1-2\frac{\alpha\gamma+\beta}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dxds\\
&+C\int_0^t(1+s)\int_0^1\left(\int_0^x\left(\frac{1}{1-c}u\right)^2dy\right)dxds+\int_0^t\int_0^1(1+s)x^{1-2\frac{\beta+\alpha\gamma}{\gamma}}A^2\left(\frac{c}{1-c}Q\right)^{2\gamma}dxds\\
&+C\int_0^t\int_0^1(1+s)\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dxds\\
&+C\int_0^t\int_0^1(1+s)x^{1-2\frac{\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}\right)^{2\gamma-\beta}Q^{2\gamma-\beta+1}\left(\int_0^1\left(\frac{1}{1-c}u\right)^2dy\right)dxds\\
\le& C+C(1+t)\int_0^1x^{1-3\alpha}(Q-Q_\infty)^2dx+C\int_0^t\int_0^1u^2dxds\\
&+C\int_0^t\int_0^1(1+s)u^2dxds+\int_0^t(1+s)\|A(s)\|_{L^\infty([0,1])}ds\int_0^1x^{3-2\frac{\beta+\alpha\gamma}{\gamma}}dx
\end{aligned}
$$

$$+\int_0^t\int_0^1 x^{2-2\frac{\alpha\gamma+\beta}{\gamma}-\frac{\theta}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\left(\frac{c}{1-c}Q\right)^\theta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\theta\right)dxds$$
$$+C\int_0^t\int_0^1(1+s)x^{3-\alpha-\frac{3\beta}{\gamma}}\left(\int_0^1 u^2dy\right)dxds$$
$$\le C, \tag{5.5}$$

where we have used the assumption (A_2), Lemma 3.1, Lemma 3.3, Lemma 5.1, Corollary 5.1 and $\theta=2(\gamma-\alpha\gamma-\beta)>0$ in Lemma 3.5 (i).

By using Young inequality and Lemma 5.1, we have

$$J_2\le\frac14\int_0^t\int_0^1(1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dxds$$
$$+C\int_0^t\int_0^1(1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dxds$$
$$\le\frac14\int_0^t\int_0^1(1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2dxds+C. \tag{5.6}$$

Then from (5.5), (5.6), we complete the proof of Corollary 5.2. □

Lemma 5.2. *Assume the conditions in Theorem 2.2 hold, we have*

$$\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)^2dx\le\frac{C}{1+t},\quad\forall\ t\ge0, \tag{5.7}$$

and

$$\int_0^\infty\int_0^1(1+t)x^{4-\frac{3\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)^2dxdt\le C. \tag{5.8}$$

Proof. Multiplying (3.27) by $(1+t)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)$, then integrating the resulting equation on $[0,1]\times[0,t]$, we get

$$\frac12(1+t)\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dx$$
$$+\gamma\rho_l\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dxds$$
$$=\frac12\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c_0}{1-c_0}Q_0\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c_0}u_0\right)^2dx$$
$$+\frac12\int_0^t\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)^\beta_x-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x+\beta\rho_l\frac{1}{1-c}u\right)^2dxds$$

$$
\begin{aligned}
&+\beta\rho_l^2\gamma\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c}u\right)\\
&\qquad\qquad\qquad\times\left(\frac{1}{1-c}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}u\right)dxds\\
&+\beta\rho_l g\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c}u\right)\\
&\qquad\qquad\qquad\times\left(1-\frac{\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}}{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta}}\right)dxds\\
&+\beta\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)\left(\left(\frac{c}{1-c}Q\right)^\beta A\right)_x dxds\\
&=\sum_{i=1}^5 J_i.
\end{aligned}
\tag{5.9}
$$

Now we estimate J_1–J_5 as follows:

First, by the assumption (A_4), Lemma 3.6, Lemma 5.2 and Cauchy-Schwarz inequality, we have

$$
J_1\le\int_0^1 x^{\frac{3\gamma-3\beta-\alpha\gamma}{2\gamma}}x^{\frac{3\gamma-\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c_0}{1-c_0}Q_0\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c_0}u_0\right)^2dx\le C,
$$

$$
J_2\le\int_0^t\int_0^1 x^{\frac{\gamma-\beta-\alpha\gamma}{2\gamma}}x^{\frac{5\gamma-3\beta-\alpha\gamma}{2\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\beta\rho_l\frac{1}{1-c}u\right)^2dxds\le C,
$$

$$
\begin{aligned}
J_3&\le\frac18\gamma\rho_l\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)^2dxds\\
&\quad+C\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\frac{1}{1-c}\right)^2u^2dxds\\
&\le\frac18\gamma\rho_l\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)^2dxds\\
&\quad+C\int_0^t\int_0^1(1+s)\left(\frac{1}{1-c}\right)^2u^2dxds\\
&\le\frac18\gamma\rho_l\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)^2dxds+C,
\end{aligned}
$$

$$
\begin{aligned}
J_4&\le\frac18\gamma\rho_l\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)^2dxds\\
&\quad+C\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\beta-\gamma}\left(1-\frac{\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}}{\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\gamma-\beta}}\right)^2dxds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{8}\gamma\rho_l \int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\frac{c}{1-c}Q\right)^{\gamma-\beta} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \frac{\beta\rho_l u}{1-c}\right)^2 dxds \\
&\quad + C\int_0^t \int_0^1 (1+s)x^{-\frac{\beta+\alpha\gamma}{\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2 dxds \\
&\leq \frac{1}{8}\gamma\rho_l \int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\frac{c}{1-c}Q\right)^{\gamma-\beta} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \frac{\beta\rho_l u}{1-c}\right)^2 dxds + C.
\end{aligned}
$$

In the following, we devote ourselves to deal with J_5, which contains the function $A(x,t)$.

$$
\begin{aligned}
J_5 = \beta &\int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \beta\rho_l \frac{1}{1-c}u\right) \\
&\qquad \times \left(\left(\frac{c}{1-c}Q\right)^\beta_x A - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x A\right) dxds \\
&+ \beta \int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x + \beta\rho_l \frac{1}{1-c}u\right) \\
&\qquad \times \left(\frac{\beta}{\gamma}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{\beta-\gamma} gA + \left(\frac{c}{1-c}Q\right)^\beta A_x\right) dxds = J_5^{(1)} + J_5^{(2)},
\end{aligned}
$$

where

$$
\begin{aligned}
J_5^{(1)} &\leq C\int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)^2 dxds \\
&\quad + C\int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \frac{1}{1-c}u \left(\left(\frac{c}{1-c}Q\right)^\beta_x A - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x A\right) dxds \\
&\leq C\int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)^2 dxds \\
&\quad + \frac{1}{8}\gamma\rho_l \int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\frac{c}{1-c}Q\right)^{\gamma-\beta} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)^2 dxds \\
&\quad + C\int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\frac{c}{1-c}Q\right)^{\beta-\gamma} \left(\frac{1}{1-c}u\right)^2 A^2 dxds \\
&\leq C + C\int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)^2 dxds \\
&\quad + \frac{1}{8}\gamma\rho_l \int_0^t \int_0^1 (1+s)x^{3-\frac{2\beta+1}{\gamma}} \left(\frac{c}{1-c}Q\right)^{\gamma-\beta} \left(\left(\frac{c}{1-c}Q\right)^\beta_x - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta_x\right)^2 dxds,
\end{aligned}
$$

$$
\begin{aligned}
J_5^{(2)} \leq & \int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)^2 dxds \\
&+C\int_0^t \|A\|_{L^\infty([0,1])}(1+s)ds\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^{2\beta-2\gamma}dx \\
&+\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)^2 dxds \\
&+C\int_0^t \|A_x\|^2_{L^\infty([0,1])}(1+s)\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\beta-\gamma}\left(\frac{c}{1-c}Q\right)^{2\beta}dxds \\
\leq & C+C\int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1 x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta+\frac{\beta\rho_l u}{1-c}\right)^2 dxds \\
&+\frac{1}{8}\gamma\rho_l\int_0^t\int_0^1(1+s)x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\frac{c}{1-c}Q\right)^{\gamma-\beta}\left(\left(\frac{c}{1-c}Q\right)_x^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)_x^\beta\right)^2 dxds.
\end{aligned}
$$

Here we have used Corollary 5.1, (3.5), (2.16) and Cauchy inequality. Substituting J_1-J_5 into (5.9), we obtain (5.7) and (5.8) immediately by using Gronwall inequality. This completes the proof of Lemma 5.2. □

Lemma 5.3. *Assume the conditions in Theorem 2.2 hold, we have*

$$\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dx \leq \frac{C}{1+t}, \quad \forall\ t\geq 0, \tag{5.10}$$

and

$$\int_0^\infty\int_0^1(1+t)u_t^2dxdt\leq C. \tag{5.11}$$

Proof. Multiplying $(2.5)_3$ by $(1+t)u_t$, integrating the resulting equation over $[0,1]\times[0,t]$, integrating by parts, we get

$$
\begin{aligned}
&\int_0^t\int_0^1(1+t)\left(\frac{1}{1-c}\right)u_t^2dxds+\frac{1}{2}(1+t)\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dx \\
=&-\int_0^t\int_0^1(1+s)\left(\frac{c}{1-c}\right)Auu_tdxds+\frac{1}{2}\int_0^1\left(\frac{c_0}{1-c_0}\right)^\beta Q_0^{1+\beta}u_{0x}^2dx \\
&+\frac{1}{2}\int_0^t\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dxds-\frac{1}{2}(1+\beta)\rho_l\int_0^t\int_0^1(1+s)\left(\frac{c}{1-c}\right)^\beta Q^{2+\beta}u_x^3dxds \\
&+\frac{1}{2}\beta\int_0^t\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2(1+s)Adx \\
&+\int_0^t\int_0^1(1+s)\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)u_{tx}dxds \\
=&\sum_{i=1}^6 K_i.
\end{aligned}
\tag{5.12}
$$

It is easy to see $K_2+K_3+K_5\leq C$ from the assumptions (A_1), (A_4), Lemma 3.1 and Lemma 5.1. By using $\|Qu_x\|_{L^\infty}\leq C$ and Lemma 5.1, we have

$$K_4\leq C\|Qu_x\|_{L^\infty}\int_0^t\int_0^1(1+s)\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dxds\leq C. \tag{5.13}$$

By using Cauchy inequality and Corollary 5.1, we have

$$
\begin{aligned}
K_1 \leq & \frac{1}{2}\int_0^t\int_0^1 (1+s)\frac{1}{1-c}u_t^2 dxds + C\int_0^t\int_0^1(1+s)\frac{1}{1-c}c^2A^2u^2dxds \\
\leq & \frac{1}{2}\int_0^t\int_0^1 (1+s)\frac{1}{1-c}u_t^2 dxds + C.
\end{aligned}
$$

In order to complete the proof of Lemma 5.3, it suffices to estimate K_6 of the right-hand side in (5.12).

$$
\begin{aligned}
K_6 = & (1+t)\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right) u_x dx \\
& -\int_0^1 \left(\left(\frac{c_0}{1-c_0}Q_0\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right) u_{0x} dx \\
& -\int_0^t\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right) u_x dxds \\
& +\gamma\rho_l\int_0^t\int_0^1 (1+s)\left(\frac{c}{1-c}\right)^\gamma Q^{1+\gamma}u_x^2 dxds \\
& -\gamma\int_0^t\int_0^1 (1+s)\left(\frac{c}{1-c}Q\right)^\gamma A dxds \\
\leq & \frac{1}{4}(1+t)\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dx \\
& +C(1+t)\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2 \left(\frac{c}{1-c}\right)^{-\beta}Q^{-1-\beta}dx \\
& +\int_0^1 \left(\left(\frac{c_0}{1-c_0}Q_0\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2 dx + \int_0^1 u_{0x}^2 dx \\
& +\int_0^t\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2 \left(\frac{c}{1-c}\right)^{-\beta}Q^{-1-\beta}dxds \\
& +\int_0^t\int_0^1 \left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dxds + \int_0^t \|A\|_{L^\infty([0,1])}(1+s)\int_0^1\left(\frac{c}{1-c}Q\right)^\gamma dxds \\
& +\gamma\rho_l\int_0^t\int_0^1 \left(\frac{c}{1-c}Q\right)^{\gamma-\beta}(1+s)\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dxds \\
\leq & C + \frac{1}{4}(1+t)\int_0^1\left(\frac{c}{1-c}\right)^\beta Q^{1+\beta}u_x^2dx \\
& +(1+t)\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2 \left(\frac{c}{1-c}\right)^{-\beta}Q^{-1-\beta}dx \\
& +\int_0^t\int_0^1 \left(\left(\frac{c}{1-c}Q\right)^\gamma - \left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2 \left(\frac{c}{1-c}\right)^{-\beta}Q^{-1-\beta}dxds.
\end{aligned}
\tag{5.14}
$$

Here, we have used the assumption (A_2), (A_3), Lemma 3.1, Lemma 3.2, (2.16) and Lemma 5.1. The rest two terms on the right-hand side of (5.14) can be estimated as follows:

$$
\begin{aligned}
&(1+t)\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2\left(\frac{c}{1-c}\right)^{-\beta}Q^{-1-\beta}dx\\
&\le C(1+t)\int_0^1(Q-Q_\infty)^2x^{2-\frac{\alpha\gamma+\beta}{\gamma}-2\alpha}dx\\
&\le C(1+t)\int_0^1(Q-Q_\infty)^2x^{1-3\alpha}dx\\
&\le C.
\end{aligned}\tag{5.15}
$$

And if we take $\theta=\gamma-\beta-\alpha\gamma>0$ in Lemma 3.5 (i), we have

$$
\begin{aligned}
&\int_0^t\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)^2\left(\frac{c}{1-c}\right)^{-\beta}Q^{-1-\beta}dxds\\
=&\int_0^t\int_0^1\frac{\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma}{\left(\frac{c}{1-c}Q\right)^\theta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\theta}\left(\frac{c}{1-c}\right)^{-\beta}Q^{-1-\beta}\\
&\qquad\times\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\left(\frac{c}{1-c}Q\right)^\theta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\theta\right)dxds\\
\le& C\int_0^t\int_0^1\left(\left(\frac{c}{1-c}Q\right)^\gamma-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\gamma\right)\left(\left(\frac{c}{1-c}Q\right)^\theta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\theta\right)dxds\\
\le& C.
\end{aligned}\tag{5.16}
$$

□

<u>*Proof of Theorem 2.2.*</u>

Choosing a positive number m large enough, which is to be determined later, applying Hölder inequality and (2.16), we have by $\left(\frac{c}{1-c}Q\right)(0,t)=\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)(0)=0$,

$$
\begin{aligned}
0\le&\left|\left(\tfrac{c}{1-c}Q\right)^\beta(x,t)-\left(\tfrac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta(x)\right|^m\\
\le& m\int_0^x\left|\left(\frac{c}{1-c}Q\right)^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right|^{m-1}\left|\left(\left(\frac{c}{1-c}Q\right)^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right)_x\right|dy\\
\le& m\left(\int_0^1\left|\left(\frac{c}{1-c}Q\right)^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right|^{2m-2}x^{-3+\frac{2\beta+\alpha\gamma}{\gamma}}dx\right)^{\frac12}\\
&\times\left(\int_0^1x^{3-\frac{2\beta+\alpha\gamma}{\gamma}}\left(\left(\left(\frac{c}{1-c}Q\right)^\beta-\left(\frac{c_\infty}{1-c_\infty}Q_\infty\right)^\beta\right)_x\right)^2\right)^{\frac12}\\
\le& C(1+t)^{-\frac12}\left(\int_0^1x^{\frac{2\beta+2\alpha\gamma-4\gamma+\beta(2m-2)}{\gamma}}\left(x^{1-3\alpha}(Q-Q_\infty)^2\right)dx\right)^{\frac12}.
\end{aligned}\tag{5.17}
$$

Taking $m=\frac{2\gamma-\alpha\gamma}{\beta}$, then $2\beta+2\alpha\gamma-4\gamma+\beta(2m-2)=0$. It follows from (5.1) and (5.17) that (2.22) holds.

For the velocity function $u(x,t)$, (2.23) follows from (4.1) and (5.10) directly. This completes the proof of Theorem 2.2.

Acknowledgements: QQL and CJZ were supported by the National Natural Science Foundation of China # 11331005, the PhD specialized grant of the Ministry of Education of China #20100144110001, and the Special Fund for Basic Scientific Research of Central Colleges #CCNU12C01001.

References

[1] C.E. Brennen, Fundamentals of Multiphase Flow, Cambridge University Press, New York, 2005.

[2] J.M. Delhaye, M. Giot, M.L. Riethmuller, Thermohydraulics of Two-Phase Systems for Industrial Design and Nuclear Engineering, Von Karman Institute, McGrawCHill, New York, 1981.

[3] Q. Duan, On the dynamics of Navier-Stokes equations for a shallow water model, J. Differential Equations, 250(2011), 2687-2714.

[4] S. Evje, Global weak solutions for a compressible gas-liquid model with well-formation interaction, J. Differential Equations, 251(2011), 2352-2386.

[5] S. Evje, Weak solutions for a gas-liquid model relevant for describing gas-kick in oil wells, SIAM J. Math. Anal., 43(2011), 1887-1922.

[6] S. Evje, K.H. Karlsen, Global existence of weak solutions for a viscous two-phase model, J. Differential Equations, 245(2008), 2660-2703.

[7] S. Evje, K.H. Karlsen, Global weak solutions for a viscous liquid-gas model with singular pressure law, Commun. Pure Appl. Anal., 8(2009), 1867-1894.

[8] H.A. Friis, S. Evje, Global weak solutions for a gas-liquid model with external forces and general pressure law, SIAM J. Appl. Math., 71(2011), 409-442.

[9] H.A. Friis, S. Evje, Asymptotic behavior of a compressible two-phase model with well-formation interaction, submitted, 2012.

[10] S. Evje, T. Flåtten, On the wave structure of two-phase flow models, SIAM J. Appl. Math., 67(2006/07), 487-511.

[11] S. Evje, T. Flåtten, H.A. Friis, Global weak solutions for a viscous liquid-gas model with transition to single-phase gas flow and vacuum, Nonlinear Anal., 70(2009), 3864-3886.

[12] L. Fan, Q.Q. Liu, C.J. Zhu, Convergence rates to stationary solutions of a gas-liquid model with external forces, Nonlinearity, 25(2012), 2875-2901.

[13] Z.H. Guo, C.J. Zhu, Remarks on one-dimensional compressible Navier-Stokes equations with density-dependent viscosity and vacuum, Acta Math. Sin., 26(2010), 2015-2030.

[14] Z.H. Guo, C.J. Zhu, Global weak solutions and asymptotic behavior to 1D compressible Navier-Stokes equations with density-dependent viscosity and vacuum, J. Differential Equations, 248(2010), 2768-2799.

[15] Z.H. Guo, Q.S. Jiu, Z.P. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, SIAM J. Math. Anal., 39(2008), 1402-1427.

[16] T. Nagasawa, On the asymptotic behavior of the one-dimensional motion of the polytropic ideal gas with stress-free condition, Quart. Appl. Math., 46(1988), 665-679.

[17] D. Hoff, Construction of solutions for compressible, isentropic Navier-Stokes equations in one space dimension with nonsmooth initial data, Proc. Roy. Soc. Edinburgh, Sect. A, 103(1986), 301-315.

[18] Q.Q. Liu, C.J. Zhu, Asymptotic behavior of a viscous liquid-gas model with mass-dependent viscosity and vacuum, J. Differential Equations, 252(2012), 2492-2519.

[19] M. Okada, Free boundary value problems for the equation of one-dimensional motion of viscous gas, Japan J. Appl. Math., 6(1989), 161-177.

[20] I. Straškraba, A. Zlotnik, Global behavior of 1d-viscous compressible barotropic fluid with a free boundary and large data, J. Math. Fluid Mech., 5(2003), 119-143.

[21] L. Yao, C.J. Zhu, Free boundary value problem for a viscous two-phase model with mass-dependent viscosity, J. Differential Equations, 247(2009), 2705-2739.

[22] L. Yao, T. Zhang, C.J. Zhu, Existence and asymptotic behavior of global weak solutions to a 2D viscous liquid-gas two-phase flow model, SIAM J. Math. Anal., 42(2010) 1874-1897.

[23] L. Yao, C.J. Zhu, Existence and uniqueness of global weak solution to a two-phase flow model with vacuum, Math. Ann., 349(2011), 903-928.

[24] L. Yao, T. Zhang, C.J. Zhu, A blow-up criterion for a 2D viscous liquid-gas two-phase flow model, J. Differential Equations, 250(2011) 3362-3378.
[25] T. Yang, Z.A. Yao, C.J. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum, Comm. Partial Differential Equations, 26(2001), 965-981.
[26] T. Yang, H.J. Zhao, A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity, J. Differential Equations, 184(2002), 163-184.
[27] T. Yang, C.J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Math. Phys., 230(2002), 329-363.
[28] T. Zhang, D.Y. Fang, Global behavior of compressible Navier-Stokes equations with a degenerate viscosity coefficient, Arch. Ration. Mech. Anal., 182(2006) 223-253.
[29] C.J. Zhu, Asymptotic behavior of compressible Navier-Stokes equations with density-dependent viscosity and vacumm, Comm. Math. Phys., 293(2010), 279-299.
[30] C.J. Zhu, R.Z. Zi, Asymptotic behavior of solutions to 1D compressible Navier-Stokes equations with gravity and vacuum, Discrete Contin. Dyn. Syst., 30(2011), 1263-1283.

(SE) University of Stavanger, NO-4068 Stavanger, Norway
E-mail address: steinar.evje@uis.no

(QQL) The Hubei Key Laboratory of Mathematical Physics, School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P. R. China
E-mail address: shuxueliuqingqing@126.com

(CJZ) The Hubei Key Laboratory of Mathematical Physics, School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P. R. China
E-mail address: cjzhu@mail.ccnu.edu.cn