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WELL-POSEDNESS OF A COMPRESSIBLE GAS-LIQUID MODEL FOR DEEPWATER OIL WELL OPERATIONS*

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Abstract The main purpose of this paper is two-fold: (i) to generalize an existence result for a compressible gas-liquid model with a friction term recently published by Friis and Evje [SIAM J. Appl. Math., 71 (2011), pp. 2014–2047]; (ii) to derive a uniqueness result for the same model. A main ingredient in the existence part is the observation that we can consider weaker assumptions on the initial liquid and gas mass, and still obtain an existence result. Compared to the above mentioned work, we rely on a more refined application of the estimates provided by the basic energy estimate. Concerning the uniqueness result, we borrow ideas from Fang and Zhang [Nonlinear Anal. TMA, 58 (2004), pp. 719–731] and derive a stability result under appropriate constraints on parameters that determine rate of decay toward zero at the boundary for gas and liquid masses, and growth rate of masses associated with the friction term and viscous coefficient.

Key words two-phase flow, well model; gas-kick; weak solutions; Lagrangian coordinates; free boundary problem; friction term; uniqueness

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1 Introduction

This work is devoted to a study of a transient gas-liquid two-phase model which, in Lagrangian variables, takes the following form:

$$\begin{aligned} \partial_t n + (n\zeta)\partial_x u &= 0, \\ \partial_t \zeta + \zeta^2 \partial_x u &= 0, \\ \partial_t u + \partial_x p(n,\zeta) &= -f\zeta^\beta u|u| + \partial_x (E(n,\zeta)\partial_x u), \quad x \in (0,1), \end{aligned} \tag{1.1}$$

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with constants $f, \beta > 0$. Here n is the gas mass, ζ the total mass (sum of gas and liquid mass), whereas u is the common fluid velocity. The pressure law, when liquid is assumed to be incompressible (ρ_l=const) and gas is treated as an ideal gas, takes the form

$$p(n,\zeta) = \left(\frac{n}{\rho_l - [\zeta - n]}\right)^{\gamma}, \qquad \gamma > 1. \tag{1.2}$$

The first term on the right hand side of the momentum equation represents wall friction where the parameter $\beta > 0$ describes the mass decay rate toward zero, whereas the second term takes into account other viscous effects and is characterized by the coefficient

$$E(n,\zeta) := \left(\frac{\zeta}{(\rho_l - [\zeta - n])}\right)^{\theta+1}, \qquad 0 < \theta < 1/2. \tag{1.3}$$

Moreover, boundary conditions are given by

$$n(0,t) = \zeta(0,t) = 0, \qquad n(1,t) = \zeta(1,t) = 0, \tag{1.4}$$

whereas initial data are

$$n(x,0) = n_0(x), \quad \zeta(x,0) = \zeta_0(x), \quad u(x,0) = u_0(x), \qquad x \in (0,1). \tag{1.5}$$

This model problem represents a natural continuation of the work [15] where an existence result was established for a similar model with inclusion of external forces like gravity and friction. In turn, this work builds upon the works [9, 10], see also [30, 31] for related interesting results.

In the recent work [16] we considered the model problem (1.1)–(1.5) for the case when the gas and liquid mass vanish at the boundary. A main concern in that work was inclusion and analysis of effects related to wall friction. The friction term is important for realistic predictions of the pressure profile along wellbore, which is crucial for a good understanding of mechanisms for safe handling of a gas-kick.

In particular, an existence result was obtained under appropriate assumptions on the parameters γ, θ, and β appearing in (1.1), (1.2), and (1.3). The heart of the matter in the analysis is the use of an appropriate variable transformation which allows writing the two-phase model in a form which naturally opens up for exploiting single-phase techniques [18, 19, 21–23, 25, 26, 29, 32–34]. It turns out that we naturally can reformulate the initial boundary value (IBV) problem (1.1)–(1.5) described in terms of the variables (n, ζ, u) into a corresponding IBV problem described in terms of the variables (c, Q, u) where $c = n/\zeta$ and $Q(c,\zeta) = \zeta/(\rho_l - [1-c]\zeta)$. The model then takes the form

$$\begin{aligned} \partial_t c &= 0 \\ \partial_t Q + \rho_l Q^2 u_x &= 0 \\ \partial_t u + \partial_x p(cQ) &= -h(c,Q)u|u| + \partial_x(E(Q)\partial_x u), \end{aligned}$$

with

$$p(cQ) = (cQ)^{\gamma}, \qquad h(c,Q) = f\rho_l^{\beta}\left(\frac{Q}{1+(1-c)Q}\right)^{\beta}, \qquad E(Q) = Q^{\theta+1}.$$

This reformulated version allows us to explore the role played by the frictional term. A main observation was that we could derive the necessary estimates by relying on assumption (2.15) which relates the β-parameter of the friction term to the θ-parameter of the viscosity term, and the parameter α, which characterizes the decay rate of initial masses toward zero at the boundaries as follows:

$$C_1\phi(x)^\alpha \le n_0(x) \le C_2\phi(x)^\alpha, \qquad D_1\phi(x)^\alpha \le m_0(x) \le D_2\phi(x)^\alpha, \qquad \alpha > 0,$$

for positive constants C_1, C_2, D_1, and D_2. As a consequence, the fraction $\frac{m_0}{n_0}$ has a positive lower and upper limit. This was directly exploited to get the estimate

$$\sup_{x\in[0,1]} c_0(x) < 1, \qquad c_0 = \frac{n_0}{\zeta_0} = \frac{n_0}{n_0 + m_0}. \tag{1.6}$$

Moreover, this upper limit was directly exploited to control the coefficient $h(c,Q)$ (written in terms of the variables (c,Q,u)) of the frictional term such that we obtained the following estimate

$$h(c,Q) = f\rho_l^\beta\left(\frac{1}{\frac{1}{Q} + (1-c)}\right)^\beta \le f\rho_l^\beta\left(\frac{1}{1-c}\right)^\beta \le C, \qquad Q \ge 0, \qquad c = c_0.$$

However, here we make use of the estimate (1.6) on c_0. Hence, the analysis of [16] depends strictly on the fact that the initial gas and liquid masses decay to zero at the boundaries at the same rate in order to obtain sufficient control of the frictional term. The novelty of the analysis of this work, as compared to [16], is as follows:

• We consider the more general assumption as described by (2.6) concerning the decay rates at the boundary

$$C_1\phi(x)^\alpha \le n_0(x) \le C_2\phi(x)^\alpha, \qquad D_1\phi(x)^\kappa \le m_0(x) \le D_2\phi(x)^\kappa, \qquad \kappa \ge \alpha > 0.$$

I.e., the liquid mass can decay faster to zero as compared to the gas mass. As a consequence, we can obtain an estimate of the coefficient $h(c,Q)$ of the friction term in the following form

$$h(c,Q) = f\rho_l^\beta\left(\frac{1}{\frac{1}{Q} + (1-c)}\right)^\beta \le f\rho_l^\beta\left(\frac{1}{1-c}\right)^\beta \le C\phi(x)^{(\alpha-\kappa)\beta}, \qquad Q \ge 0.$$

Under appropriate assumptions on the internal relation between γ, θ, α, κ, we can then derive pointwise control (upper and lower bounds) on Q similar to what was done in [16]. This gives the basis for deriving the existence result stated in Theorem 2.1.

• We provide a uniqueness result for the model problem (1.1)–(1.5). The friction term produces new terms that must be properly handled compared to previous analysis for single-phase models. This requires more restrictions on different parameters, e.g., the β parameter. See Theorem 2.2 for details.

The rest of the paper is structured as follows: In Section 2 we first give the assumptions on initial data and other important parameters like γ, β, and θ. We then give precise statements of the main results of this paper, i.e., existence of weak solutions and a uniqueness result within a suitable subclass of the weak solutions. Section 3 contains the estimates, ranging from basic energy estimate to pointwise upper and lower limits of masses n, ζ, and velocity u, as well as various higher order regularity estimates. As a result, the existence proof follows. Finally, Section 4 presents the proof of the uniqueness result.

2 Main Results

In this section we give the assumptions and present the main results of the paper. Note that the various assumptions we impose are essentially similar to those used in [16]. The main difference is assumption (2.6) which is more general than the one in [16] which requires that $\alpha = \kappa$. The estimate given in (2.10) below is crucial for obtaining an estimate of the rate of growth of the friction coefficient h as explained in the beginning of Section 3.2. We also need the assumption (2.16) on the new parameter κ.

Before we specify the assumptions, we introduce some basic notation. We let $L^p(K,B)$ with norm $\|\cdot\|_{L^p(K,B)}$ denote the space of all strongly measurable, pth-power integrable functions from K to B where K typically is subset of $\mathbb{R}$ and B is a Banach space. We also use D_T to represent the domain $D_T = [0,1]\times[0,T]$.

Assumptions In this paper we use a weight function $\phi(x)$, which is assumed to fulfill

$$0 < \phi(x) < 1, \quad \text{for} \quad 0 < x < 1, \quad \phi(0) = \phi(1) = 0, \tag{2.1}$$

$$\phi'(x) \in L^\infty(I), \tag{2.2}$$

$$(x(1-x)) \le C\phi(x). \tag{2.3}$$

From (2.3), it follows that $\phi(x)^a \in L^1([0,1])$ for every $a > -1$. Furthermore, the above model is subject to the following assumptions:

$$0 < \theta < \frac{1}{2}, \tag{2.4}$$

$$\gamma > 1. \tag{2.5}$$

For the initial masses n_0, m_0 it is assumed that there are constants $C_1, C_2, D_1, D_2 > 0$ and parameters $\alpha, \kappa > 0$, which are characterized more precisely below, such that

$$C_1\phi(x)^\alpha \le n_0(x) \le C_2\phi(x)^\alpha, \qquad D_1\phi(x)^\kappa \le m_0(x) \le D_2\phi(x)^\kappa, \qquad \kappa \ge \alpha, \tag{2.6}$$

where $D_2 < \rho_l$. Consequently, we have that

$$\frac{D_1}{C_2}\phi(x)^{\kappa-\alpha} \le \frac{m_0}{n_0}(x) \le \frac{D_2}{C_1}\phi(x)^{\kappa-\alpha}, \qquad \kappa \ge \alpha.$$

For $c_0 = \frac{n_0}{n_0+m_0} = \frac{1}{1+\frac{m_0}{n_0}}$ it follows that

$$\sup_{x\in[0,1]} c_0(x) \le 1, \qquad \inf_{x\in[0,1]} c_0(x) > 0, \qquad (c_0)_x \in L^\infty([0,1]). \tag{2.7}$$

Moreover, it follows that

$$[n_0 + m_0](x) \le (C_2 + D_2)\phi(x)^\alpha, \tag{2.8}$$

since $\phi(x)^\kappa \le \phi(x)^\alpha$, and

$$\left(\frac{1}{1+\frac{D_2}{C_1}}\right)\frac{D_1}{C_2}\phi(x)^{\kappa-\alpha} \le [1-c_0(x)] := \frac{\frac{m_0}{n_0}(x)}{1+\frac{m_0}{n_0}(x)} \le \frac{m_0}{n_0}(x) \le \frac{D_2}{C_1}\phi(x)^{\kappa-\alpha}, \tag{2.9}$$

which implies that

$$\frac{1}{1-c_0(x)} \le A^{-1}\phi(x)^{\alpha-\kappa}, \qquad \kappa \ge \alpha, \qquad A = \left(\frac{1}{1+\frac{D_2}{C_1}}\right)\frac{D_1}{C_2}. \tag{2.10}$$

Concerning the initial fluid velocity u_0 we assume that

$$u_0(x) \in L^\infty([0,1]). \tag{2.11}$$

For $Q_0 = \frac{n_0+m_0}{\rho_l - m_0}$ we assume that

$$(Q_0^{1+\theta}u_{0x}(x))_x \in L^{2n}([0,1]), \qquad n \in \mathbb{N}. \tag{2.12}$$

Now, let $\alpha > 0$ introduced in (2.6) satisfy the following relation

$$\frac{19}{20} + \frac{1}{10}\theta \le \alpha \le \frac{1}{2\theta}, \tag{2.13}$$

and let $\nu > 0$ be defined by

$$\nu = \left(\frac{1}{2} - \theta\right)\left(1 + \frac{\theta}{10}\right). \tag{2.14}$$

The following restriction is then assumed for β

$$\beta > \max\left(\frac{\nu}{\alpha} + \theta, \frac{1}{2} + \frac{\theta}{2}\right) > 0, \tag{2.15}$$

whereas the following restriction is assumed for κ

$$2\beta(\alpha - \kappa) > -1, \qquad \kappa \ge \alpha. \tag{2.16}$$

Let $k_1 > 0$ satisfy

$$2\nu < k_1 < \min\left((2\gamma - 3\theta + 1)\alpha, \frac{60(1-2\theta)}{11(1+3\theta)} - 2\nu, \frac{40(1-2\theta)}{11(1+\theta)} - 2\nu\right), \tag{2.17}$$

and, moreover

$$k_1 < \begin{cases} 1 + (1-3\theta)\alpha & \text{if } 0 < \theta < \dfrac{1}{3}, \\ \dfrac{20(1-2\theta)}{9-7\theta} + \dfrac{22(1-3\theta)}{9-7\theta}\nu & \text{if } \dfrac{1}{3} \le \theta < \dfrac{1}{2}. \end{cases} \tag{2.18}$$

The following control for $Q_0 = \frac{n_0+m_0}{\rho_l - m_0}$ is then required (the first one is just a consequence of (2.8)):

$$0 \le Q_0(x) \le C\phi^\alpha(x), \tag{2.19}$$

$$\phi^\nu(x)(Q_0^\theta(x))_x \in L^2([0,1]), \tag{2.20}$$

$$\phi^{k_1}(x)Q_0^{2\theta-2}(x) \in L^1([0,1]), \tag{2.21}$$

and

$$(Q_0^\gamma(x))_x \in L^{2n}([0,1]), \qquad n \in \mathbb{N}. \tag{2.22}$$

Now we can state the global existence result.

Theorem 2.1 (Main Result) Given the assumptions (2.4)–(2.22), the initial-boundary problem (1.1)–(1.5) possesses a global weak solution (n, ζ, u) in the sense that for any $T > 0$,

(A) we have the following regularity:

$$\begin{aligned} &n,\zeta,u\in L^\infty([0,1]\times[0,T])\cap C^1([0,T];L^2([0,1])),\\ &E(n,\zeta)u_x\in L^\infty([0,1]\times[0,T])\cap C^{\frac12}([0,T];L^2([0,1])). \end{aligned}$$

In particular for $k_1>2\nu$ and $k_2=\nu+\frac{k_1}{2}$, the following pointwise estimates holds for $\mu>0$:

$$\begin{aligned} \Big(\inf_{x\in[0,1]}c_0\Big)\frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}}\le n(x,t)\le \min\Big\{\rho_l C(T)\phi(x)^\alpha,\frac{\rho_l-\mu}{1-\sup_{[0,1]}c}\Big\},\\ \frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}}\le \zeta(x,t)\le \min\Big\{\rho_l C(T)\phi(x)^\alpha,\frac{\rho_l-\mu}{1-\sup_{[0,1]}c}\Big\}, \end{aligned}$$

$\forall(x,t)\in[0,1]\times[0,T]$, where the positive constant μ only depends on time T and the regularity of the initial data as stated in the assumptions.

(B) Moreover, the following equations hold,

$$\begin{aligned} &\int_0^\infty\int_0^1\Big[n\varphi_t-n\zeta u_x\varphi\Big]\mathrm{d}x\mathrm{d}t+\int_0^1 n_0(x)\varphi(x,0)\mathrm{d}x=0,\\ &\int_0^\infty\int_0^1\Big[\zeta\psi_t-\zeta^2u_x\psi\Big]\mathrm{d}x\mathrm{d}t+\int_0^1 \zeta_0(x)\psi(x,0)\mathrm{d}x=0,\\ &\int_0^\infty\int_0^1\Big[u\omega_t+\Big(p(n,\zeta)-E(n,\zeta)u_x\Big)\omega_x-f\zeta^\beta u|u|\omega\Big]\mathrm{d}x\mathrm{d}t+\int_0^1 u_0(x)\omega(x,0)\mathrm{d}x=0, \end{aligned}\tag{2.23}$$

for any test function $\varphi(x,t),\psi(x,t),\omega(x,t)\in C_0^\infty(D)$, with $D:=\{(x,t)\,|\,0\le x\le 1,\ t\ge 0\}$.

The proof of Theorem 2.1 is based on a series of priori estimates for approximate solutions of (1.1)–(1.5) and a corresponding limit procedure.

Moreover, we also obtain the following uniqueness result:

Theorem 2.2 (Uniqueness Result) Under the assumptions (2.4)–(2.22), and for $k_2<\frac{10}{11}(1-2\theta)\frac{\alpha}{\theta}$, $k_1<\frac{15(1-2\theta)}{11\theta}-2\nu$, and $\beta\ge1$, the initial-boundary problem (1.1)–(1.5) possesses a unique weak solution.

3 A Priori Estimates

In order to obtain the necessary estimates it is convenient to introduce a shift of variables as follows:

3.1 Transformed models

We introduce the variable

$$c=\frac{n}{\zeta},\tag{3.1}$$

and see from the first two equations of (1.1) that

$$\partial_t c=\frac{1}{\zeta}n_t-\frac{n}{\zeta^2}\zeta_t=-\frac{n\zeta}{\zeta}u_x+\frac{n\zeta^2}{\zeta^2}u_x=0.$$

Consequently, the model (1.1)–(1.5) then can be written in terms of the variables (c,ζ,u) in the form

$$\begin{aligned} &\partial_t c=0,\\ &\partial_t\zeta+\zeta^2\partial_x u=0,\\ &\partial_t u+\partial_x p(c,\zeta)=-f\zeta^\beta u|u|+\partial_x(E(c,\zeta)\partial_x u),\quad x\in(0,1), \end{aligned}\tag{3.2}$$

with

$$p(c,\zeta)=\left(\frac{c\zeta}{\rho_l-[1-c]\zeta}\right)^{\gamma} \tag{3.3}$$

and

$$E(c,\zeta)=\frac{\zeta^{\theta+1}}{(\rho_l-[1-c]\zeta)^{\theta+1}}, \qquad 0<\theta<1/2. \tag{3.4}$$

Moreover, boundary conditions are given by

$$\begin{aligned} &\zeta(0,t)=0, && \zeta(1,t)=0, \\ &c(0,t)=c_0(0), && c(1,t)=c_0(1), \qquad t\geq 0, \end{aligned} \tag{3.5}$$

whereas initial data are

$$c(x,0)=c_0(x), \quad \zeta(x,0)=\zeta_0(x), \quad u(x,0)=u_0(x), \qquad x\in(0,1). \tag{3.6}$$

We introduce the quantity $Q(c,\zeta)=\frac{\zeta}{\rho_l-(1-c)\zeta}$ and deduce a reformulated model in terms of the variables (c,Q,u). That is, we introduce the variable

$$Q(c,\zeta)=\frac{\zeta}{\rho_l-(1-c)\zeta}, \qquad \text{(which implies that } \ \zeta=\rho_l\frac{Q}{1+(1-c)Q}\text{)}, \tag{3.7}$$

implicitly assuming $\zeta\geq 0$ and $\zeta<\frac{\rho_l}{1-c}$, and observe that

$$\begin{aligned} Q(c,\zeta)_t &= \left(\frac{\zeta}{\rho_l-(1-c)\zeta}\right)_t=\left(\frac{1}{\rho_l-(1-c)\zeta}+\frac{(1-c)\zeta}{(\rho_l-(1-c)\zeta)^2}\right)\zeta_t \\ &= \frac{\rho_l}{(\rho_l-(1-c)\zeta)^2}\zeta_t=-\rho_l\frac{\zeta^2}{(\rho_l-(1-c)\zeta)^2}u_x=-\rho_l Q(c,\zeta)^2u_x, \end{aligned}$$

in view of the second equation of (3.2). Consequently, we rewrite the model (3.2) in the form

$$\begin{aligned} &\partial_t c=0, \\ &\partial_t Q+\rho_l Q^2 u_x=0, \\ &\partial_t u+\partial_x p(cQ)=-h(c,Q)u|u|+\partial_x(E(Q)\partial_x u), \quad x\in(0,1), \end{aligned} \tag{3.8}$$

with

$$p(cQ)=(cQ)^{\gamma}, \tag{3.9}$$

and

$$h(c,Q)=f\rho_l^{\beta}\left(\frac{Q}{1+(1-c)Q}\right)^{\beta}, \tag{3.10}$$

and

$$E(Q)=Q^{\theta+1}, \qquad 0<\theta<1/2. \tag{3.11}$$

This model is then subject to the boundary conditions

$$\begin{aligned} &Q(0,t)=0, && Q(1,t)=0, \\ &c(0,t)=c_0(0), && c(1,t)=c_0(1), \qquad t\geq 0, \end{aligned} \tag{3.12}$$

In addition, we have the corresponding initial data

$$c(x,0)=c_0(x), \quad Q(x,0)=\frac{\zeta_0(x)}{\rho_l-(1-c_0(x))\zeta_0(x)}, \quad u(x,0)=u_0(x), \quad x\in(0,1). \tag{3.13}$$

In particular, the first equation of (3.8) gives that

$$c(x,t)=c_0(x),\qquad t>0. \tag{3.14}$$

3.2 A priori estimates

We are now ready to establish some important estimates. We let C and $C(T)$ denote a generic positive constant depending only on the initial data and the given time T, respectively. We also note that a constant C can change from one line to another in a sequence of calculations. In particular, we note from (3.10) and the estimate (2.10) that for $\beta>0$

$$h(c,Q)=f\rho_l^\beta\left(\frac{1}{\frac{1}{Q}+(1-c)}\right)^\beta\le f\rho_l^\beta\left(\frac{1}{1-c}\right)^\beta\le C\phi(x)^{(\alpha-\kappa)\beta},\quad Q\ge 0. \tag{3.15}$$

This estimate is weaker than the one obtained in [16]. In particular, for $\kappa>\alpha$ it gives a rate how fast $h(c,Q)$ will blow up at the boundary. Nevertheless, we will see that we can bound Q by upper and lower limits as in [16]. This is achieved by a more refined use of the energy estimate that now follows.

Lemma 3.1 (Energy estimate) Under the assumptions of Theorem 2.1 we have the basic energy estimate

$$\begin{aligned}&\int_0^1\left(\frac12u^2+\frac{c^\gamma}{\gamma-1}Q^{\gamma-1}\right)\mathrm{d}x+\int_0^t\int_0^1Q^{1+\theta}u_x^2\mathrm{d}x\mathrm{d}s+\int_0^t\int_0^1h(c,Q)u^2|u|\mathrm{d}x\mathrm{d}s\\&=\int_0^1\left(\frac12u_0^2+\frac{c_0^\gamma}{\gamma-1}Q_0^{\gamma-1}\right)\mathrm{d}x\le C,\qquad\forall t\in[0,T].\end{aligned} \tag{3.16}$$

Proof Start by summing equation (3.8)(b) multiplied by $\frac{c^\gamma Q^\gamma}{\rho_lQ^2}$ with equation (3.8)(c) multiplied by u to obtain

$$\frac{c^\gamma Q^\gamma Q_t}{\rho_lQ^2}+c^\gamma Q^\gamma u_x+uu_t+u(c^\gamma Q^\gamma)_x=u(Q^{1+\theta}u_x)_x-h(c,Q)u^2|u|. \tag{3.17}$$

Then rewrite equation (3.17) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac12u^2+\int_0^Q\frac{c^\gamma\xi^\gamma}{\rho_l\xi^2}d\xi\right)+(c^\gamma Q^\gamma u)_x=u(Q^{1+\theta}u_x)_x-h(c,Q)u^2|u|, \tag{3.18}$$

and integrate it over $[0,1]\times[0,t]$ to yield

$$\begin{aligned}&\int_0^1\left(\frac12u^2+\frac{c^\gamma}{\gamma-1}\frac{Q^{\gamma-1}}{\rho_l}\right)\mathrm{d}x+\int_0^t\int_0^1Q^{1+\theta}u_x^2\mathrm{d}x\mathrm{d}s\\&=\int_0^1\left(\frac12u_0^2+\frac{c_0^\gamma}{\gamma-1}\frac{Q_0^{\gamma-1}}{\rho_l}\right)\mathrm{d}x+\int_0^t(Q^{1+\theta}uu_x)\Big|_{x=0}^{x=1}\mathrm{d}s\\&\quad-\int_0^t(c^\gamma Q^\gamma u)\Big|_{x=0}^{x=1}\mathrm{d}s-\int_0^t\int_0^1h(c,Q)u^2|u|\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.19}$$

Now invoking the boundary conditions (3.12) and the assumptions on the initial data we arrive at the conclusion (3.16). □

Now, we derive a pointwise upper bound on Q. We first present an upper bound which does not depend on the weighting function $\phi(x)$. This estimate is obtained by making use of the estimate (3.15) combined with the control of the term $\int\int h(c,Q)u^2|u|\mathrm{d}x\mathrm{d}s$ provided by (3.16), see (3.27) below. Note that this step is different from what was done in [16]. Then, in Corollary 3.2 we present a more refined upper bound by making use of the higher order regularity of u as given by Lemma 3.3.

Lemma 3.2 Under the assumptions of Theorem 2.1 we have the pointwise upper bound

$$Q(x,t) \le C(T), \qquad \forall (x,t) \in [0,1]\times[0,T]. \tag{3.20}$$

Proof Multiplying equation (3.8)(b) by $\theta Q^{\theta-1}$, we observe that

$$(Q^\theta)_t = -\rho_l \theta Q^{1+\theta} u_x. \tag{3.21}$$

We then integrate equation (3.21) over $[0,t]$ and, moreover, equation (3.8)(c) over $[0,x]$ (or alternatively over $[x,1]$), which gives

$$Q^\theta(x,t) = Q_0^\theta(x) - \rho_l\theta \int_0^t (Q^{1+\theta}u_x)(x,s)\mathrm{d}s \tag{3.22}$$

and

$$\begin{aligned} Q^{1+\theta}u_x &= (cQ)^\gamma + \int_0^x u_t \mathrm{d}y + \int_0^x h(c,Q)u|u|\mathrm{d}y \\ &= (cQ)^\gamma - \int_x^1 u_t \mathrm{d}y - \int_x^1 h(c,Q)u|u|\mathrm{d}y. \end{aligned} \tag{3.23}$$

Putting $x=1$ in this last equation, using the boundary conditions, and integrating in time over $[0,t]$ reveals that

$$\begin{aligned} &\int_0^x (u_0 - u(y,t))\mathrm{d}y - \int_0^t\int_0^x h(c,Q)u|u|\mathrm{d}y \\ &= -\int_x^1 (u_0 - u(y,t))\mathrm{d}y + \int_0^t\int_x^1 h(c,Q)u|u|\mathrm{d}y, \end{aligned} \tag{3.24}$$

a fact which will be used in the following. We further substitute equation (3.23) into equation (3.22), and exploit the boundary conditions such that

$$\begin{aligned} &Q^\theta(x,t) + \rho_l\theta\int_0^t (cQ)^\gamma(x,s)\mathrm{d}s \\ &= Q_0^\theta(x) + \rho_l\theta\Big(\int_0^x u_0(y)\mathrm{d}y - \int_0^x u(y,t)\mathrm{d}y\Big) - \rho_l\theta\int_0^t\int_0^x h(c,Q)u|u|\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.25}$$

We can then estimate $Q^\theta(x,t)$ as follows

$$Q^\theta(x,t) \le Q_0^\theta + C\int_0^x |u_0(y)|\mathrm{d}y + C\int_0^x |u(y,t)|\mathrm{d}y + C\int_0^t\int_0^x h(c,Q)u^2\mathrm{d}y\mathrm{d}s \tag{3.26}$$

for $0 < x < 1$, $0 < t \le T$. Using the estimate (3.15) and Lemma 3.1, application of Cauchy inequality allows us to estimate the friction term as follows

$$\int_0^t\int_0^1 h(c,Q)u^2\mathrm{d}x\mathrm{d}s \le \frac{1}{2}\int_0^t\int_0^1 h(c,Q)|u|\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t\int_0^1 h(c,Q)u^2|u|\mathrm{d}x\mathrm{d}s$$

$$\begin{aligned}
&\le C + \frac{1}{4}\int_0^t\int_0^1 h(c,Q)^2\mathrm{d}x\mathrm{d}s + \frac{1}{4}\int_0^t\int_0^1 |u|^2\mathrm{d}x\mathrm{d}s \\
&\le C(T) + \int_0^t\int_0^1 h(c,Q)^2\mathrm{d}x\mathrm{d}s \\
&\le C(T) + C\int_0^t\int_0^1 \phi(x)^{2\beta(\alpha-\kappa)}\mathrm{d}x\mathrm{d}s \le C(T)
\end{aligned} \tag{3.27}$$

subject to the condition that $2\beta(\alpha-\kappa) > -1$, in accordance with assumption (2.16). Again, using assumption (2.11), Lemma 3.1 and the Hölder inequality, we find that (3.26), in view of (3.27), can be estimated as follows

$$Q^\theta(x,t) \le Q_0^\theta + Cx + C\bigg(\int_0^1 u^2(y,t)\mathrm{d}y\bigg)^{\frac{1}{2}} x^{\frac{1}{2}} + C(T) \le Q_0^\theta + Cx^{\frac{1}{2}} + C(T), \tag{3.28}$$

where we have used that $x \le x^{1/2}$ for $x \in [0,1]$. However, using equation (3.24) in equation (3.25) we can similarly deduce that

$$Q^\theta(x,t) \le Q_0^\theta + C(1-x)^{\frac{1}{2}} + C(T). \tag{3.29}$$

Finally, combining (3.28) and (3.29) and exploiting the fact that $\min(x, 1-x) \le 2x(1-x)$ (for $0 < x < 1$) lead us to the following estimate

$$Q^\theta(x,t) \le Q_0^\theta + C(x(1-x))^{\frac{1}{2}} + C(T) \le C\phi(x)^{\alpha\theta} + C(x(1-x))^{\frac{1}{2}} + C(T), \tag{3.30}$$

where we use assumption (2.19) on the initial data Q_0. Clearly, we can conclude that the estimate (3.20) holds. □

Now, we can obtain a stronger control on the friction term $h(c,Q)$ as expressed by the following corollary.

Corollary 3.1 Under the assumptions of Theorem 2.1 we have the pointwise upper bound

$$h(c,Q) \le C(T), \qquad \forall (x,t) \in [0,1]\times[0,T]. \tag{3.31}$$

Proof It follows directly from (3.10), using the newly obtained result of Lemma 3.2, that

$$h(c,Q) \le f\rho_l^\beta Q^\beta \le C(T).$$ □

As a consequence of estimate (3.31) we are back in the same setting as in [16]. Consequently, the following lemma can be proved.

Lemma 3.3 Under the assumptions of Theorem 2.1 we have the following higher order estimate for any integer m

$$\int_0^1 u^{2m}\mathrm{d}x + m(2m-1)\int_0^t\int_0^1 u^{2m-2}Q^{1+\theta}u_x^2\mathrm{d}x\mathrm{d}s + 2m\int_0^t\int_0^1 h(c,Q)u^{2m}|u|\mathrm{d}x\mathrm{d}s \le C(T). \tag{3.32}$$

We omit the proof of Lemma 3.3 for brevity. It can be proved using similar arguments as in [15]. A key step is that we make use of the pointwise upper bound of Q given by (3.20).

However, equipped with the higher order control on u as given by Lemma 3.3, we can derive a more refined upper bound for Q that depends on $\phi(x)$.

Corollary 3.2 Under the assumptions of Theorem 2.1 we have the pointwise upper bound

$$Q(x,t) \le C(T)\phi^{\alpha}(x), \qquad \forall (x,t) \in [0,1]\times[0,T]. \tag{3.33}$$

Proof We only have to revisit the last term of (3.26), which is the friction related term. We can estimate this term as follows, by referring to Corollary 3.1 and Lemma 3.3 and using the Hölder inequality

$$\int_0^t \int_0^x h(c,Q)u^2 \mathrm{d}y\mathrm{d}s \le C(T)x^{\frac12}\int_0^t \left(\int_0^1 u^4\mathrm{d}x\right)^{\frac12}\mathrm{d}s \le C(T)x^{\frac12}.$$

Following the same arguments as used in Lemma 3.2, we conclude that (3.30) is refined to

$$Q^{\theta}(x,t) \le C\phi(x)^{\alpha\theta} + C(T)(x(1-x))^{\frac12}. \tag{3.34}$$

But, since $0<\alpha\le \frac{1}{2\theta}$, according to (2.13), the conclusion (3.33) follows. □

Also, we can prove the following set of lemmas and corollaries. However, we omit their proofs, since they all can be proved along the same lines as in [16], thanks to the estimate (3.31).

Lemma 3.4 Under the assumptions of Theorem 2.1 and for $2\nu=(1-2\theta)(1+\frac{\theta}{10})$ we have the following upper bound

$$\int_0^1 \phi^{2\nu}Q^{2\theta-2}Q_x^2\mathrm{d}x \le C(T). \tag{3.35}$$

Lemma 3.5 Under the assumptions of Theorem 2.1 where k_1 is characterized by (2.17), for any integer $m>0$ and for $\alpha_1=(1-\frac{1}{2^m})(\theta-1)<0$, we have the following upper bound

$$\int_0^1 \phi^{k_1}Q^{\alpha_1}u^2\mathrm{d}x + \int_0^t\int_0^1 \phi^{k_1}Q^{1+\theta+\alpha_1}u_x^2\mathrm{d}x\mathrm{d}s + \int_0^t\int_0^1 \phi^{k_1}h(c,Q)Q^{\alpha_1}u^2|u|\mathrm{d}x\mathrm{d}s \le C(T). \tag{3.36}$$

Lemma 3.6 Under the assumptions of Theorem 2.1 and for any integer $m>0$ and for $\beta_1=(2-\frac{1}{2^m})(\theta-1)<0$, we have

$$\int_0^1 \phi^{k_1}Q^{\beta_1}\mathrm{d}x \le C(T). \tag{3.37}$$

Lemma 3.7 Under the assumptions of Theorem 2.1, and for $k_2=\nu+\frac{k_1}{2}$ where $k_1>2\nu$, we have the following pointwise lower bound on Q

$$Q(x,t) \ge C(T)\phi^{\frac{11k_2}{10(1-2\theta)}}(x), \qquad \forall (x,t)\in[0,1]\times[0,T]. \tag{3.38}$$

Corollary 3.3 We have the upper and lower bounds

$$C(T)\phi^{\frac{11k_2}{10(1-2\theta)}} \le Q(x,t) \le C(T)\phi(x)^{\alpha}, \tag{3.39}$$

$$\frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}} \le \zeta(x,t) \le \min\left\{\rho_l C(T)\phi(x)^{\alpha}, \frac{\rho_l-\mu}{1-\sup_{[0,1]}c}\right\}, \tag{3.40}$$

$$\left(\inf_{x\in[0,1]} c_0\right)\frac{\rho_l C(T)}{1+C(T)}\phi(x)^{\frac{11k_2}{10(1-2\theta)}} \le n(x,t) \le \min\left\{\rho_l C(T)\phi(x)^{\alpha}, \frac{\rho_l-\mu}{1-\sup_{[0,1]}c}\right\}, \tag{3.41}$$

where $\mu > 0$ is a small constant.

Corollary 3.4 We have the estimates

$$\int_0^1 |\partial_x \zeta| \mathrm{d}x \le C(T), \qquad \int_0^1 |\partial_x n| \mathrm{d}x \le C(T), \tag{3.42}$$

for a suitable constant $C(T)$.

Lemma 3.8 For a given integer $n > 0$, and under the assumptions of Theorem 2.1, we can prove that

$$\int_0^1 u_t^{2n} \mathrm{d}x + n(2n-1) \int_0^t \int_0^1 Q^{\theta+1} u_{xt}^2 u_t^{2n-2} \mathrm{d}x \mathrm{d}s \le C(T). \tag{3.43}$$

Lemma 3.9 Under the assumptions of Theorem 2.1 we have the estimates

$$\|Q^{\theta+1} u_x\|_{L^\infty(D_T)} \le C(T), \tag{3.44}$$

$$\int_0^1 |(Q^{\theta+1} u_x)_x| \mathrm{d}x \le C(T), \tag{3.45}$$

$$\int_0^1 |Q_x| \mathrm{d}x \le C(T), \tag{3.46}$$

for a suitable constant $C(T)$ and where $D_T = [0,1] \times [0,T]$.

Lemma 3.10 Under the assumptions of Theorem 2.1, we have the following estimates for the velocity u

$$\int_0^1 |u_x(x,t)| \mathrm{d}x \le C(T), \tag{3.47}$$

$$|u(x,t)| \le C(T). \tag{3.48}$$

Lemma 3.11 Under the assumptions of Theorem 2.1, we have for $0 < s < t \le T$ that

$$\int_0^1 |Q(x,t) - Q(x,s)|^2 \mathrm{d}x \le C(T)|t-s|^2, \tag{3.49}$$

$$\int_0^1 |\zeta(x,t) - \zeta(x,s)|^2 \mathrm{d}x \le C(T)|t-s|^2, \tag{3.50}$$

$$\int_0^1 |n(x,t) - n(x,s)|^2 \mathrm{d}x \le C(T)|t-s|^2, \tag{3.51}$$

$$\int_0^1 |u(x,t) - u(x,s)|^2 \mathrm{d}x \le C(T)|t-s|^2, \tag{3.52}$$

$$\int_0^1 |(Q^{\theta+1} u_x)(x,t) - (Q^{\theta+1} u_x)(x,s)|^2 \mathrm{d}x \le C(T)|t-s|. \tag{3.53}$$

3.3 Proof of Theorem 2.1

In order to construct weak solutions to the initial-boundary problem (IBVP) (1.1)–(1.5), we apply the line method [26] where a system of ODEs is derived that can approximate the original model. For the details we refer to [10], which in turn is based on single-phase works like [13]. Semi-discrete version of the various lemmas can be obtained, and in combination with Helly's theorem, the result of Theorem 2.1 follows, see [18, 19, 22, 23, 25, 26, 29, 32–34] and references therein for details.

4 Uniqueness of Weak Solutions

The following uniqueness proof is inspired by [14]. However, obviously we have to deal properly with the additional friction term appearing in the momentum equation. In particular, we observe that we have to enforce a stronger assumption on the β parameter associated with the friction term by requiring that $\beta \geq 1$.

We start with the following lemma, which will be needed in the following, and which also extends the result (3.44) of Lemma 3.9.

Lemma 4.1 Under the assumptions of Theorem 2.1, for λ given by

$$\lambda = 1 + \theta - \frac{10(2n-1)(1-2\theta)}{22nk_2}, \qquad k_2 = \nu + \frac{k_1}{2}, \qquad k_1 < \frac{15(1-2\theta)}{11\theta} - 2\nu, \tag{4.1}$$

and for a sufficiently large positive integer n, we have the estimate

$$\|(Q^\lambda u_x)(x,t)\|_{L^\infty(D_T)} \leq C(T), \tag{4.2}$$

for a suitable constant $C(T)$ and where $D_T = [0,1] \times [0,T]$.

Proof First, note from (4.1) that when $k_1 < \frac{15(1-2\theta)}{11\theta} - 2\nu$, we have that $\theta < \lambda < 1$ for a sufficiently large positive integer n. It is sufficient to observe that

$$\frac{10}{22}(2-1/n)(1-2\theta)\frac{1}{k_2} < \frac{20}{22}\frac{1}{(1+\theta/10)} < 1,$$

since $k_2 > 2\nu = (1-2\theta)(1+\theta/10)$, and

$$\frac{10}{22}(2-1/n)(1-2\theta)\frac{1}{k_2} > \frac{10}{22}(2-1/n)(1-2\theta)\frac{22\theta}{15(1-2\theta)} > \frac{2}{3}(2-1/n)\theta > \frac{4}{3}\theta$$

for sufficiently large n. Also note that we then have $\lambda - 1 - \theta < 0$ and $\gamma + \lambda - 1 - \theta > 0$.

Now using the Hölder inequality with $p = 2n$ and $q = \frac{2n}{2n-1}$, assumption (2.7), Corollary 3.1 and 3.2, Lemmas 3.3, 3.7 and 3.8, it follows from (3.23) that

$$\begin{aligned} Q^\lambda u_x &= c^\gamma Q^{\gamma+\lambda-1-\theta} + Q^{\lambda-1-\theta}\left[\int_0^x u_t \mathrm{d}x + \int_0^x h(c,Q)u|u|\mathrm{d}x\right] \\ &\leq C(T) + C(T)Q^{\lambda-1-\theta}\left[\left(\int_0^1 u_t^{2n}\mathrm{d}x\right)^{\frac{1}{2n}}\Big(x(1-x)\Big)^{\frac{2n-1}{2n}} \right. \\ &\quad \left. + \left(\int_0^1 u^{4n}\mathrm{d}x\right)^{\frac{1}{2n}}\Big(x(1-x)\Big)^{\frac{2n-1}{2n}}\right] \\ &\leq C(T) + C(T)\Big(x(1-x)\Big)^{\left(\frac{2n-1}{2n} + \frac{11k_2(\lambda-1-\theta)}{10(1-2\theta)}\right)}. \end{aligned} \tag{4.3}$$

This proves Lemma 4.1, since it follows from (4.1) that

$$\frac{2n-1}{2n} + \frac{11k_2(\lambda-1-\theta)}{10(1-2\theta)} = 0. \tag{4.4}$$

□

Proof of Theorem 2.2

Proof Assume that (c_1, Q_1, u_1) and (c_2, Q_2, u_2) are two (possibly different) global weak solutions of the problem as defined in Theorem 2.1, but with the same initial conditions. It then immediately follows from (3.8) a) that $c_1(x,t) = c_2(x,t) = c_0(x)$. We then proceed by introducing the new variables

$$\hat{Q}(x,t) = (Q_1 - Q_2)(x,t), \tag{4.5}$$

and

$$\omega(x,t) = \int_0^x (u_1 - u_2)(y,t)\mathrm{d}y. \tag{4.6}$$

We note that

$$\hat{Q}(0,t) = \hat{Q}(1,t) = 0, \tag{4.7}$$

and moreover, that

$$\omega(0,t) = 0, \quad \omega_t(0,t) = 0, \quad \omega_t(1,t) = \int_0^1 \Big(h(c_0,Q_2)u_2|u_2| - h(c_0,Q_1)u_1|u_1|\Big)\mathrm{d}x. \tag{4.8}$$

Using these new variables, the second and third equation of (3.8) can be rewritten as follows

$$\left(\frac{\hat{Q}}{Q_1Q_2}\right)_t + \rho_l\omega_{xx} = 0, \tag{4.9}$$

and

$$\begin{aligned} &\omega_t + \left(\frac{p(c_0Q_1) - p(c_0Q_2)}{Q_1 - Q_2}\right)\hat{Q} \\ &= Q_1^{1+\theta}\omega_{xx} + \left(\frac{Q_1^{1+\theta} - Q_2^{1+\theta}}{Q_1 - Q_2}\right)\hat{Q}u_{2x} + \int_0^x \Big(h(c_0,Q_2)u_2|u_2| - h(c_0,Q_1)u_1|u_1|\Big)\mathrm{d}y. \end{aligned} \tag{4.10}$$

Moreover, multiplying (4.9) by $Q_1^{\theta}Q_2^{-1}\hat{Q}$ and reorganizing using i.a. integration by parts we obtain

$$\begin{aligned} &(Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2)_t + \rho_l\theta Q_1^{\theta}Q_2^{-2}\hat{Q}^2u_{1x} - \rho_l Q_1^{-1+\theta}Q_2^{-1}\hat{Q}^2u_{2x} + \rho_l Q_1^{1+\theta}Q_2^{-2}\hat{Q}u_{1x} \\ &-\rho_l Q_1^{-1+\theta}\hat{Q}u_{2x} + \rho_l Q_1^{\theta}Q_2^{-1}\hat{Q}\omega_{xx} = 0. \end{aligned} \tag{4.11}$$

Further manipulation gives the following form

$$(Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2)_t + \rho_l(\theta+1)Q_1^{\theta}Q_2^{-2}\hat{Q}^2u_{1x} + 2\rho_l Q_1^{\theta}Q_2^{-1}\hat{Q}\omega_{xx} = 0. \tag{4.12}$$

We now integrate equation (4.12) in x over $[0,1]$ to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x \;&=\; -\rho_l(\theta+1)\int_0^1 Q_1^{\theta}Q_2^{-2}\hat{Q}^2u_{1x}\mathrm{d}x - 2\rho_l\int_0^1 Q_1^{\theta}Q_2^{-1}\hat{Q}\omega_{xx}\mathrm{d}x \\ &=: I_A + I_B. \end{aligned} \tag{4.13}$$

Estimating the quantities I_A and I_B can be done as follows: First, we have

$$\begin{aligned} I_A &= -\rho_l(\theta+1)\int_0^1 Q_1^{\theta}Q_2^{-2}\hat{Q}^2u_{1x}\mathrm{d}x \le C(T)\int_0^1 Q_1^{\theta-\lambda}Q_2^{-2}\hat{Q}^2|Q_1^{\lambda}u_{1x}|\mathrm{d}x \\ &\le C(T)\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x, \end{aligned} \tag{4.14}$$

due to Corollary 3.2 (and the fact that $0 < 1-\lambda$) and Lemma 4.1, Next, using the Cauchy inequality we can estimate as follows

$$I_B = -2\rho_l \int_0^1 Q_1^{\theta} Q_2^{-1} \hat{Q} \omega_{xx} \mathrm{d}x \le C(T) \int_0^1 Q_1^{-1+\theta} Q_2^{-2} \hat{Q}^2 \mathrm{d}x + \frac{1}{4} \int_0^1 Q_1^{1+\theta} \omega_{xx}^2 \mathrm{d}x. \tag{4.15}$$

Consequently, (4.13) corresponds to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 Q_1^{-1+\theta} Q_2^{-2} \hat{Q}^2 \mathrm{d}x \le C(T) \int_0^1 Q_1^{-1+\theta} Q_2^{-2} \hat{Q}^2 \mathrm{d}x + \frac{1}{4} \int_0^1 Q_1^{1+\theta} \omega_{xx}^2 \mathrm{d}x. \tag{4.16}$$

Furthermore, multiplying equation (4.10) by ω_{xx}, using $\omega_t \omega_{xx} = (\omega_t \omega_x)_x - \frac{1}{2}(\omega_x^2)_t$, and integrating it in x over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{1}{2} \omega_x^2 \mathrm{d}x + \int_0^1 Q_1^{1+\theta} \omega_{xx}^2 \mathrm{d}x \\
&= \int_0^1 \left(\frac{p(c_0 Q_1) - p(c_0 Q_2)}{Q_1 - Q_2} \right) \hat{Q} \omega_{xx} \mathrm{d}x + \omega_t \omega_x \Big|_{x=1} \\
&\quad + \int_0^1 \left[\int_0^x (h(c_0, Q_1) u_1 |u_1| - h(c_0, Q_2) u_2 |u_2|) \mathrm{d}y \right] \omega_{xx} \mathrm{d}x \\
&\quad - \int_0^1 \left(\frac{Q_1^{1+\theta} - Q_2^{1+\theta}}{Q_1 - Q_2} \right) \hat{Q} u_{2x} \omega_{xx} \mathrm{d}x,
\end{aligned} \tag{4.17}$$

where we have used (4.8). We focus on the second line of (4.17) and set

$$U = \int_0^x \Big(h(c_0, Q_1) u_1 |u_1| - h(c_0, Q_2) u_2 |u_2| \Big) \mathrm{d}y, \qquad V_x = \omega_{xx}.$$

That is,

$$U_x = \Big(h(c_0, Q_1) u_1 |u_1| - h(c_0, Q_2) u_2 |u_2| \Big), \qquad V = \omega_x.$$

Consequently, in light of (4.8) we have $U|_{x=1} = -\omega_t|_{x=1}$, which implies that

$$\begin{aligned}
&\int_0^1 \left[\int_0^x (h(c_0, Q_1) u_1 |u_1| - h(c_0, Q_2) u_2 |u_2|) \mathrm{d}y \right] \omega_{xx} \mathrm{d}x \\
&= \int_0^1 U V_x \mathrm{d}x = UV \Big|_0^1 - \int_0^1 U_x V \mathrm{d}x \\
&= -\omega_t \omega_x \Big|_{x=1} - \int_0^1 \Big(h(c_0, Q_1) u_1 |u_1| - h(c_0, Q_2) u_2 |u_2| \Big) \omega_x \mathrm{d}x.
\end{aligned}$$

This yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{1}{2} \omega_x^2 \mathrm{d}x + \int_0^1 Q_1^{1+\theta} \omega_{xx}^2 \mathrm{d}x \\
&= \int_0^1 \left(\frac{p(c_0 Q_1) - p(c_0 Q_2)}{Q_1 - Q_2} \right) \hat{Q} \omega_{xx} \mathrm{d}x - \int_0^1 \Big(h(c_0, Q_1) u_1 |u_1| - h(c_0, Q_2) u_2 |u_2| \Big) \omega_x \mathrm{d}x \\
&\quad - \int_0^1 \left(\frac{Q_1^{1+\theta} - Q_2^{1+\theta}}{Q_1 - Q_2} \right) \hat{Q} u_{2x} \omega_{xx} \mathrm{d}x \\
&=: J_A + J_B + J_C.
\end{aligned} \tag{4.18}$$

We now seek estimates of the quantities J_A, J_B and J_C. Using Cauchy inequality, Corollary 3.2 and Lemma 3.7, we can estimate as follows

$$\begin{aligned} J_A &= \int_0^1 \frac{p(c_0Q_1)-p(c_0Q_2)}{Q_1-Q_2}\hat{Q}\omega_{xx}\mathrm{d}x \le C(T)\int_0^1 |\hat{Q}||\omega_{xx}|\mathrm{d}x \\ &\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\int_0^1 Q_1^{-1-\theta}\hat{Q}^2\mathrm{d}x \\ &\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\max_{x\in[0,1]}(Q_1^{-2\theta}Q_2^2)\Big(\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x\Big) \\ &\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\max_{x\in[0,1]}\Big[\big(x(1-x)\big)^{(2\alpha-\frac{22\theta k_2}{10(1-2\theta)})}\Big]\Big(\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x\Big) \\ &\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x, \end{aligned} \tag{4.19}$$

since we can choose $\alpha \ge \frac{11\theta k_2}{10(1-2\theta)}$, as assumed in the statement of Theorem 2.2. Furthermore, note from (3.10) that

$$h_Q(c,Q) = f\beta\rho_l^\beta\Big(\frac{Q}{1+(1-c)Q}\Big)^{\beta-1}\Big(\frac{1}{1+(1-c)Q}\Big)^2. \tag{4.20}$$

Then using Corollaries 3.1, 3.2, and Lemma 3.10, we can estimate as follows by means of Cauchy inequality:

$$\begin{aligned} J_B &= \int_0^1 \Big(h(c_0,Q_1)u_1|u_1| - h(c_0,Q_2)u_2|u_2|\Big)\omega_x\mathrm{d}x \\ &= \int_0^1 h(c_0,Q_1)\Big(u_1|u_1|-u_2|u_2|\Big)\omega_x\mathrm{d}x + \int_0^1 u_2|u_2|\Big(h(c_0,Q_1)-h(c_0,Q_2)\Big)\omega_x\mathrm{d}x \\ &\le C(T)\int_0^1 \Big|\frac{u_1|u_1|-u_2|u_2|}{u_1-u_2}\Big|\omega_x^2\mathrm{d}x + C(T)\int_0^1\Big|\frac{h(c_0,Q_1)-h(c_0,Q_2)}{Q_1-Q_2}\Big||\hat{Q}\omega_x|\mathrm{d}x \\ &\le C(T)\max_{u\in\mathrm{int}(u_1,u_2)}|u|\int_0^1\omega_x^2\mathrm{d}x + C(T)\max_{Q\in\mathrm{int}(Q_1,Q_2)}|h_Q(c_0,Q)|\int_0^1|\hat{Q}||\omega_x|\mathrm{d}x \\ &\le C(T)\int_0^1\omega_x^2\mathrm{d}x + C(T)\max_{x\in[0,1]}(Q_1^{1-\theta}Q_2^2)\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x \\ &\le C(T)\int_0^1\omega_x^2\mathrm{d}x + C(T)\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x. \end{aligned} \tag{4.21}$$

Here we have used (4.20) in combination with the requirement that $\beta \ge 1$ to bound $|h_Q(c_0,Q)|$.

Using Cauchy inequality, Corollary 3.2, Lemma 4.1 and 3.7, we can estimate as follows:

$$\begin{aligned} J_C &= -\int_0^1 \frac{Q_1^{1+\theta}-Q_2^{1+\theta}}{Q_1-Q_2}\hat{Q}u_{2x}\omega_{xx}\mathrm{d}x \le C(T)\int_0^1 (Q_1^\theta+Q_2^\theta)|\hat{Q}u_{2x}\omega_{xx}|\mathrm{d}x \\ &\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\int_0^1 (Q_1^{2\theta}+Q_2^{2\theta})Q_1^{-1-\theta}\hat{Q}^2u_{2x}^2\mathrm{d}x \\ &\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\Big(\max_{x\in[0,1]}[(Q_1^{2\theta}+Q_2^{2\theta})(Q_1^{-2\theta}Q_2^{2-2\lambda})]\Big) \\ &\quad\times\max_{x\in[0,1]}\Big[(Q_2^\lambda u_{2x})^2\Big]\Big(\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x\Big) \end{aligned}$$

$$\begin{aligned}&\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\Big(x(1-x)\Big)^{((2\theta+2-2\lambda)\alpha-\frac{22\theta k_2}{10(1-2\theta)})}\Big(\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x\Big)\\ &\le \frac{1}{8}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x, \end{aligned}\tag{4.22}$$

since it follows from (4.1), that for $k_2 < \frac{10}{11}(1-2\theta)\sqrt{\frac{\alpha}{\theta}}$, we can get $(2\theta+2-2\lambda)\alpha \ge \frac{22\theta k_2}{10(1-2\theta)}$ for sufficiently large n.

As a consequence of (4.18), (4.19), (4.21), and (4.22), we can then establish the inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 \frac{1}{2}\omega_x^2\mathrm{d}x\Big) + \int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x\\ &\le \frac{1}{4}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x + C(T)\Big(\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x + \int_0^1 \frac{1}{2}\omega_x^2\mathrm{d}x\Big). \end{aligned}\tag{4.23}$$

Finally, combining the results from (4.16) and (4.23) we arrive at the inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x + \int_0^1 \frac{1}{2}\omega_x^2\mathrm{d}x\Big) + \frac{1}{2}\int_0^1 Q_1^{1+\theta}\omega_{xx}^2\mathrm{d}x\\ &\le C(T)\Big(\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x + \int_0^1 \frac{1}{2}\omega_x^2\mathrm{d}x\Big). \end{aligned}\tag{4.24}$$

Gronwall's inequality then implies that

$$\int_0^1 Q_1^{-1+\theta}Q_2^{-2}\hat{Q}^2\mathrm{d}x + \int_0^1 \frac{1}{2}\omega_x^2\mathrm{d}x = 0. \tag{4.25}$$

Obviously, this means that $\hat{Q} = \omega_x = 0$, and the uniqueness proof is completed since $Q_1 = Q_2$ and $u_1 = u_2$ a.e. □

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