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ANALYSIS OF A COMPRESSIBLE GAS-LIQUID MODEL MOTIVATED BY OIL WELL CONTROL OPERATIONS*

Dedicated to Professor Constantine M. Dafermos on the occasion of his 70th birthday

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Abstract We are interested in a viscous two-phase gas-liquid mixture model relevant for modeling of well control operations within the petroleum industry. We focus on a simplified mixture model and provide an existence result within an appropriate class of weak solutions. We demonstrate that upper and lower limits can be obtained for the gas and liquid masses which ensure that transition to single-phase regions do not occur. This is used together with appropriate a prior estimates to obtain convergence to a weak solution for a sequence of approximate solutions corresponding to mollified initial data. Moreover, by imposing an additional regularity condition on the initial masses, a uniqueness result is obtained. The framework herein seems useful for further investigations of more realistic versions of the gas-liquid model that take into account different flow regimes.

Key words gas-liquid two-phase model; weak solution; existence; uniqueness

2000 MR Subject Classification

1 Introduction

We are interested in a one-dimensional two-phase liquid (ℓ) and gas (g) model in the form

$$
\begin{aligned}
&m_t + (v_\ell m)_x = 0,\\
&n_t + (v_{\mathrm{g}} n)_x = 0,\\
&(mv_\ell + nv_{\mathrm{g}})_t + (mv_\ell^2 + nv_{\mathrm{g}}^2)_x + p(m,n)_x = q_F + q_G + \mu(v_{\mathrm{mix}})_{xx},
\end{aligned}
\tag{1.1}
$$

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where $\mu > 0$ and

$$m = \alpha_\ell \rho_\ell, \qquad n = \alpha_{\rm g}\rho_{\rm g},$$
$$q_F = -f v_{\rm mix}, \qquad q_G = -g\rho_{\rm mix},$$
$$v_{\rm mix} = \alpha_\ell v_\ell + \alpha_{\rm g} v_{\rm g}, \qquad \rho_{\rm mix} = \alpha_\ell \rho_\ell + \alpha_{\rm g}\rho_{\rm g},$$

where f and g are nonnegative constants. The variables involved are as follows: $\alpha_\ell, \alpha_{\rm g}$ are volume fractions, $\rho_\ell, \rho_{\rm g}$ are fluid densities, and $v_\ell, v_{\rm g}$ are fluid velocities. This model is supplemented with the following constraints (algebraic relations):

$$p = C\rho_{\rm g}^\gamma, \qquad \rho_l = \text{constant}, \quad \gamma \geq 1, \tag{1.2}$$
$$\alpha_\ell + \alpha_{\rm g} = 1, \tag{1.3}$$
$$v_{\rm g} = K v_{\rm mix} + S, \qquad K, S \ \text{ are constants.} \tag{1.4}$$

The model (1.1)–(1.4) is often referred to as the drift-flux model. Note that by combining (1.2) and (1.3), we get $\rho_{\rm g} = \rho_l \frac{n}{\rho_l - m}$. Consequently,

$$p = C\rho_l^\gamma \left(\frac{n}{\rho_l - m}\right)^\gamma = p(m,n). \tag{1.5}$$

The drift-flux model is highly relevant in modeling of various well operations [7]. In particular, the model has been used for the study of drilling operations. Currently, there is much focus on development of safe and optimal drilling methods in the context of deepwater wells. In this setting a typical model problem involves an interesting but complicated interaction of different physical mechanisms like the balance between the pressure gradient induced by frictional forces q_F and the hydrostatic pressure q_G, transition from mixture to single-phase regions, free gas-liquid interface behavior, and various compressible effects like compression and decompression. One aspect that requires special attention is the possibility of having a gas-kick. A gas-kick refers to a situation where gas flows into the well from the surrounding formation. As this gas ascends in the well it will typically experience a lower pressure. This leads to decompression of the gas, which can provoke blowout-like scenarios. Clearly, for the study of such flow scenarios we need a two-phase gas-liquid model that takes into account compressible effects.

The purpose of this paper is to focus on one aspect of the model (1.1) by making several simplifying assumptions. More precisely, we first neglect the acceleration terms in the mixture momentum equation. That is, we consider the simplified momentum balance given by the following static force balance

$$p(m,n)_x = q_F + q_G + \mu(v_{\rm mix})_{xx}. \tag{1.6}$$

The resulting "vanishing Mach number" model, often with $\mu = 0$ and sometimes called the no-pressure wave model, has been demonstrated to be highly relevant in the context of well flow scenarios, see [19] and [18] and references therein. The main effect of using the simpler momentum equation (1.6) is that pressure (acoustic) waves are neglected. However, for many applications the main interest is the slow transport of volume fraction waves (mass waves), and not a detailed study of the pressure waves. In the present model the pressure waves are approximated by infinite velocity waves.

In this paper we want to explore the viscous dominated model where we also neglect the friction term q_F and assume horizontal flow, i.e., $q_G = 0$. In addition, we restrict to the special flow case of no slip between the two phases, i.e., $v_\ell = v_{\rm g} = u$. This represents a situation where the gas is dispersed in the liquid phase such that the two-phase mixture moves with the same velocity, more or less.

Previous work on the model (1.1)

Development of good discrete methods for solving the compressible gas-liquid model (1.1) has been a topic for many papers during the last decade [1–3, 6, 7, 11–13, 18–21]. However, it is only recently that the mathematical properties of this model have been investigated. In [8] a simplified version of (1.1) was studied. More precisely, it was assumed that the gas and liquid velocities were equal, no external forces were taken into account in the momentum equation, and certain gas terms were neglected in the momentum equation taking advantage of the fact that $\rho_l/\rho_g \gg 1$. The existence of global weak solutions was then obtained under suitable assumptions on the initial data. In particular, the result showed that when the initial masses n_0 and m_0 do not vanish or blow up ($n = \alpha_g \rho_g$ and $m = \alpha_l \rho_l$), then n and m remain bounded:

$$C_T^{-1} \leq n(t,\cdot), m(t,\cdot) \leq C_T, \qquad t \in [0,T],\, T > 0,$$

for a positive constant C_T. The 1D results [8] are extended to a 2D version of the model in [25]. The main assumption in [25] is that the initial energy is small in a certain sense. The provided estimates are also strong enough to give the large time asymptotic behavior of the solution. We also refer to [26] for a result on the blow-up behavior of the 2D gas-liquid model in Eulerian coordinates.

Studies have also been carried out with the model (1.1) considered in Lagrangian variables with free boundaries and a viscosity term depending on the masses. A first work in this direction can be found in [9, 10]. More recently, these studied have been extended to include the possibility of different fluid velocities [4], well-reservoir interaction [5], and external forces like friction and gravity [14].

Contributions of the present work

The main purpose of this paper is to explore some other aspects of the two-phase model (1.1) concerning the existence and uniqueness within an appropriate class of weak solutions. The model we investigate becomes different than the one studied in [8, 25, 26] since we consider a steady state mixture momentum equation (1.6). A main motivation for this work is to establish a framework that possibly can allow for inclusion of important physical mechanisms that currently are neglected in the model studied in the works [8, 25, 26]. Such investigations are left for future work.

An important feature of the model we study is that although we apply a simplified linear EOS for the gas phase (isothermal flow) by choosing $\gamma = 1$ in (1.2), the resulting pressure law $p(m,n)$ for the two-phase mixture (1.5) becomes a nonlinear function. This reflects some of the additional complexity represented by two-phase over single-phase modeling. Clearly, a potential difficulty with the model we consider is the singularity $m = \rho_l$ (i.e. $\alpha_\ell = 1$) in the pressure law (1.5). This corresponds to a situation where transition to single-phase liquid flow occur. A main observation is that the assumption of no-slip condition implies that the two masses m and n are related as $\frac{n}{m} = s$ with s controlled. As a consequence, we obtain pointwise control on m and n (lower and upper bounds) which allows us to verify that the pressure $p(m,n)$ is well-defined.

Equipped with pointwise control on m,n we derive several a priori estimates in L^p and Sobolev spaces of a sequence of approximate solutions $\{m_k,n_k,u_k\}$ obtained by applying a regularization of initial data m_0,n_0. These estimates yield some basic (weak) convergence results. However, strong convergence of m_k,n_k is required to recover the nonlinear pressure law $p(m,n)$. This is obtained by studying various renormalizations of the approximate solutions and corresponding defect measures.

An interesting aspect of the model we study is that a (presumably natural) viscous approximation of (2.1) is not easy to analyze using weak compactness and renormalization arguments. This appears to be due to the fact that our analysis relies heavily on using the simple equation satisfied by the quantity $s:=\frac{n}{m}$, an equation that is not available in the context of uniformly parabolic problems. A similar difficulty arises when attempting to prove strong convergence of upwind-type difference schemes, cf. the discussion in Section 6.

The remaining part of this paper is organized as follows: In Section 2 we present the compressible gas-liquid model and state the main results. Section 3 contains the analysis yielding pointwise control on the masses as well as various L^p estimates. In Section 4 the compactness (convergence) of a sequence of approximate solutions is established. Moreover, the limit functions are identified as weak solutions of the two-phase model in question. In Section 5, a uniqueness result is derived under some additional regularity on the initial masses. Finally, we make some concluding remarks in Section 6.

2 A Viscous Two-phase Model

We focus on a two-phase model in the following form

$$\begin{aligned} &m_t+(um)_x=0, \quad n_t+(un)_x=0,\\ &p(m,n)_x=\mu u_{xx}, \qquad \mu>0, \end{aligned} \tag{2.1}$$

where the pressure function is given by

$$p(m,n)=C\left(\frac{n}{\rho_l-m}\right)^{\gamma}, \qquad \gamma\ge 1. \tag{2.2}$$

In what follows, we set $C=1$, $\gamma=1$, and $\mu=1$. We may restate the model as

$$m_t+um_x=-mp(m,n), \qquad n_t+un_x=-np(m,n).$$

The main purpose of this work is to establish the global-in-time existence of weak solutions to the initial-boundary value problem

$$\begin{aligned} &m_t+um_x=-mp(m,n), \quad t>0,\, x\in\mathbb{R}^+,\\ &n_t+un_x=-np(m,n), \quad t>0,\, x\in\mathbb{R}^+,\\ &u_x=p(m,n), \quad p(m,n)=\frac{n}{\rho_l-m},\\ &u(t,x)|_{x=0}=0, \quad m|_{t=0}=m_0(x), \quad n|_{t=0}=n_0(x), \end{aligned} \tag{2.3}$$

where $m_0, n_0 \in L^\infty(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)$ for $p > 1$ and $\mathbb{R}^+ := (0, \infty)$.

We mention that this system (2.3), along with its analysis, shares some resemblence with the so-called Hunter-Saxton model

$$v_t + uv_x = -\frac{1}{2}v^2, \qquad u_x = v,$$

see for example [22–24] and references therein.

Definition 1 (Weak solution) We call $(m(t,x), n(t,x), u(t,x))$ a weak solution of (2.3) provided the following conditions hold:

(i) $m(t,x), n(t,x) \in L^\infty_{\text{loc}}(\mathbb{R}^+, L^p(\mathbb{R}^+))$, $u(t,x) \in C([0,\infty) \times \mathbb{R}^+)$;

(ii) $m_t + (um)_x = 0$, $n_t + (un)_x = 0$, $p(m,n) = u_x$ in the sense of distributions;

(iii) The function $u(t,x)$ is equal to zero at $x = 0$ as a continuous function. The function $m(t,x)$ and $n(t,x)$ take on the initial values $m_0(x)$ and $n_0(x)$ in the sense $C([0,\infty), L^1(\mathbb{R}^+))$.

To establish an existence result for weak solutions, the main challenge is to pass to the limit in a sequence of approximate solutions in the nonlinear pressure function without relying on BV or Sobolev-type of estimates.

Theorem 1 (Existence result) We assume that

(i) $m_0(x), n_0(x) \in L^\infty([0,\infty)) \cap L^p([0,\infty))$ for some appropriate p;

(ii) there are positive constants A_1, A_2 and B_1, B_2 such that

$$0 < A_1 \le m_0(x) \le A_2 < \rho_l, \qquad 0 < B_1 \le n_0(x) \le B_2. \tag{2.4}$$

Moreover, m_0 and n_0 have compact support;

Then problem (2.3) possesses a global weak solution $(m(t,x), n(t,x), u(t,x))$ in the sense of Definition 1. In addition, for any finite time $T > 0$, there are positive constants $C_1 = C_1(T)$, $C_2 = C_2(T)$ such that

$$\begin{aligned} &0 < C_1^{-1} \le m(t,x) \le C_1 < \rho_l < \infty, \\ &0 < C_2^{-1} \le n(t,x) \le C_2 < \infty. \end{aligned}$$

Remark As reflected by the assumptions, we consider a situation where the initial state represents a true gas-liquid mixture. In other words, there exists no pure gas or liquid regions at time zero.

In what follows, we will make use of the following notations:

$$Q_T = [0,T] \times [0,\infty), \quad Q_\infty = [0,\infty) \times [0,\infty).$$

By assuming some more regularity on the initial masses m_0 and n_0, we can supply a uniqueness result in the $BV \cap L^\infty$ class.

Theorem 2 (Uniqueness result) Fix any $T > 0$. For $i = 1, 2$, let (m^i, n^i, u^i) be a weak solution on Q_T of the system

$$\begin{aligned} &m_t^i + (u^i m^i)_x = 0, \quad n_t^i + (u^i n^i)_x = 0, \\ &u^i(t,x) = \int_0^x p(m^i(t,y), n^i(t,y))\mathrm{d}y, \end{aligned} \tag{2.5}$$

with initial data $m^i|_{t=0} = m_0^i \in L^\infty \cap BV$, $n^i|_{t=0} = n_0^i \in L^\infty \cap BV$ satisfying

$$0 \le m_0^i(\cdot) \le C_{m^i} < \rho_l, \quad 0 \le n^i(\cdot) \le C_{n^i},$$

for some finite constants C_{m^i}, C_{n^i}. Then for any $t \in (0,T)$,

$$\left\|m^1(t,\cdot) - m^2(t,\cdot)\right\|_{L^1} + \left\|n^1(t,\cdot) - n^2(t,\cdot)\right\|_{L^1} \le C_T \left(\left\|m_0^1 - m_0^2\right\|_{L^1} \left\|n_0^1 - n_0^2\right\|_{L^1} \right).$$

In particular, there is uniqueness of weak solutions of (2.3) in the $BV \cap L^\infty$ class.

3 Estimates

First, we solve (2.3) with smooth initial data and obtain a priori estimates. In particular, we assume for the initial data m_0, n_0 that $m_0, n_0 \in C_c^\infty(\mathbb{R})$ and $\operatorname{supp}(m_0)$, $\operatorname{supp}(n_0) \subset [0,1)$ and they are nonnegative. We apply the method of characteristics and rewrite (2.3) in the following form

$$\begin{aligned}
&\frac{\mathrm{d}X_t(x)}{\mathrm{d}t} = u(t, X_t(x)) = \int_0^{X_t(x)} [p(m,n)](t,y)\mathrm{d}y, \qquad X_0(x) = x, \\
&\frac{\mathrm{d}}{\mathrm{d}t} m(t, X_t(x)) = -[mp(m,n)](t, X_t(x)), \\
&\frac{\mathrm{d}}{\mathrm{d}t} n(t, X_t(x)) = -[np(m,n)](t, X_t(x)), \\
&m(t, X_t(x))|_{t=0} = m_0(x), \qquad n(t, X_t(x))|_{t=0} = n_0(x).
\end{aligned} \tag{3.1}$$

Clearly, the right hand side of this system of ODEs is Lipschitz continuous for an open set U, such that m is bounded away from ρ_l for all $(m,n) \in U$. Hence, for each $(\bar{m}, \bar{n}) \in U$, where we also clearly can assume that $(m_0, n_0) \in U$ in light of (2.4), there exists a unique solution of (3.1) which can be continued up to the boundary of U [15]. Now, we can study the behavior of m and n along the characteristics. We have

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -mp(m,n), \qquad \frac{\mathrm{d}n}{\mathrm{d}t} = -np(m,n),$$

from which we get, since $m, n > 0$,

$$\frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t},$$

or

$$\frac{\mathrm{d}\log(m)}{\mathrm{d}t} = \frac{\mathrm{d}\log(n)}{\mathrm{d}t}.$$

In other words,

$$\log(m(t, X_t(x))) = \log(n(t, X_t(x))) + C, \quad \text{i.e.,} \quad m(t, X_t(x)) = Cn(t, X_t(x)).$$

In view of the initial data we get the following solution

$$m(t, X_t(x)) = n(t, X_t(x)) \frac{m_0(x)}{n_0(x)}, \tag{3.2}$$

where

$$A_1 B_2^{-1} \le \frac{m_0(x)}{n_0(x)} \le A_2 B_1^{-1}. \tag{3.3}$$

Along the characteristics the first equation of (2.3) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t,X_t(x)) = -\left[m\,p\Big(m,\frac{n_0}{m_0}m\Big)\right](t,X_t(x)) = -\Big(\frac{n_0}{m_0}\Big)\frac{m^2(t,X_t(x))}{\rho_l - m(t,X_t(x))},$$

since $\rho_l - m(t,X_t(x)) > 0$, which implies that

$$\log(m(t,X_t(x))) + \frac{\rho_l}{m(t,X_t(x))} = \Big(\frac{n_0}{m_0}\Big)t + C(m_0), \quad C(m_0) = \log(m_0) + \frac{\rho_l}{m_0}. \tag{3.4}$$

Define $\phi(m)$ by

$$\phi(m) \stackrel{\text{def}}{:=} \frac{m}{m\log(m)+\rho_l}.$$

Then we have from (3.4)

$$\phi(m(t,X_t(x)) = \frac{1}{\frac{n_0}{m_0}t + C(m_0)}. \tag{3.5}$$

Moreover, we find that

$$\phi'(m) = \frac{\rho_l - m}{(m\log(m)+\rho_l)^2},$$
$$\phi''(m) = -\frac{2\log(m)\rho_l - m\log(m) + 3\rho_l - 2m}{(m\log(m)+\rho_l)^3}.$$

According to (2.4), we have that

$$0 < \inf_{[0,1]} m_0(x) \le m_0(x) \le \sup_{[0,1]} m_0(x) < \rho_l.$$

Hence, $C(m_0) > 0$ so we can conclude that the right hand side of (3.5) is always positive and does not blow up for any $t > 0$. Moreover, We can check that $\phi(m)$ is strictly increasing in $[0,\rho_l]$, $\phi(0) = 0$, $\phi'(\rho_l) = 0$, $\phi''(\rho_l) = -\frac{1}{(\rho_l \log\rho_l + \rho_l)^2} < 0$ such that $m = \rho_l$ is a maximum point, and $\phi(\rho_l) = \frac{1}{\log(\rho_l)+1}$. From this we get for $0 < t_1 < t_2$,

$$\phi(m(t_1,X_{t_1}(x))) = \frac{1}{\frac{n_0}{m_0}t_1 + C(m_0)} > \frac{1}{\frac{n_0}{m_0}t_2 + C(m_0)} = \phi(m(t_2,X_{t_2}(x))),$$

which implies that

$$m_0(x) > m(t_1,X_{t_1}(x)) > m(t_2,X_{t_2}(x)).$$

In other words, $m(t,X_t(x))$ is decreasing in time (along its characteristics). Since $\sup_{[0,1]} m_0 < \rho_l$ we conclude that $m(t,X_t(x)) \in [0,\rho_l)$ for all $t > 0$, and we have the relation

$$\rho_l > \sup_{[0,1]} m_0 > m(t_1,X_{t_1}(x)) > m(t_2,X_{t_2}(x)) \ge 0.$$

It is of interest to quantify the rate of the decrease in m as time is running. For that purpose, we may argue as follows: Clearly ϕ^{-1} exists on $[0,\sup_{[0,1]} m_0]$ and we find that

$$m(t,X_t(x)) = \phi^{-1}\Big(\frac{1}{\frac{n_0}{m_0}t + C(m_0)}\Big).$$

We may find $\psi(x) = Kx$ for some constant K such that

$$\phi^{-1}(x) \le \psi(x) = Kx, \qquad x \in [0, M], \qquad M = \phi(\sup_{[0,1]} m_0) < \frac{1}{\log(\rho_l) + 1}.$$

Thus, we may conclude that

$$m(t, X_t(x)) \le \frac{K}{\frac{n_0}{m_0} t + C(m_0)} \le \frac{K m_0}{n_0 t} \le \frac{K A_2}{B_1 t}, \tag{3.6}$$

in view of (3.3). Consequently, the rate of dissipation directly depends on the bounds of the initial data m_0 and n_0.

Next, we see from the first equation of (3.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\partial_x X_t(x)] = [p(m, n)](t, X_t(x)) \cdot [\partial_x X_t(x)].$$

In other words (using that $\partial_x X_0(x) = 1$), we get

$$\log[\partial_x X_t(x)] - \log(1) = \int_0^t [p(m, n)](s, X_s(x)) \mathrm{d}s.$$

That is,

$$\begin{aligned} 1 \le \partial_x X_t(x) &= \exp\left(\int_0^t [p(m, n)](s, X_s(x)) \mathrm{d}s \right) \\ &= \exp\left(\frac{n_0(x)}{m_0(x)} \int_0^t \left[\frac{m}{\rho_l - m} \right] (s, X_s(x)) \mathrm{d}s \right) \\ &= \exp\left(\frac{n_0(x)}{m_0(x)} \int_0^t g(m)(s, X_s(x)) \mathrm{d}s \right), \qquad g(m) =: \frac{m}{\rho_l - m}, \\ &= \exp\left(\frac{n_0(x)}{m_0(x)} \int_0^t g\left(\phi^{-1}\left(\frac{1}{\frac{n_0}{m_0} s + C(m_0)} \right) \right) \mathrm{d}s \right) \\ &\le C(t, m_0(x), n_0(x)), \end{aligned} \tag{3.7}$$

for an appropriate constant $C(t, m_0, n_0)$ where we employ the fact that $g(m)$ is bounded since m is bounded away from ρ_l. Thus, we conclude that $X_t(x)$ can be inverted for all $x \in \mathbb{R}^+$ and $t > 0$ since it is strictly increasing in x; we denote the inverse mapping by $X_t^{-1}(x)$. Using this in (3.2) we see that

$$\frac{n(t, x)}{m(t, x)} = \left(\frac{n_0}{m_0} \right) (X_t^{-1}(x)) =: s(t, x). \tag{3.8}$$

Moreover, in view of (3.5),

$$\phi(m(t, x)) = \frac{1}{\frac{n_0(X_t^{-1}(x))}{m_0(X_t^{-1}(x))} t + C(m_0(X_t^{-1}(x)))} = \frac{1}{s(t, x)t + C(m_0(X_t^{-1}(x)))}. \tag{3.9}$$

Clearly, in view of (2.4) we have the estimate

$$0 < B_1 A_2^{-1} \le s(t, x) \le B_2 A_1^{-1} < \infty. \tag{3.10}$$

For further use we apply (3.8) and define

$$p(m, n) = p(m, ms) = s \frac{m}{\rho - m} = s g(m) \overset{\text{def}}{:=} P(m, s). \tag{3.11}$$

As far as the velocity u is concerned, we estimate as follows:

$$\begin{aligned} u(t,x) &= \int_0^x P(m,s)\mathrm{d}y \le \sup_{x\in\mathbb{R}^+} s(t,x) \int_0^x g(m(t,y))\mathrm{d}y \\ &\le B_2 A_1^{-1} \int_0^x g(m(t,y))\mathrm{d}y \le C B_2 A_1^{-1} \int_0^x m(t,y)\mathrm{d}y \\ &\le K C B_2 A_1^{-1} \int_0^x \frac{1}{\frac{n_0(X_t^{-1}(y))}{m_0(X_t^{-1}(y))} t + C(m_0(X_t^{-1}(y)))} \mathrm{d}y \\ &\le C_u(t). \end{aligned}$$

Here we have taken advantage of (3.10) and the fact that

$$g(m) = \frac{m}{\rho_l - m} \le \frac{m(t,x)}{\rho_l - \sup(m_0(x))} \le C m(t,x),$$

for an appropriate choice of C and by an application of assumption (2.4). We also have used (3.6). Consequently, we can conclude that

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} \le C_u(t), \qquad t \ge 0, \tag{3.12}$$

and hence we have proved

Lemma 1 For $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and under the assumptions (2.4) we have the following pointwise estimates:

$$\begin{aligned} &0 \le m(t,x) \le \sup(m_0(x)) < \rho_l, \\ &0 \le n(t,x) \le \sup(s)\sup(m_0(x)) \le A_1^{-1} A_2 B_2, \\ &|u(t,x)| \le C_u(t). \end{aligned}$$

Note that this is sufficient to guarantee that $p(n,m) = P(m,s)$ is well-defined. We may sharpen the above estimates for m and n as follows when we consider a finite time $T > 0$.

Corollary 1 For any finite time T, there is an $\varepsilon = \varepsilon(T) > 0$ such that

$$0 < \varepsilon \le m(t,x), \qquad 0 < \varepsilon B_1 A_2^{-1} \le n(t,x),$$

for all $x \in [0,\infty)$ and $t \in [0,T]$.

Proof To see this, assume (for a fixed $T > 0$) that this is not the case. This implies that there exist points $\{x_k, t_k\}$ where $t_k \in [0,T]$ and a time $t \in [0,T]$ such that

$$m_k = m(t_k, x_k) \to 0, \qquad t_k \to t, \qquad \text{as } k \to \infty.$$

In light of (3.9), this implies that $t = \infty$, which produces a contradiction. The lower estimate of $n(t,x)$ follows from the relation $n(t,x) = s(t,x)m(t,x)$ given by (3.8) and its lower and upper bounds (3.10).

Remark 2 An interesting and relevant aspect of the model (1.1) (and simplified variants of this) is to obtain a clearer understanding when vacuum (transition to single-phase flow) might appear. The above calculation shows that for the model (2.1) and (2.2), we are guaranteed that vacuum will not occur in finite time. However, it's not clear to us to what extent this will be the case for more general slip relations and/or pressure laws.

Remark 3 If transition to single-phase liquid flow occurred in an appropriately modified two-phase model of the form (2.1) with pressure law (2.2), then this would correspond to a situation where m becomes equal to ρ_l, i.e., $p(m,n)=u_x$ blows up, say at time t^*. In this case we would have to describe precisely how the solution should be extended beyond the blow-up time t^*. Note that a similar phenomenon is observed for the Hunter-Saxton equation. This ambiguity gives rise to different notions of weak solution, see [22–24] and references therein.

We now consider the case with general initial data m_0, n_0 with compact support. Let $j_k(x)$, $k=1,2,\cdots$, be a standard Friedrich's mollifier satisfying

$$j_k(x)=kj(kx), \quad j(x)\in C_c^\infty(\mathbb{R}), \quad j(x)\geq 0, \quad \int_{\mathbb{R}} j(x)\mathrm{d}x=1.$$

We assume $1\leq m_0(\cdot)<\rho_l$, $0\leq n_0(\cdot)$, and $m_0,n_0\in L^p(\mathbb{R})$. Without loss of generality, we assume that $\mathrm{supp}(m_0)$, $\mathrm{supp}(n_0)\subset[0,1)$. Set

$$m_0^k(x)=j_k\star m_0(x), \qquad n_0^k(x)=j_k\star n_0(x).$$

For each fixed k, by the preceding calculation we find that (2.3) has a global smooth solution $m_k(t,x)$, $n_k(t,x)$ with initial data $m_0^k(x)$ and $n_0^k(x)$, respectively. Clearly, from the third equation of (2.3)

$$u_k=\int_0^x p(m_k,n_k)\mathrm{d}y.$$

We also note that in view of (3.12) we have

$$\|u_k(t,\cdot)\|_{L^\infty(\mathbb{R}^+)}\leq C_u, \qquad \mathrm{supp}(m_k(t,\cdot))\subset[0,K_m(t)),$$

where $K_m(t)$ is determined from the first equation of (3.1) which implies that $X_t(x)\leq C_u t$ so that $K_m(t)=C_u(t)t+1$ will do the job.

The plan now is to prove that the approximations $\{m_k(t,x),n_k(t,x)\}_{k\geq 1}$ are compact in $L^q([0,T]\times\mathbb{R}^+)$ for any $T>0$, $q>1$.

According to (3.8) we have

$$s_k(t,x)=\frac{n_k(t,x)}{m_k(t,x)},$$

and from (3.11) it follows

$$p(m_k,n_k)=\frac{n_k}{\rho_l-m_k}=s_k g(m_k)=P(m_k,s_k).$$

We have the following estimates:

Lemma 2 The approximate solutions $\{m_k(t,x)\}$ satisfy the following estimates:

$$\|m_k(t,\cdot)\|_{L^p(\mathbb{R}^+)}\leq\|m_0\|_{L^p(\mathbb{R}^+)}, \qquad p\geq 1, \tag{3.13}$$

$$\|m_k^p P(m_k,s_k)\|_{L^1(Q_\infty)}\leq\frac{1}{p-1}\|m_0\|^p_{L^p(\mathbb{R}^+)}, \qquad p>1, \tag{3.14}$$

$$\|P(m_k(t,\cdot),s_k(t,\cdot))\|_{L^1(\mathbb{R}^+)}\leq\|P(m_0,s_0)\|_{L^1(\mathbb{R}^+)}, \tag{3.15}$$

$$\|s_k^{-1}P(m_k,s_k)^3\|_{L^1(Q_\infty)}\leq\|P(m_0,s_0)\|_{L^1(\mathbb{R}^+)}, \tag{3.16}$$

$$\|P(m_k(t,\cdot),s_k(t,\cdot))^q\|_{L^1(\mathbb{R}^+)}\leq\|P(m_0,s_0)^q\|_{L^1(\mathbb{R}^+)}, \qquad q\geq 1, \tag{3.17}$$

$$\|P(m_k,s_k)^{q+1}\|_{L^1(Q_\infty)}\leq\frac{1}{q-1}\|P(m_0,s_0)^q\|_{L^1(\mathbb{R}^+)}, \qquad q>1, \tag{3.18}$$

$$\|s_k^{-1}P(m_k,s_k)^{q+2}\|_{L^1(Q_\infty)}\leq\frac{1}{q}\|P(m_0,s_0)^q\|_{L^1(\mathbb{R}^+)}, \qquad q\geq 1. \tag{3.19}$$

Proof Multiplying the first equation of (2.3) by $f'(m_k)$ we get

$$f(m_k)_t + u_k f(m_k)_x = -f'(m_k)m_k P(m_k, s_k).$$

This can also be written in the form

$$f(m_k)_t + (u_k f(m_k))_x = [f(m_k) - f'(m_k)m_k]P(m_k, s_k).$$

First, by using the choice $f(m) = m^p$ for $p \geq 1$ we get

$$(m_k^p)_t + (u_k m_k^p)_x = -(p-1)m_k^p P(m_k, s_k).$$

That is, by using that the support of $m_k(t,x)$ is contained in $[0, K(T)]$ for $t \in [0,T]$ we get

$$\int_0^t \int_0^{K(T)} \Big[(m_k^p)_t + (p-1)m_k^p P(m_k, s_k)\Big] \mathrm{d}x\mathrm{d}t = 0, \qquad p \geq 1.$$

From this, we get

$$\int_0^{K(T)} (m_k^p)\mathrm{d}x + \int_0^t \int_0^{K(T)} (p-1)m_k^p P(m_k, s_k)\mathrm{d}x\mathrm{d}t = \int_0^{K(T)} (m_0^p)\mathrm{d}x.$$

In particular, since both terms on the left hand side are non-negative,

$$\int_0^{K(T)} m_k^p(t,x)\mathrm{d}x \leq \int_{\mathbb{R}^+} m_0(x)^p \mathrm{d}x = \|m_0\|^p_{L^p(\mathbb{R}^+)}$$

and

$$(p-1)\int_0^T \int_0^{K(T)} m_k^p P(m_k, s_k)\mathrm{d}x\mathrm{d}t \leq \|m_0\|^p_{L^p(\mathbb{R}^+)}.$$

Thus, by letting $T \to \infty$, the estimates (3.13) and (3.14) follow. Next, we are interested in deriving an equation for the choice

$$f(m_k, s_k) = P(m_k, s_k) = s_k g(m_k) = s_k \frac{m_k}{\rho - m_k}.$$

Noting that

$$m_k g'(m_k) = g(m_k) + g(m_k)^2, \tag{3.20}$$

and from the two mass equations of (2.3) it follows that

$$(s_k)_t + u_k(s_k)_x = 0, \qquad s_k(t,x) = \frac{n_k(t,x)}{m_k(t,x)}. \tag{3.21}$$

Now we can calculate as follows:

$$\begin{aligned}
P(m_k, s_k)_t &= (s_k g(m_k))_t \\
&= s_k g'(m_k)(m_k)_t + g(m_k)(s_k)_t \\
&= -s_k g'(m_k)(m_k u_k)_x - g(m_k)u_k(s_k)_x \\
&= -(s_k)^2 g'(m_k)m_k g(m_k) - u_k s_k g(m_k)_x - u_k g(m_k)(s_k)_x \\
&= -(s_k)^2 g'(m_k)m_k g(m_k) - u_k(g(m_k)s_k)_x \\
&= -(s_k)^2[g(m_k) + g(m_k)^2]g(m_k) - u_k P(m_k, s_k)_x \\
&= -P(m_k, s_k)^2 - (s_k)^2 g(m_k)^3 - u_k P(m_k, s_k)_x,
\end{aligned} \tag{3.22}$$

and

$$(u_kP(m_k,s_k))_x=(u_k)_xP(m_k,s_k)+u_kP(m_k,s_k)_x=P(m_k,s_k)^2+u_kP(m_k,s_k)_x. \tag{3.23}$$

Here we have repeatedly used the third equation of (2.3) together with (3.20) and (3.21). Combining (3.22) and (3.23) we arrive at the following equation

$$P(m_k,s_k)_t+(u_kP(m_k,s_k))_x=-s_k^{-1}P(m_k,s_k)^3. \tag{3.24}$$

This implies the identity

$$\int_0^t\int_0^{K(T)}\Big[P(m_k,s_k)_t+s_k^{-1}P(m_k,s_k)^3\Big]\mathrm{d}x\mathrm{d}t=0,$$

from which we get

$$\int_0^{K(T)}P(m_k,s_k)\mathrm{d}x+\int_0^t\int_0^{K(T)}s_k^{-1}P(m_k,s_k)^3\mathrm{d}x\mathrm{d}t=\int_0^{K(T)}P(m_0,s_0)\mathrm{d}x.$$

From this equality the estimates (3.15) and (3.16) follow. Finally, consider the choice $f(m_k,s_k)=P(m_k,s_k)^q$ for $q\ge 1$. Then we calculate as follows:

$$\begin{aligned}(P(m_k,s_k)^q)_t&=qP(m_k,s_k)^{q-1}P(m_k,s_k)_t\\&=qP(m_k,s_k)^{q-1}\Big[-s_k^{-1}P(m_k,s_k)^3-(u_kP(m_k,s_k))_x\Big]\\&=qP(m_k,s_k)^{q-1}\Big[-s_k^{-1}P(m_k,s_k)^3-P(m_k,s_k)^2-u_kP(m_k,s_k)_x\Big],\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}(u_kP(m_k,s_k)^q)_x&=(u_k)_xP(m_k,s_k)^q+u_kqP(m_k,s_k)^{q-1}P(m_k,s_k)_x\\&=P(m_k,s_k)^{q+1}+u_kqP(m_k,s_k)^{q-1}P(m_k,s_k)_x.\end{aligned} \tag{3.26}$$

Adding (3.25) and (3.26) gives

$$(P(m_k,s_k)^q)_t+(u_kP(m_k,s_k)^q)_x=-(q-1)P(m_k,s_k)^{q+1}-qs_k^{-1}P(m_k,s_k)^{q+2}.$$

Hence, for $q\ge 1$,

$$\int_0^t\int_0^{K(T)}\Big[(P(m_k,s_k)^q)_t+(q-1)P(m_k,s_k)^{q+1}+qs_k^{-1}P(m_k,s_k)^{q+2}\Big]\mathrm{d}x\mathrm{d}t=0,$$

and the estimates (3.17), (3.18), and (3.19) follow.

By making use of the relation $n=sm$ and the estimates (3.10), we obtain the following k-uniform estimates for $n_k(t,x)$:

Corollary 2 The following estimates hold for the approximate solutions $n_k(t,x)$:

$$\|n_k(t,\cdot)\|_{L^p(\mathbb{R}^+)}\le B_2A_1^{-1}\|m_0\|_{L^p(\mathbb{R}^+)}.$$

4 Compactness

We begin by establishing strong compactness of the velocity $\{u_k(t,x)\}$. By Morrey's inequality,

$$W^{1,q}_{\mathrm{loc}}(Q_\infty) \hookrightarrow C^{1-\frac{2}{q}}_{\mathrm{loc}}(Q_\infty), \qquad \text{for } 2 < q \le \infty.$$

Consequently, we want to bound $\{u_k\}$ in $W^{1,q}_{\mathrm{loc}}(Q_\infty)$ for some $q > 2$. We have

$$u_k(t,x) = \int_0^x p(m_k,n_k)(y,t)\mathrm{d}y = \int_0^x P(m_k,s_k)(y,t)\mathrm{d}y.$$

In view of the pressure equation (3.24) we get

$$\begin{aligned}\partial_t u_k &= \int_0^x \partial_t P(m_k,s_k)(y,t)\mathrm{d}y \\ &= -\int_0^x (u_k P(m_k,s_k))_y \mathrm{d}y - \int_0^x s_k^{-1} P(m_k,s_k)^3 \mathrm{d}y.\end{aligned}$$

Consequently, for $(t,x) \in [0,T] \times \mathbb{R}^+$ (using that $u_k(t,0) = 0$)

$$\begin{aligned}|\partial_t u_k| &\le \max(|u_k|)P(m_k,s_k) + \int_0^{K_m(T)} s_k^{-1} P(m_k,s_k)^3 \mathrm{d}y \\ &\le C_u P(m_k,s_k) + \max(s_k^{-1})\|P(m_0,s_0)^3\|_{L^1(\mathbb{R}^+)},\end{aligned}$$

where we have applied (3.17) with $q = 3$. This implies that

$$|\partial_t u_k|^q \le C\Big(P(m_k,s_k)^q + \|P(m_0,s_0)^3\|^q_{L^1(\mathbb{R}^+)}\Big), \qquad q > 2.$$

That is, by (3.18) with $q-1$ as the exponent we get (note that $q > 2$)

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^+} |\partial_t u_k|^q \mathrm{d}x\mathrm{d}t \\ &\le C\bigg(\int_0^T \int_{\mathbb{R}^+} P(m_k,s_k)^q \mathrm{d}x\mathrm{d}t + K_m(T)T\|P(m_0,s_0)^3\|^q_{L^1(\mathbb{R}^+)} \bigg) \\ &\le C\bigg(\frac{1}{q-2}\|P(m_0,s_0)^{q-1}\|_{L^1(\mathbb{R}^+)} + K_m(T)T\|P(m_0,s_0)^3\|^q_{L^1(\mathbb{R}^+)} \bigg).\end{aligned}$$

Since $u_x = P(m,s)$, then (3.18) also implies that

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^+} |\partial_x u_k|^q \mathrm{d}x\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^+} P(m_k,s_k)^q \mathrm{d}x\mathrm{d}t \\ &\le \frac{1}{q-2}\|P(m_0,s_0)^{q-1}\|_{L^1(\mathbb{R}^+)}, \qquad q > 2.\end{aligned}$$

Hence, we can conclude (by Ascoli-Arzela and Banach-Sakes theorems) that there is some $u(t,x) \in W^{1,q}_{\mathrm{loc}}(Q_\infty)$ for $q > 2$ and a subsequence of $\{u_k(t,x)\}$ such that $u_k(t,x)$ converges to $u(t,x)$ uniformly on any compact subset of Q_∞. Furthermore, $\partial_x u_k(t,x) = P(m_k(t,x), s_k(t,x))$ converges weakly to a limit function

$$v(t,x) = \overline{P(m(t,x),s(t,x))} \quad \text{in } L^q(Q_\infty) \text{ for } q > 2.$$

Furthermore, it is clear that

$$1 \le \partial_x X_t^k(x) \le C, \qquad |\partial_t X_t^k(x)| \le C.$$

The first estimate follows from (3.7) and the upper bound on $P(m_k, s_k)$ (cf. Lemma 1). Similarly, the second estimate follows from the first equation of (3.1) and the pointwise upper bound on u_k, also guaranteed by Lemma 1. Consequently, we have uniform Hölder continuity in space and time for $\{X_t^k(x)\}$, i.e., the sequence converges to a limit function $X_t(x)$ uniformly on compact sets in Q_∞. Clearly, the same properties hold for the inverse $Y_t^k(\cdot) = (X_t^k)^{-1}(\cdot)$, such that $Y_t^k(\cdot) \to Y_t(\cdot)$ uniformly on compact sets in Q_∞ where for each t, and $Y_t(\cdot) = X_t^{-1}(\cdot)$.

Since $s_k(t,x) = \frac{n_0^k}{m_0^k}((X_t^k)^{-1}(x))$, see (3.8), we conclude that $s_k(t,x)$ converges a.e. to the limit function $s(t,x) = \frac{n_0}{m_0}(X_t^{-1}(x))$.

To sum up, we have the following lemma.

Lemma 3 (Compactness) Regarding the initial data m_0 and n_0, we assume

$$\begin{aligned} &0 < \varepsilon \le m_0(x) < \rho_l, \\ &0 < \varepsilon \le n_0(x) < \infty, \\ &m_0 \in L^2(\mathbb{R}^+), \qquad P(m_0, s_0) = p(m_0, n_0) \in L^2(\mathbb{R}^+), \end{aligned}$$

for some $\varepsilon > 0$. We have the following basic convergence result towards limit functions (m, n, u, v, w) as $k \to \infty$:

$$\begin{aligned} &u_k \to u \text{ uniformly in } [0,R] \times [0,T] \text{ for each } R > 0 \text{ and pointwise in } \overline{Q_T} \\ &\text{and the limit function } u \text{ belongs to } W^{1,q}(\overline{Q_T}) \hookrightarrow C_{\rm loc}^{1-\frac{2}{q}}(\overline{Q_T}) \text{ for } q > 2; \end{aligned} \tag{4.1}$$

$$\begin{aligned} &s_k \to s \text{ a.e. in } Q_\infty \text{ and} \\ &s(t,x) = \frac{n_0}{m_0}(X_t^{-1}(x)), \text{ i.e., } \frac{\rm d}{{\rm d}t} s(t, X_t(x)) = 0, \ \text{ that is, } s_t + u s_x = 0; \end{aligned} \tag{4.2}$$

$$\begin{aligned} &m_k \rightharpoonup m \text{ in } L^p(Q_T), \qquad p \ge 1; \\ &n_k \rightharpoonup n \text{ in } L^p(Q_T), \qquad p \ge 1; \end{aligned} \tag{4.3}$$

$$\begin{aligned} v_k &:= P(m_k, s_k) = s_k g(m_k) = \partial_x u_k \\ &\rightharpoonup \partial_x u = v = s\overline{g(m)} \text{ in } L^q(Q_T), \quad q \ge 1; \end{aligned} \tag{4.4}$$

$$(v_k)^3 = (P(m_k, s_k))^3 = (s_k)^3 g(m_k)^3 \rightharpoonup s^3 w \text{ in } L^1(Q_T); \tag{4.5}$$

$$\begin{aligned} &u_k m_k \rightharpoonup um \text{ in } L^p(Q_T), \quad p \ge 1; \\ &u_k n_k \rightharpoonup un \text{ in } L^p(Q_T), \qquad p \ge 1. \end{aligned} \tag{4.6}$$

Finally, the limit functions m, v, s, w are related by the inequalities

$$P(m, s) = s g(m) \le v, \qquad v^3 \le s^3 w, \tag{4.7}$$

or equivalently,

$$g(m) \le \overline{g(m)}, \qquad \Big(\overline{g(m)}\Big)^3 \le w.$$

Proof The limit operations (4.1) and (4.2) follow from the above discussion, whereas (4.3), (4.4), (4.5), (4.6) follow from the estimates of Lemma 2 and Corollary 2. The relations (4.7) rely on the convexity and continuity properties of $g(\cdot)$ and $(\cdot)^3$.

We are now in a position to prove strong convergence of m_k, n_k by analyzing a particular renormalization (in the sense of Diperna-Lions) of the approximate solutions and their limits. Strong convergence ensures that the weak limit functions m and n solve the original equations.

Lemma 4 (Limit equations) The limit functions (m, n, s, u, v, w) from Lemma 3 satisfy

$$m_t + (um)_x = 0, \qquad n_t + (un)_x = 0, \qquad s_t + us_x = 0, \qquad u_x = v, \tag{4.8}$$

in the sense of distributions on Q_T, and

$$m \in C([0,T]; L^p(\mathbb{R}^+)), \qquad \lim_{t\to 0} \|m(\cdot,t) - m_0\|_{L^p(\mathbb{R}^+)} = 0,$$

$$n \in C([0,T]; L^p(\mathbb{R}^+)), \qquad \lim_{t\to 0} \|n(\cdot,t) - n_0\|_{L^p(\mathbb{R}^+)} = 0,$$

for any $p \geq 1$. Moreover,

$$v_t + (uv)_x = -s^{-1}[s^3 w] \tag{4.9}$$

in the sense of distributions on Q_T and

$$\lim_{t\to 0} \int_0^\infty \big(v(t,x) - P(m_0(x), s_0(x))\big)\mathrm{d}x = 0.$$

Proof The approximate solutions (m_k, n_k, u_k) satisfy the system

$$\begin{aligned}
&\partial_t m_k + \partial_x(u_k m_k) = 0,\\
&\partial_x u_k = p(m_k, n_k) = P(m_k, s_k),\\
&n_k(t,x) = m_k(t,x)\frac{n_0(X_{k,t}^{-1}(x))}{m_0(X_{k,t}^{-1}(x))} = m_k(t,x)s_k(t,x).
\end{aligned}$$

In view of Lemma 3, it follows that (4.8) holds. Similarly, (4.9) follows from the pressure equation (3.24).

Lemma 5 (Identification) Suppose that

(i) $u(t,x)$ is bounded and continuous in $\overline{Q_T}$ with $u(0,t) = 0$ for $t \in [0,T]$, $m \in L^\infty((0,T); L^p(\mathbb{R}^+))$, and $m \geq 0$ a.e. in Q_T;

(ii) $v \in L^\infty((0,T); L^p(\mathbb{R}^+))$ and $P(m,s) \leq v$ a.e. in Q_T;

(iii) $w \in L^\infty((0,T); L^1(\mathbb{R}^+))$, and $v^3 \leq s^3 w$ a.e. in Q_T;

(iv) As $t \to 0$,

$$\int_0^\infty \big(v(t,x) - P(m(t,x), s(t,x))\big)\mathrm{d}x \to 0; \tag{4.10}$$

(v) The limit functions u, m, n, s, v, w satisfy the system

$$m_t + (um)_x = 0, \quad n_t + (un)_x = 0, \quad s_t + us_x = 0, \quad u_x = v, \tag{4.11}$$

$$v_t + (uv)_x = -s^{-1}[s^3 w], \tag{4.12}$$

in the sense of distributions on Q_T.

Then $P(m,s)=sg(m)=v$ a.e. in Q_T.

Proof The proof follows along standard lines in the theory of renormalized solutions. We set $m^\varepsilon=m\star\omega^\varepsilon$, $n^\varepsilon=n\star\omega^\varepsilon$, $s^\varepsilon=s\star\omega^\varepsilon$, $v^\varepsilon=v\star\omega^\varepsilon$, and $w^\varepsilon=w\star\omega^\varepsilon$ where ω^ε is a standard mollifier acting on the spatial variable. In view of (4.11) together with an application of the Diperna-Lions lemma we get

$$m_t^\varepsilon+um_x^\varepsilon=-v^\varepsilon m^\varepsilon+R^\varepsilon, \tag{4.13}$$

where $R^\varepsilon=u(m^\varepsilon)_x-(um_x)\star\omega^\varepsilon+v^\varepsilon m^\varepsilon-(vm)\star\omega^\varepsilon$ and

$$R^\varepsilon\to 0\quad \text{in } L^p(Q_T) \text{ for any } p\geq 1.$$

Having this regularized version of the first equation in (4.11), the plan is now to derive from this an equation that contains information about $P(m,s)$. First, we multiply (4.13) with $g'(m^\varepsilon)$ and rewrite (using $u_x=v$) such that we get

$$g(m^\varepsilon)_t+ug(m^\varepsilon)_x=-v^\varepsilon m^\varepsilon g'(m^\varepsilon)+R^\varepsilon g'(m^\varepsilon).$$

Then we multiply by s^ε and get

$$[s^\varepsilon g(m^\varepsilon)]_t+u[s^\varepsilon g(m^\varepsilon)]_x-g(m^\varepsilon)[s_t^\varepsilon+us_x^\varepsilon]=-v^\varepsilon m^\varepsilon s^\varepsilon g'(m^\varepsilon)+R^\varepsilon g'(m^\varepsilon)s^\varepsilon,$$

or

$$P(m^\varepsilon,s^\varepsilon)_t+(uP(m^\varepsilon,s^\varepsilon))_x-g(m^\varepsilon)[s_t^\varepsilon+us_x^\varepsilon]=vP(m^\varepsilon,s^\varepsilon)-v^\varepsilon m^\varepsilon s^\varepsilon g'(m^\varepsilon)+R^\varepsilon g'(m^\varepsilon)s^\varepsilon.$$

Sending $\varepsilon\to 0$ we get

$$P(m,s)_t+(uP(m,s))_x=vP(m,s)-vmsg'(m),$$

in the sense of distributions. Using (3.20) and $P(m,s)=sg(m)$, we get

$$\begin{aligned}P(m,s)_t+(uP(m,s))_x&=vP(m,s)-vs[g(m)+g(m)^2]\\&=-vsg(m)^2=-s^{-1}vP(m,s)^2.\end{aligned} \tag{4.14}$$

Taking the difference between (4.12) and (4.14) we get

$$\begin{aligned}\partial_t[v-P(m,s)]+\partial_x(u[v-P(m,s)])&=s^{-1}(vs^2g(m)^2-s^3w)\\&=s^{-1}(vP(m,s)^2-s^3w)\\&\leq s^{-1}(v^3-s^3w)\leq 0,\end{aligned} \tag{4.15}$$

using the relations $P(m,s)=sg(m)\leq v$ and $v^3\leq s^3w$, see (4.7). Recalling that (4.15) holds in the sense of distributions we can choose a test function $\psi(t,x)=\omega_1(t)\omega_2(x)$ and then let $\omega_1(t)$ be a smooth approximation to $\chi_{[t_1,t_2]}$ for $t_1<t_2$ whereas $\omega_2(x)=1$. Then (4.15) simplifies to

$$-\int_0^T\int_0^\infty\Big(v(\tau,x)-P(m(\tau,x),s(\tau,x))\Big)\omega_1'(\tau)\mathrm{d}x\,\mathrm{d}\tau\leq 0,$$

that is,

$$
\begin{aligned}
&-\int_0^\infty \Big(v(t_1,x) - P(m(t_1,x), s(t_1,x))\Big)\mathrm{d}x \\
&+\int_0^\infty \Big(v(t_2,x) - P(m(t_2,x), s(t_2,x))\Big)\mathrm{d}x \le 0.
\end{aligned}
$$

Letting $t_1 \to 0$ and $t_2 \to t$ and comparing with (4.10), we get

$$
v(t,x) - P(m(t,x), s(t,x)) \le 0 \quad \text{for a.e. } (t,x) \in Q_T,
$$

which implies that $v = P(m,s)$ a.e. in Q_T, in view of (4.7).

Proof of Theorem 1 The existence result of Theorem 1 now follows as a result of Lemmas 3, 4, and 5.

5 A Uniqueness Result in the $L^\infty \cap BV$ Class

It was proved in Section 4 that the smooth solutions (m_k, n_k, u_k) converge to a weak solution (m, n, u) of (2.3). Now we want to prove that this weak solution, which satisfies

$$
0 \le m(\cdot,\cdot) \le C_m < \rho_l, \quad 0 \le n(\cdot,\cdot) \le C_n, \quad |u(\cdot,\cdot)| \le C_u, \tag{5.1}
$$

possesses spatial BV regularity provided the initial data do so; more precisely,

$$
m_0, n_0 \in BV(\mathbb{R}) \Longrightarrow m, n \in L^\infty(0,T; BV(\mathbb{R})), \quad T > 0. \tag{5.2}
$$

To this end, it is sufficient to establish an estimate of the form

$$
\int_{\mathbb{R}} |\partial_x m_k(t,x)| + |\partial_x n_k(t,x)| \,\mathrm{d}x \le C_T, \qquad t \in (0,T),
$$

for some constant C_T that is independent of k.

Set

$$
q_k^m = \partial_x m_k, \qquad q_k^n = \partial_x n_k.
$$

Then

$$
\begin{aligned}
&\partial_t q_k^m + u \partial_x q_k^m + p(m_k, n_k) q_k^m \\
&= -q^m P(m_k, n_k) - m_k \left(p_m(m_k, n_k) q_k^m + p_n(m_k, n_k) q_k^n\right)
\end{aligned}
$$

Multiplying by $\mathrm{sgn}(q^m)$ yields

$$
\begin{aligned}
&|q_k^m|_t + u_k \,|q_k^m|_x + p(m_k, n_k)\,|q_k^m| \\
&= -\,|q_k^m|\, p(m_k, n_k) - m_k p_m(m_k, n_k)\,|q_k^m| - m_k p_n(m_k, n_k) q_k^n \mathrm{sgn}(q_k^m) \\
&\le \|m\|_{L^\infty} \|p_n(m,n)\|_{L^\infty} |q^n| \le C\,|q^n|,
\end{aligned}
$$

where we have used that

$$
p = \frac{n}{\rho_l - m}, \quad p_m = \frac{n}{(\rho_l - m)^2}, \quad p_n = \frac{1}{\rho_l - m}
$$

are all (nonnegative) bounded quantities. In divergence form, this equation inequality reads

$$|q_k^m|_t + (u_k\,|q_k^m|)_x \le C_m\,|q_k^n|\,,$$

where the constant C_m is independent of k.

Similarly, it follows that

$$|q_k^n|_t + (u_k\,|q_k^n|)_x \le C_n\,|q_k^m|\,,$$

for some constant C_n that does not depend on k.

Adding the inequalities for $|q_k^m|$ and $|q_k^n|$ yields

$$(|q_k^n| + |q_k^n|)_t + (u_k\,(|q_k^m| + |q_k^n|))_x \le C\,(|q_k^n| + |q_k^m|)\,, \qquad C := C_m + C_n,$$

and thus by Gronwall's inequality,

$$\int_{\mathbb{R}} \Big(\,|\partial_x m_k(t,x)| + |\partial_x n_k(t,x)|\,\Big)\mathrm{d}x \le e^{Ct}\int_{\mathbb{R}} \Big(\,|\partial_x m_k(0,x)| + |\partial_x n_k(0,x)|\,\Big)\mathrm{d}x,$$

and the claims follows.

Let us now turn to the uniqueness of weak solutions in the $L^\infty \cap BV$ class, i.e., the proof of Theorem 2. In view of the assumptions, the solutions (m^i, n^i, u^i) satisfy (5.1) and (5.2). By the DiPerna-Lions regularization lemma, one can prove that the weak solutions (m_i, n_i, u_i) of (2.5) are entropy solutions, i.e., for any convex $\eta : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned}
&\eta(m^i) + \big(u^i\eta(m^i)\big)_x + p(m^i,n^i)\left[m^i\eta'(m^i) - \eta(m^i)\right] \le 0,\\
&\eta(n^i) + \big(u^i\eta(n^i)\big)_x + p(m^i,n^i)\left[n^i\eta'(n^i) - \eta(n^i)\right] \le 0,
\end{aligned}$$

in the sense of distributions.

Now the uniqueness of weak solutions is an immediate consequence of a result proved in [17] regarding continuous dependence of entropy solutions with respect to the flux function. We may apply Theorem 1.3 in [17] to conclude that there exists a constant C such that

$$\begin{aligned}
\left\|m^1(t,\cdot) - m^2(t,\cdot)\right\|_{L^1} &\le \left\|m_0^1 - m_0^2\right\|_{L^1} + \int_0^t\!\!\int \left|u_x^1(s,x) - u_x^2(s,x)\right|\mathrm{d}x\mathrm{d}s\\
&\quad + \int_0^t \left\|u^1(s,\cdot) - u^2(s,\cdot)\right\|_{L^\infty}\left|m^2(s,\cdot)\right|_{BV}\mathrm{d}s\\
&\le \left\|m_0^1 - m_0^2\right\|_{L^1} + \int_0^t\!\!\int_{\mathbb{R}} \left|p(m^1,n^1) - p(m^2,n^2)\right|\mathrm{d}x\mathrm{d}s\\
&\quad + \tilde{C}\int_0^t\!\!\int \left|p(m^1,n^1) - p(m^2,n^2)\right|\mathrm{d}x\mathrm{d}s\\
&\le \left\|m_0^1 - m_0^2\right\|_{L^1} + C_m\int_0^t\!\!\int \Big(\left|m^1 - m^2\right| + \left|n^1 - n^2\right|\Big)\mathrm{d}x\mathrm{d}s. \qquad (5.3)
\end{aligned}$$

Similarly,

$$\left\|n^1(t,\cdot) - n^2(t,\cdot)\right\|_{L^1} \le \left\|n_0^1 - n_0^2\right\|_{L^1} + C_n\int_0^t\!\!\int_{\mathbb{R}} \Big(\left|m^1 - m^2\right| + \left|n^1 - n^2\right|\Big)\mathrm{d}x\mathrm{d}s. \qquad (5.4)$$

By adding the two inequalities (5.3), (5.4) and, following this, applying the Gronwall inequality, we arrive at

$$\begin{aligned}&\left\|m^1(t,\cdot)-m^2(t,\cdot)\right\|_{L^1}+\left\|n^1(t,\cdot)-n^2(t,\cdot)\right\|_{L^1}\\ &\le e^{Ct}\left(\left\|m_0^1-m_0^2\right\|_{L^1}\left\|n_0^1-n_0^2\right\|_{L^1}\right), \qquad C:=C_m+C_n.\end{aligned}$$

This concludes the proof of Theorem 2.

6 Concluding Remarks

In this work we have investigated a simplified no-pressure gas-liquid model which is composed of two continuity equations for the two phases and a steady state momentum equation which represents the balance between the pressure gradient and a viscous term. We derive pointwise upper and lower bounds on the masses which guarantee that the initial two-phase mixture remains a two-phase mixture, i.e., no transition to single-phase flow occurs in finite time. Existence of weak solutions is shown under minimal regularity on the initial masses. Moreover, a uniqueness result is derived by requiring that the the initial masses are BV bounded.

Interesting extensions of the model studied in this work would be to take into account that the two phases can move with different fluid velocities, consider inclusion of more general pressure laws, as well as take into account terms representing external forces like gravity and friction.

It is difficult to find solutions of the system (2.3) without resorting to numerical methods. Fortunately, it is possible to devise very simple finite difference schemes for computing approximate solutions of (2.3). To this end, introduce the spatial grid cells $I_j=[x_{j-1/2},x_{j+1/2})$, where $x_{j\pm1/2}=x_j\pm\Delta x/2$, $j\in\mathbb{N}_0:=\{0,1,\cdots\}$. The forward/backward difference operators are denoted by D_+/D_-, respectively. Let $\left\{m_j^0\right\}_{j\in\mathbb{N}_0}$, $\left\{n_j^0\right\}_{j\in\mathbb{N}_0}$ be discrete initial data satisfying

$$0\le m_j^0(x)<\rho_l, \qquad 0\le n_j^0\le \text{Const}<\infty, \qquad j\in\mathbb{N}_0.$$

For $j\in\mathbb{N}_0$ and $t\in[0,T]$, let $\left\{(m_j(t),n_j(t),u_j(t))\right\}_{j\in\mathbb{N}_0}$ satisfy the finite system of ordinary differential equations

$$\begin{aligned}&m_j'(t)+u_j(t)D_-m_j(t)=-m_j(t)p(m_j(t),n_j(t)),\\ &n_j'(t)+u_j(t)D_-n_j(t)=-n_j(t)p(m_j(t),n_j(t)),\\ &D_+u_j(t)=p(m_j(t),n_j(t)), \qquad u_0(t)=0,\\ &m_j|_{t=0}=m_j^0, \qquad n_j|_{t=0}=n_j^0,\end{aligned} \tag{6.1}$$

It follows that

$$u_j(t)=\Delta x\sum_{i=0}^{j-1}p(m_i(t),n_i(t)), \qquad \text{for } j\in\mathbb{N},\ t\in[0,T].$$

By properly adapting the BV arguments from Section 5 to the discrete setting, one can prove that the numerical scheme (6.1) converges strongly in L^p, $p<\infty$, to the unique $L^\infty\cap BV$ weak solution of (2.3). The details will be presented elsewhere.

The numerical scheme is of upwind type, which means that the finite differencing of the transport terms um_x, un_x are biased in the direction of incoming waves. The use of upwind schemes is quite natural, since they dissipate energy and hence generate dissipative solutions in the limit as $\Delta x \to 0$. We stress that the convergence result is not obvious. Indeed, whereas the BV approach works, it is not clear how to apply the "weak convergence" argument used in the existence proof to the numerical approximations; the reason seems to be that the argument relies too heavily on the non-dissipative nature of the approximate solutions.

The semi-discrete method (6.1) amounts to a system of ordinary differential equations which must be solved by some numerical method. There are a variety of time-discretization methods available. The simplest and most obvious one is an explicit discretization, i.e., we replace the time derivatives in (6.1) by forward differences,

$$m_j'(t) \to \frac{m_j^{\ell+1} - m_j^{\ell}(t)}{\Delta t}, \qquad n_j'(t) \to \frac{n_j^{\ell+1} - n_j^{\ell}(t)}{\Delta t},$$

while the remaining part of (6.1) are evaluated at $t^\ell := \ell\Delta t$, $\ell = 0, 1, \cdots$. It can be shown that this fully discrete difference scheme also converges to the $L^\infty \cap BV$ weak solution of (2.3).

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