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ON THE WAVE STRUCTURE OF TWO-PHASE FLOW MODELS*

STEINAR EVJE† AND TORE FLÅTTEN‡

Abstract. We explore the relationship between two common two-phase flow models, usually denoted as the *two-fluid* and *drift-flux* models. They differ in their mathematical description of momentum transfer between the phases. In this paper we provide a framework in which these two model formulations are unified. The drift-flux model employs a mixture momentum equation and treats interphasic momentum exchange indirectly through the *slip relation*, which gives the relative velocity between the phases as a function of the flow parameters. This closure law is in general highly complex, which makes it difficult to analyze the model algebraically. To facilitate the analysis, we express the quasi-linear formulation of the drift-flux model as a function of three parameters: the derivatives of the slip with respect to the vector of unknown variables. The wave structure of this model is investigated using a perturbation technique. Then we rewrite the drift-flux model with a general slip relation such that it is expressed in terms of the canonical two-fluid form. That is, we replace the mixture momentum equation and the slip relation with equivalent evolution equations for the momentums of each phase. We obtain a mathematically equivalent formulation in terms of a two-fluid model and by this bridge some of the gap between the drift-flux model and the two-fluid model. Finally, the effect of the various exchange terms on the wave structure of the two-fluid model is investigated.

Key words. hyperbolic system of conservation laws, two phase flow, drift-flux model, two-fluid model, perturbation method, eigenvalues, interface friction

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1. Introduction.

In general, multiphase flows exhibit a complex dynamical behavior, where depending on the physical parameters several different *flow regimes* may occur. Flow regimes are commonly divided into *separated* (stratified, annular) and *mixed* (bubbly, dispersed) flows.

There exists no simple model formulation able to describe all these phenomena adequately. Rather, a variety of different models have been suggested with different applications in mind; see, for instance, [6, 8, 23, 24].

A classical way to obtain tractable models is to average in space. Of such models, two particular strategies have attracted considerable interest in the petroleum industry: the *two-fluid* [4, 20] and *drift-flux* [22] models. These models, described in sections 2 and 3, are the focus of the current paper.

The models contain a significant amount of additional closure laws. These closure relations typically depend on the flow structure and represent the main difficulty in the model formulation.

As noted by Bouré [9], the effect of closure relations may be viewed on two different levels:

1. Their *physical magnitude* affects the predicted values of the flow parameters.

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†International Research Institute of Stavanger (IRIS), Prof. Olav Hanssensvei 15, NO-4091 Stavanger, Norway (steinar.evje@irisresearch.no).

‡Centre of Mathematics for Applications (CMA), 1053 Blindern, NO-0316 Oslo, Norway (tore.flatten@cma.uio.no).

2. Their *mathematical form* affects the propagation properties of the flow model. That is, differential closure terms affect the velocities and composition of the predicted waves, whereas nondifferential terms do not.

The drift-flux model and its closure relations are commonly formulated to model *mixed* flow regimes. Depending on the closure relations, the two-fluid model has more general validity. In its most basic form, it is nevertheless best suited for a description of *separated* flows. These different domains of applicability manifest themselves through the different wave structures of the common formulation of the two models.

The purpose of this paper is twofold.

(I) Primary aim. To demonstrate how *nondifferential* closure relations for the drift-flux model may be transformed into corresponding *differential* relations in the two-fluid framework. By this transformation, we obtain a two-fluid model whose underlying mathematical structure is identical to the original drift-flux model. Hence it becomes possible to alternate between the two formulations within a unified framework.

(II) Secondary aim. To demonstrate how the wave structure of the drift-flux model may be investigated by a perturbation technique, first applied to two-phase flows by Toumi and coworkers [27, 28], who considered the two-fluid model.

The paper is organized as follows: In sections 2 and 3 we describe the two-fluid and drift-flux models in question. Section 4 is dedicated to the secondary aim of the paper; here we investigate the wave structure of the drift-flux model.

In section 5 we confront the primary aim of our paper, writing the drift-flux model in the framework of a two-fluid model. A main result is equation (118), the explicit form of the interface friction that makes the two-fluid model mathematically equivalent to a general drift-flux model.

Armed with a thorough understanding of the mathematical structure of both models, we demonstrate in section 6 how the wave velocities of the two-fluid model gradually change by addition of the different terms of (118). This illustrates the *physical effects* of the different closure terms on the wave phenomena inherent in the models.

2. Two-fluid model. To be consistent with the dynamical behavior of the flow physics, the two-phase models we consider must describe the following wave phenomena:

- *Sonic waves*, conveying rapid variations in the pressure and the associated velocity fields. They are a consequence of the compressibility of the flow.
- *Material waves*, conveying large scale variations in the volumetric phase fractions and mixture density. They are responsible for the dynamics corresponding to mass transport.
- *Entropy waves*, representing thermodynamic properties transported along the flow.

As noted, for instance, by [9, 27], the entropy waves are uncoupled from the remaining wave structure. Phasic entropies are simply advected with the fluid velocities.

Hence the structure of the sonic and material waves may be studied with no loss of generality by considering only *isentropic* flow models. Such models are based on the physical principle of conservation of the mass and momentum variables, neglecting dynamic energy transfers.

Supplemented by proper closure relations, the models hence consist of mass and momentum balance equations, expressed in the form of *partial differential* equations.

2.1. Model formulation. For a gas (g) and a liquid (ℓ) phase, the isentropic two-fluid model may be written as follows:

- Conservation of mass

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}) + \frac{\partial}{\partial x}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho_{\ell}\alpha_{\ell}) + \frac{\partial}{\partial x}(\rho_{\ell}\alpha_{\ell}v_{\ell}) = 0, \tag{2}$$

- Momentum balances

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}) + \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}^2\right) + \frac{\partial}{\partial x}(\alpha_{\mathrm{g}}p_{\mathrm{g}}) - p^{\mathrm{i}}\frac{\partial}{\partial x}(\alpha_{\mathrm{g}}) = Q_{\mathrm{g}} + M_{\mathrm{g}}^{\mathrm{i}}, \tag{3}$$

$$\frac{\partial}{\partial t}(\rho_{\ell}\alpha_{\ell}v_{\ell}) + \frac{\partial}{\partial x}\left(\rho_{\ell}\alpha_{\ell}v_{\ell}^2\right) + \frac{\partial}{\partial x}(\alpha_{\ell}p_{\ell}) - p^{\mathrm{i}}\frac{\partial}{\partial x}(\alpha_{\ell}) = Q_{\ell} + M_{\ell}^{\mathrm{i}}. \tag{4}$$

Here α_k is the volume fraction of phase k with

$$\alpha_{\mathrm{g}} + \alpha_{\ell} = 1, \tag{5}$$

where ρ_k, p_k, and v_k denote the density, pressure, and fluid velocity of phase k, respectively, and p^{i} is the pressure at the gas-liquid interface. M_k^{i} represents interphasic momentum exchange terms with $M_{\mathrm{g}}^{\mathrm{i}} + M_{\ell}^{\mathrm{i}} = 0$. Momentum sources acting on each phase separately, such as wall friction or gravitational forces, are represented by the terms Q_k.

2.2. Closure relations. The closure relations needed to complete the model may be divided into three groups.

2.2.1. Thermodynamic submodels. For each phase k, the thermodynamic *state relation*

$$p_k = p(\rho_k, S_k) \tag{6}$$

must be specified. Here S_k is the entropy of phase k. Furthermore, the interface pressure p^{i} must be expressed as a function of the phasic pressures:

$$p^{\mathrm{i}} = p^{\mathrm{i}}(p_{\mathrm{g}}, p_{\ell}). \tag{7}$$

When the flows are separated due to gravitational forces, the relationships between the pressures p^{i}, p_{g}, and p_{ℓ} are commonly chosen to model the effects of *hydrostatics*. In this case, the two-fluid model is able to describe travelling surface waves on the gas-liquid interface; see, for instance, [2].

2.2.2. Phase-specific source terms. The main momentum sources acting on each phase separately are the following:

- *Gravity.*
  The effect of gravitational acceleration is expressed by

$$Q_k = -\rho_k\alpha_k g\sin\theta, \tag{8}$$

  where θ is the angle of the flow direction with respect to the horizontal.

- *Wall friction.*
  For separated flows, the wall friction for each phase is commonly expressed in terms of *friction factors* as follows:

$$Q_k = -f_k \frac{\rho_k |v_k| v_k}{2}. \tag{9}$$

  The Blasius equation is commonly used for calculating f_k; see, for instance, [1, 25]. According to [7], most *mixed* flow regimes may be modeled to acceptable accuracy by using friction factors corresponding to one-phase liquid flow ($f_{\mathrm{g}} = 0$).

2.2.3. Interphasic momentum exchange terms. The interactions between the phases are highly complex and different in character for each flow regime. Hence these terms are notoriously difficult to derive from theoretical considerations. Nor are they easily determined from experimental data, as their effects are only indirectly visible. We here briefly describe two of the most common approaches for modeling the interphasic momentum exchange, applied to separated and mixed flows, respectively.

- *Stratified flows.*
  For stratified flows it is common [1, 25] to express the interphasic momentum exchange in nondifferential form, as a function of a *friction factor* f_{i}:

$$M_\ell^{\mathrm{i}} = -M_{\mathrm{g}}^{\mathrm{i}} = f_{\mathrm{i}} \frac{\rho_{\mathrm{g}} |v_{\mathrm{g}} - v_\ell| (v_{\mathrm{g}} - v_\ell)}{2}. \tag{10}$$

  Andritsos and Hanratty [1] noted that waves existing on the gas-liquid interface have a significant effect on the magnitude of f_{i}. They suggested that for sufficiently small gas flow rates $\alpha_{\mathrm{g}} v_{\mathrm{g}} < U_{\mathrm{crit}}$, such that no waves are generated at the interface,

$$f_{\mathrm{i}} \approx f_{\mathrm{g}}. \tag{11}$$

  For $\alpha_{\mathrm{g}} v_{\mathrm{g}} > U_{\mathrm{crit}}$ they developed a correlation where $f_{\mathrm{i}}/f_{\mathrm{g}}$ increases linearly with $\alpha_{\mathrm{g}} v_{\mathrm{g}}$.
- *Bubbly flows.* For a two-phase mixture of gas dispersed within the liquid, the momentum transfer induced by a gas bubble *accelerating* with respect to the surrounding fluid must be taken into account. This effect, denoted as the *virtual mass force*, has been analyzed by Drew, Cheng, and Lahey [10]. By imposing the condition that this interface friction is invariant under a change of reference frame, they derived the expression

$$\begin{aligned} M_{\mathrm{g}}^{\mathrm{i}} = \alpha_{\mathrm{g}} \rho_\ell C_{\mathrm{vm}} \Big(& \partial_t (v_{\mathrm{g}} - v_\ell) + v_{\mathrm{g}} \partial_x (v_{\mathrm{g}} - v_\ell) \\ & + (v_{\mathrm{g}} - v_\ell) ((\lambda - 2) \partial_x v_{\mathrm{g}} + (1 - \lambda) \partial_x v_\ell) \Big), \end{aligned} \tag{12}$$

  where λ and C_{vm} (the coefficient of virtual mass) are volume fraction dependent parameters. The value of C_{vm} is expected to be 1/2 for noninteracting spheres and smaller for bubbles of other shapes.
  The wave structure of the two-fluid model with virtual mass force included has been analyzed in [18, 19, 28]. In particular, Lahey [19] discusses similarities between such a two-fluid model and the drift-flux model.

2.3. Canonical formulation. The multitude of possible closure relations gives rise to a large class of slightly different models, all falling under the heading of *two-fluid models*. In the following, we will find it useful to base our analyses on some common formulation of these models. By neglecting the phasic pressure difference ($p = p_{\mathrm{g}} = p_\ell$) and writing

$$\tau_{\mathrm{i}} = \left(p - p^{\mathrm{i}}\right) \frac{\partial \alpha_{\mathrm{g}}}{\partial x} - M_{\mathrm{g}}^{\mathrm{i}} = -\left(p - p^{\mathrm{i}}\right) \frac{\partial \alpha_\ell}{\partial x} + M_\ell^{\mathrm{i}}, \tag{13}$$

we arrive at the following *canonical* two-fluid model:

- Conservation of mass

$$\frac{\partial}{\partial t}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}\right) + \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}\right) = 0, \tag{14}$$

$$\frac{\partial}{\partial t}\left(\rho_\ell\alpha_\ell\right) + \frac{\partial}{\partial x}\left(\rho_\ell\alpha_\ell v_\ell\right) = 0, \tag{15}$$

- Momentum balances

$$\frac{\partial}{\partial t}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}\right) + \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}^2\right) + \alpha_{\mathrm{g}}\frac{\partial}{\partial x}(p) + \tau_{\mathrm{i}} = Q_{\mathrm{g}}, \tag{16}$$

$$\frac{\partial}{\partial t}\left(\rho_\ell\alpha_\ell v_\ell\right) + \frac{\partial}{\partial x}\left(\rho_\ell\alpha_\ell v_\ell^2\right) + \alpha_\ell\frac{\partial}{\partial x}(p) - \tau_{\mathrm{i}} = Q_\ell, \tag{17}$$

where the interfacial momentum exchange term τ_{i} may or may not contain differential operators.

3. Drift-flux model. A strategy to avoid the modeling difficulties associated with the momentum exchange terms, as mentioned in the previous section, is to reformulate the model such that these terms no longer directly appear. This is precisely the idea of the *drift-flux* formulation of two-phase flow. By making the simplifying assumption

$$p = p_{\mathrm{g}} = p_\ell, \tag{18}$$

and adding the two momentum equations (3) and (4), we obtain the conservation equation for the *mixture* momentum:

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}} + \rho_\ell\alpha_\ell v_\ell) + \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}^2 + \rho_\ell\alpha_\ell v_\ell^2 + p\right) = Q_{\mathrm{g}} + Q_\ell. \tag{19}$$

Note that (18) is consistent with the assumption of a *mixed* flow regime, which is the situation for which the drift-flux model is commonly applied.

The phasic momentums must satisfy a *slip relation* in the functional form

$$v_{\mathrm{g}} - v_\ell = \Phi(p, \alpha_{\mathrm{g}}, v_{\mathrm{g}}). \tag{20}$$

Hence the two momentum *evolution equations* (16)–(17) of the two-fluid model are replaced by one evolution equation (19) and one *functional relation* (20). Bouré [9] discusses *generalized* drift-flux models where Φ may also contain differential operators.

3.1. Model formulation. In summary, using the nomenclature

$$m_{\mathrm{g}} = \rho_{\mathrm{g}}\alpha_{\mathrm{g}}, \tag{21}$$

$$m_{\ell} = \rho_{\ell}\alpha_{\ell}, \tag{22}$$

$$I_{\mathrm{g}} = m_{\mathrm{g}}v_{\mathrm{g}}, \tag{23}$$

$$I_{\ell} = m_{\ell}v_{\ell}, \tag{24}$$

$$I = I_{\mathrm{g}} + I_{\ell}, \tag{25}$$

$$Q = Q_{\mathrm{g}} + Q_{\ell}, \tag{26}$$

we may express the drift-flux model as

$$\frac{\partial m_{\mathrm{g}}}{\partial t} + \frac{\partial I_{\mathrm{g}}}{\partial x} = 0, \tag{27}$$

$$\frac{\partial m_{\ell}}{\partial t} + \frac{\partial I_{\ell}}{\partial x} = 0, \tag{28}$$

$$\frac{\partial I}{\partial t} + \frac{\partial}{\partial x}\left(I_{\mathrm{g}}v_{\mathrm{g}} + I_{\ell}v_{\ell} + p\right) = Q, \tag{29}$$

supplemented with the following functional relations:

- *Thermodynamics:* $p = p(\rho_{\mathrm{g}}) = p(\rho_{\ell})$.
- *Slip relation:* $v_{\mathrm{g}} - v_{\ell} = \Phi(m_{\mathrm{g}}, m_{\ell}, v_{\mathrm{g}})$.

3.2. Quasi-linear formulation. The model (27)–(29) may be written in the following *quasi-linear form*:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{Q}(\mathbf{U}), \tag{30}$$

where

$$\mathbf{U} = \begin{bmatrix} m_{\mathrm{g}} \\ m_{\ell} \\ I \end{bmatrix} \tag{31}$$

and

$$\mathbf{Q}(\mathbf{U}) = \begin{bmatrix} 0 \\ 0 \\ Q \end{bmatrix}. \tag{32}$$

In the following, we will derive an expression for the Jacobi matrix $\mathbf{A}$. Towards this aim, we will follow the common practice of thermodynamics and take

$$\left(\frac{\partial X}{\partial Y}\right)_{a,b} \tag{33}$$

to mean the partial derivative of X with respect to Y under the assumption of constant a and b.

3.2.1. Some definitions. We now define the following basic abbreviations:

$$\mu_{\rm g} = \left(\frac{\partial \Phi}{\partial m_{\rm g}}\right)_{m_\ell, v_{\rm g}}, \tag{34}$$

$$\mu_\ell = \left(\frac{\partial \Phi}{\partial m_\ell}\right)_{m_{\rm g}, v_{\rm g}}, \tag{35}$$

$$\mu_{\rm v} = \left(\frac{\partial \Phi}{\partial v_{\rm g}}\right)_{m_{\rm g}, m_\ell}, \tag{36}$$

$$\zeta = \left(\frac{\partial v_\ell}{\partial v_{\rm g}}\right)_{m_{\rm g}, m_\ell}. \tag{37}$$

We further define the *pseudomass* $\hat{\rho}$:

$$\hat{\rho} = m_{\rm g} + \zeta m_\ell. \tag{38}$$

Remark 1. We observe that by writing (20) as

$$d\Phi = dv_{\rm g} - dv_\ell, \tag{39}$$

we obtain from (36) and (37) the basic relation

$$\mu_{\rm v} = 1 - \zeta. \tag{40}$$

We may now derive the following useful differentials.

Differential 1 (gas velocity). *We may expand dI as*

$$dI = m_{\rm g}\, dv_{\rm g} + v_{\rm g}\, dm_{\rm g} + v_\ell\, dm_\ell + m_\ell\, dv_\ell. \tag{41}$$

Using (39) *and*

$$d\Phi = \mu_{\rm g}\, dm_{\rm g} + \mu_\ell\, dm_\ell + \mu_{\rm v}\, dv_{\rm g}, \tag{42}$$

we obtain

$$dv_{\rm g} = \frac{1}{\hat{\rho}}\left(dI + (m_\ell \mu_{\rm g} - v_{\rm g})\ dm_{\rm g} + (m_\ell \mu_\ell - v_\ell)\ dm_\ell\right). \tag{43}$$

Differential 2 (gas momentum). *Using*

$$dI_{\rm g} = m_{\rm g}\, dv_{\rm g} + v_{\rm g}\, dm_{\rm g}, \tag{44}$$

we obtain from (43)

$$dI_{\rm g} = \frac{1}{\hat{\rho}}\left(m_{\rm g}\, dI + (m_{\rm g} m_\ell \mu_{\rm g} + \zeta m_\ell v_{\rm g})\ dm_{\rm g} + (m_{\rm g} m_\ell \mu_\ell - m_{\rm g} v_\ell)\ dm_\ell\right). \tag{45}$$

Differential 3 (liquid momentum). *Using*

$$dI = dI_{\rm g} + dI_\ell, \tag{46}$$

we obtain from (45)

$$dI_\ell = \frac{1}{\hat{\rho}}\left(\zeta m_\ell\, dI - (m_{\rm g} m_\ell \mu_{\rm g} + \zeta m_\ell v_{\rm g})\ dm_{\rm g} - (m_{\rm g} m_\ell \mu_\ell - m_{\rm g} v_\ell)\ dm_\ell\right). \tag{47}$$

Differential 4 (pressure). *Writing* $\alpha_{\mathrm{g}} + \alpha_\ell = 1$ *as*

$$\frac{m_{\mathrm{g}}}{\rho_{\mathrm{g}}(p)} + \frac{m_\ell}{\rho_\ell(p)} = 1, \tag{48}$$

we obtain by differentiation

$$dp = \kappa \left(\rho_\ell \, dm_{\mathrm{g}} + \rho_{\mathrm{g}} \, dm_\ell \right), \tag{49}$$

where

$$\kappa = \frac{1}{(\partial \rho_{\mathrm{g}} / \partial p) \, \rho_\ell \alpha_{\mathrm{g}} + (\partial \rho_\ell / \partial p) \, \rho_{\mathrm{g}} \alpha_\ell}. \tag{50}$$

Differential 5 (gas momentum convection). *We have*

$$d \left(I_{\mathrm{g}} v_{\mathrm{g}} \right) = I_{\mathrm{g}} \, dv_{\mathrm{g}} + v_{\mathrm{g}} \, dI_{\mathrm{g}}. \tag{51}$$

Hence from (43) *and* (45) *we obtain*

$$\begin{aligned} d \left(I_{\mathrm{g}} v_{\mathrm{g}} \right) = \frac{1}{\hat{\rho}} \Big(& 2 m_{\mathrm{g}} v_{\mathrm{g}} \, dI + \left(2 m_{\mathrm{g}} m_\ell v_{\mathrm{g}} \mu_{\mathrm{g}} + (\zeta m_\ell - m_{\mathrm{g}}) v_{\mathrm{g}}^2 \right) \, dm_{\mathrm{g}} \\ & + \left(2 m_{\mathrm{g}} m_\ell v_{\mathrm{g}} \mu_\ell - 2 m_{\mathrm{g}} v_{\mathrm{g}} v_\ell \right) \, dm_\ell \Big). \end{aligned} \tag{52}$$

Differential 6 (liquid momentum convection). *We have*

$$dv_\ell = dv_{\mathrm{g}} - d\Phi = \zeta \, dv_{\mathrm{g}} - \mu_{\mathrm{g}} \, dm_{\mathrm{g}} - \mu_\ell \, dm_\ell. \tag{53}$$

From (43) *we obtain*

$$dv_\ell = \frac{1}{\hat{\rho}} \left(\zeta \, dI - (m_{\mathrm{g}} \mu_{\mathrm{g}} + \zeta v_{\mathrm{g}}) \, dm_{\mathrm{g}} - (m_{\mathrm{g}} \mu_\ell + \zeta v_\ell) \, dm_\ell \right). \tag{54}$$

Hence from

$$d \left(I_\ell v_\ell \right) = I_\ell \, dv_\ell + v_\ell \, dI_\ell \tag{55}$$

we obtain

$$\begin{aligned} d \left(I_\ell v_\ell \right) = \frac{1}{\hat{\rho}} \Big(& 2 \zeta m_\ell v_\ell \, dI - \left(2 m_{\mathrm{g}} m_\ell v_\ell \mu_{\mathrm{g}} + 2 \zeta m_\ell v_{\mathrm{g}} v_\ell \right) \, dm_{\mathrm{g}} \\ & - \left(2 m_{\mathrm{g}} m_\ell v_\ell \mu_\ell + (\zeta m_\ell - m_{\mathrm{g}}) v_\ell^2 \right) \, dm_\ell \Big). \end{aligned} \tag{56}$$

3.2.2. The Jacobi matrix. With the aid of these differentials we can more or less directly write down the Jacobi matrix

$$\mathbf{A}(\mathbf{U}) = \frac{1}{\hat{\rho}} \begin{bmatrix} m_{\mathrm{g}} m_\ell \mu_{\mathrm{g}} + \zeta m_\ell v_{\mathrm{g}} & m_{\mathrm{g}} m_\ell \mu_\ell - m_{\mathrm{g}} v_\ell & m_{\mathrm{g}} \\ -(m_{\mathrm{g}} m_\ell \mu_{\mathrm{g}} + \zeta m_\ell v_{\mathrm{g}}) & m_{\mathrm{g}} v_\ell - m_{\mathrm{g}} m_\ell \mu_\ell & \zeta m_\ell \\ A_{31} & A_{32} & 2(m_{\mathrm{g}} v_{\mathrm{g}} + \zeta m_\ell v_\ell) \end{bmatrix}, \tag{57}$$

where

$$A_{31} = \kappa \hat{\rho} \rho_\ell + 2 m_{\mathrm{g}} m_\ell \mu_{\mathrm{g}} (v_{\mathrm{g}} - v_\ell) + (\zeta m_\ell - m_{\mathrm{g}}) v_{\mathrm{g}}^2 - 2 \zeta m_\ell v_{\mathrm{g}} v_\ell \tag{58}$$

and

$$A_{32} = \kappa \hat{\rho} \rho_{\mathrm{g}} + 2 m_{\mathrm{g}} m_\ell \mu_\ell (v_{\mathrm{g}} - v_\ell) - (\zeta m_\ell - m_{\mathrm{g}}) v_\ell^2 - 2 m_{\mathrm{g}} v_{\mathrm{g}} v_\ell. \tag{59}$$

4. Wave structure analysis. As is well known from the theory of hyperbolic conservation laws, the velocities of the inherent wave phenomena of the system (30) are given by the eigenvalues of $\mathbf{A}$.

These eigenvalues satisfy the characteristic equation

$$\begin{aligned}(\lambda - v_{\rm g})(\lambda - v_\ell)(\hat{\rho}\lambda - m_{\rm g}v_{\rm g} - \zeta m_\ell v_\ell) + m_{\rm g}m_\ell \left(\mu_\ell(\lambda - v_{\rm g})^2 - \mu_{\rm g}(\lambda - v_\ell)^2\right)\\ + \kappa\rho_{\rm g}\rho_\ell \left(\alpha_{\rm g}\alpha_\ell(\rho_{\rm g}\mu_{\rm g} - \rho_\ell\mu_\ell) - \alpha_{\rm g}(\lambda - v_\ell) - \zeta\alpha_\ell(\lambda - v_{\rm g})\right) = 0.\end{aligned} \tag{60}$$

Remark 2 (eigenvectors). The eigenvector equation for $\mathbf{A}$ is

$$\mathbf{A}\omega = \lambda\omega. \tag{61}$$

From (57) we obtain

$$\omega = \begin{bmatrix} m_{\rm g}\left(m_\ell\mu_\ell + (\lambda - v_\ell)\right) \\ \zeta m_\ell(\lambda - v_{\rm g}) - m_{\rm g}m_\ell\mu_{\rm g} \\ \lambda\left(\hat{\rho}\lambda - m_{\rm g}m_\ell(\mu_{\rm g} - \mu_\ell) - m_{\rm g}v_\ell - \zeta m_\ell v_{\rm g}\right) \end{bmatrix}. \tag{62}$$

The eigenvalue equation (60), being a third-order polynomial, can in principle be solved exactly to yield algebraic expressions for the eigenvalues λ. However, as tools for understanding the wave structure of the drift-flux model, these exact solutions are of limited value due to their high degree of complexity. In practice, one would often prefer making some simplifying assumptions and study the resulting *approximate* eigenvalues.

4.1. The Zuber–Findlay relation. A very important special case is the Zuber–Findlay slip relation [30], which can be written in the following simple analytical form:

$$v_{\rm g} = K\left(\alpha_{\rm g}v_{\rm g} + \alpha_\ell v_\ell\right) + S \tag{63}$$

or equivalently

$$\Phi = \frac{(K-1)v_{\rm g} + S}{K\alpha_\ell}. \tag{64}$$

This expression was derived from continuity considerations by Zuber and Findlay [30], where two different effects are taken into account:

1. The effect of nonuniform velocity and concentration profiles. The *shape factor* K is defined as

   $$K = \frac{\langle(\alpha_{\rm g}v_{\rm g} + \alpha_\ell v_\ell)\alpha_{\rm g}\rangle}{\langle\alpha_{\rm g}v_{\rm g} + \alpha_\ell v_\ell\rangle\langle\alpha_{\rm g}\rangle}, \tag{65}$$

   where

   $$\langle\cdot\rangle = \frac{1}{A}\int_A (\cdot)(x,y,z)\,dA. \tag{66}$$

   Here A is the cross-sectional area in the (y,z)-plane.
2. The effect of local relative velocity. The *drift velocity* S is defined as the terminal velocity of a single gas bubble rising through the liquid.

The Zuber–Findlay relation (63) has been experimentally established for a broad range of parameters for both bubbly and slug flows [3, 15].

This particular drift-flux model has been extensively studied by Théron [26] and Benzoni-Gavage [5]. By making some simplifying assumptions (most notably constant K and S as well as an incompressible liquid phase) they obtained the eigenvalues

- *sonic waves*

$$\lambda_{\mathrm{s}} = v_\ell \pm \sqrt{\frac{p}{\rho_\ell \alpha_{\mathrm{g}} \left(1 - K\alpha_{\mathrm{g}}\right)}}, \tag{67}$$

- *material wave*

$$\lambda_{\mathrm{m}} = v_{\mathrm{g}}. \tag{68}$$

Benzoni-Gavage [5] demonstrated that the sonic characteristic fields are genuinely nonlinear, whereas the material field is linearly degenerate. Provided the liquid phase is incompressible, Gavrilyuk and Fabre [16] have demonstrated that under a suitable variable transformation, the drift-flux model with slip relation (63) is mathematically similar to the Euler equations of gas dynamics.

In the following sections, we demonstrate how the drift-flux model may be analyzed more generally using a perturbation technique suggested by Toumi and coworkers [27, 28, 29]. In particular, we allow the liquid to be compressible and recover the above results of [26, 5] as the low-order limit in the perturbation parameter.

4.2. A simplifying assumption. In the following, we will assume that the slip relation can be expressed in the Zuber–Findlay form (63). Here we allow the parameters K and S to be expressed as general functions:

$$K = K(p, v_{\mathrm{g}}), \tag{69}$$

$$S = S(p, v_{\mathrm{g}}). \tag{70}$$

Equivalently, this can be expressed as a differential equation:

$$\alpha_\ell \left(\frac{\partial \Phi}{\partial \alpha_\ell}\right)_p + \Phi = 0. \tag{71}$$

From (34), (35), and (48) we may derive the following identity:

$$\left(\frac{\partial \Phi}{\partial \alpha_\ell}\right)_p \equiv \rho_\ell \mu_\ell - \rho_{\mathrm{g}} \mu_{\mathrm{g}}. \tag{72}$$

Hence from (71) we obtain

$$\mu_\ell = \frac{\rho_{\mathrm{g}}}{\rho_\ell} \mu_{\mathrm{g}} + \frac{v_\ell - v_{\mathrm{g}}}{m_\ell} \tag{73}$$

and the eigenvalue equation (60) simplifies to

$$(\lambda - v_{\mathrm{g}})(\lambda - v_\ell)(\hat{\rho}\lambda - m_{\mathrm{g}} v_{\mathrm{g}} - \zeta m_\ell v_\ell) + m_{\mathrm{g}} m_\ell \left(\mu_\ell (\lambda - v_{\mathrm{g}})^2 - \mu_{\mathrm{g}} (\lambda - v_\ell)^2\right) - \kappa \rho_{\mathrm{g}} \rho_\ell \left(\alpha_{\mathrm{g}} + \zeta \alpha_\ell\right)(\lambda - v_{\mathrm{g}}) = 0. \tag{74}$$

4.3. Dimensionless formulation. By making the substitution

$$\lambda = v_{\mathrm{g}} + a\sigma \tag{75}$$

we will achieve some simplification, where a now plays the role of the unknown. We may now write (74) as

$$\begin{aligned} &a\sigma(v_{\mathrm{g}} - v_\ell + a\sigma)\left(\hat{\rho} a\sigma + \zeta m_\ell (v_{\mathrm{g}} - v_\ell)\right) + m_{\mathrm{g}} m_\ell \left(\mu_\ell a^2\sigma^2 - \mu_{\mathrm{g}}(v_{\mathrm{g}} - v_\ell + a\sigma)^2\right) \\ &- \kappa\rho_{\mathrm{g}}\rho_\ell \left(\alpha_{\mathrm{g}} + \zeta\alpha_\ell\right) a\sigma = 0. \end{aligned} \tag{76}$$

Now defining σ as

$$\sigma^2 = \kappa\hat{\rho}\left(\alpha_{\mathrm{g}} + \zeta\alpha_\ell\right) \tag{77}$$

and introducing the *dimensionless* variables

$$\varepsilon = \frac{v_{\mathrm{g}} - v_\ell}{\sigma}, \tag{78}$$

$$z = \frac{m_{\mathrm{g}}\alpha_\ell}{\sigma}\mu_{\mathrm{g}}, \tag{79}$$

$$\psi = \frac{\rho_{\mathrm{g}}}{\hat{\rho}}, \tag{80}$$

$$\varphi = \frac{\rho_\ell}{\hat{\rho}}, \tag{81}$$

the eigenvalue equation (76) may correspondingly be written in dimensionless form

$$a(\varepsilon + a)(a + \zeta\alpha_\ell\varphi\varepsilon) + z\psi a^2 - z\varphi(\varepsilon + a)^2 - \alpha_{\mathrm{g}}\psi\varepsilon a^2 - \varphi\psi a = 0. \tag{82}$$

Now introducing the *pseudoliquid fraction*

$$\hat{\alpha} = \zeta\alpha_\ell \tag{83}$$

and noting that

$$\alpha_{\mathrm{g}}\psi + \zeta\alpha_\ell\varphi = 1, \tag{84}$$

the eigenvalue equation (82) simplifies to

$$a^3 + \left(2\hat{\alpha}\varphi\varepsilon - z(\varphi - \psi)\right) a^2 + \varphi\left(\hat{\alpha}\varepsilon^2 - 2z\varepsilon - \psi\right) a - z\varphi\varepsilon^2 = 0. \tag{85}$$

4.4. A power series approximation. We may now write a as a power series expansion

$$a = \sum_{i=0}^{\infty} \beta_i \chi^i \tag{86}$$

for some perturbation parameter χ. Now several choices for χ are available through (78)–(81), depending on the values of the physical variables. In the following, we will use as our starting point the *incompressible* limit and obtain eigenvalues accurate to the lowest orders of compressibility.

Towards this aim, we observe that σ given by (77) will have a magnitude in the order of the phasic sound velocities (which tend to infinity in the incompressible limit). Hence, for subsonic flows, we expect

$$\varepsilon \ll 1. \tag{87}$$

Consequently we write

$$a = \sum_{i=0}^{\infty} \beta_i \varepsilon^i \tag{88}$$

and obtain the coefficients β_i by repeatedly solving (85) to the corresponding order in ε.

4.4.1. Material wave. From (85) we obtain

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -z/\psi \\ 2z^2/\psi^2 \\ \vdots \end{bmatrix}, \tag{89}$$

which translates into the eigenvalue

$$\lambda^{\mathrm{m}} = v_{\mathrm{g}} - \frac{\alpha_{\mathrm{g}}\alpha_\ell}{\alpha_{\mathrm{g}} + \zeta\alpha_\ell}\mu_{\mathrm{g}}\frac{(v_{\mathrm{g}} - v_\ell)^2}{\kappa} + \mathcal{O}(\varepsilon^3) \tag{90}$$

by the relations of section 4.3.

4.4.2. Sonic waves. We will find it convenient to introduce the shorthand

$$w = \sqrt{z^2(\psi - \varphi)^2 + 4\varphi\psi}. \tag{91}$$

From (85) we obtain

- *downstream pressure wave*

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \left((\varphi - \psi)z + w\right)/2 \\ 2\varphi\left(z - \hat{\alpha}\beta_0\right)/w \\ \beta_1\left(4\varphi\psi(1 - \hat{\alpha}\varphi) - z^2(\varphi^2 - \psi^2) - 2\varphi w z\right)/(2\beta_0 w^2) \\ \vdots \end{bmatrix}, \tag{92}$$

- *upstream pressure wave* (obtained from (85))

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \left((\varphi - \psi)z - w\right)/2 \\ -2\varphi\left(z - \hat{\alpha}\beta_0\right)/w \\ \beta_1\left(4\varphi\psi(1 - \hat{\alpha}\varphi) - z^2(\varphi^2 - \psi^2) + 2\varphi w z\right)/(2\beta_0 w^2) \\ \vdots \end{bmatrix}. \tag{93}$$

Now by writing the sonic eigenvalues in the form

$$\lambda^{\mathrm{p}} = \bar{v}^{\mathrm{p}} \pm c, \tag{94}$$

the coefficients (92)–(93) yield after some manipulation

$$\bar{v}^{\mathrm{p}} = \frac{m_{\mathrm{g}}v_{\mathrm{g}} + \zeta m_\ell v_\ell}{m_{\mathrm{g}} + \zeta m_\ell} + m_{\mathrm{g}}\alpha_\ell\mu_{\mathrm{g}}\frac{\rho_\ell - \rho_{\mathrm{g}}}{2\hat{\rho}} + \frac{\alpha_{\mathrm{g}}\alpha_\ell}{\alpha_{\mathrm{g}} + \zeta\alpha_\ell}\mu_{\mathrm{g}}\frac{(v_{\mathrm{g}} - v_\ell)^2}{2\kappa} + \mathcal{O}(\varepsilon^3), \tag{95}$$

as well as the sonic velocity c:

$$c = \frac{1}{2}w\sigma + \frac{z\varphi}{w}\left(2 - \hat{\alpha}(\varphi - \psi)\right)(v_{\mathrm{g}} - v_\ell) + \mathcal{O}(\varepsilon^2). \tag{96}$$

Remark 3. Given that

$$\text{trace}(\mathbf{A}) = \sum_i \lambda_i, \tag{97}$$

the following *exact* relation between $\bar{v}^{\mathrm{p}}$ and λ^{m} is satisfied:

$$2\bar{v}^{\mathrm{p}} + \lambda^{\mathrm{m}} = \frac{1}{\hat{\rho}}\left(m_{\mathrm{g}} m_\ell \mu_{\mathrm{g}} \left(1 - \frac{\rho_{\mathrm{g}}}{\rho_\ell}\right)\right) + v_{\mathrm{g}} + 2\frac{m_{\mathrm{g}} v_{\mathrm{g}} + \zeta m_\ell v_\ell}{\hat{\rho}}. \tag{98}$$

Remark 4. Although these eigenvalue expressions have been obtained under the assumption (71), similar techniques may be applied to solve (60) for other slip relations not satisfying (71). However, some knowledge of the relationship between the parameters μ_{g}, μ_ℓ, and μ_{v} will be useful for simplifying the calculations and determining a good choice of perturbation parameter.

4.5. Zuber–Findlay revisited. We now revisit the special case of the Zuber–Findlay slip relation (63)

$$v_{\mathrm{g}} = K(\alpha_{\mathrm{g}} v_{\mathrm{g}} + \alpha_\ell v_\ell) + S, \tag{99}$$

but we now consider K and S to be *constants*, which depend on the flow regime. This further simplification of (71) is often used for practical calculations [11, 15, 30].

4.5.1. Slip derivatives. By differentiation, we obtain the following explicit expressions for the slip parameters (34)–(37):

$$\mu_{\mathrm{v}} = \frac{K-1}{K\alpha_\ell}, \tag{100}$$

$$\mu_{\mathrm{g}} = (v_{\mathrm{g}} - v_\ell)\kappa \frac{\partial \rho_\ell}{\partial p}, \tag{101}$$

$$\mu_\ell = -(v_{\mathrm{g}} - v_\ell)\kappa \frac{\alpha_{\mathrm{g}}}{\alpha_\ell}\frac{\partial \rho_{\mathrm{g}}}{\partial p}, \tag{102}$$

and

$$\zeta = \frac{1 - K\alpha_{\mathrm{g}}}{K\alpha_\ell}. \tag{103}$$

Asymptotic expressions for the eigenvalues could now be obtained by substituting (100)–(103) into the previously calculated expressions (90) and (95)–(96). Equivalently, we may also substitute (100)–(103) into (76) and repeat the power series analysis. This will greatly simplify the calculations, a point that will be demonstrated in the following.

4.5.2. Eigenvalue equation. From (101) we note that z, as defined by (79), may be written as

$$z = \eta\varepsilon, \tag{104}$$

where

$$\eta = m_{\mathrm{g}} \alpha_\ell \kappa \frac{\partial \rho_\ell}{\partial p}. \tag{105}$$

Substituting in (85) we obtain the eigenvalue equation

$$a^3 + \varepsilon\left(2\hat{\alpha}\varphi - \eta(\varphi - \psi)\right)a^2 + \varphi\left(\hat{\alpha}\varepsilon^2 - 2\eta\varepsilon^2 - \psi\right)a - \eta\varphi\varepsilon^3 = 0. \tag{106}$$

4.5.3. Material wave. Solving (106) to powers of ε yields the following coefficients:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\eta/\psi \\ 0 \\ \eta(2\eta - \hat{\alpha})/\psi^2 \\ \vdots \end{bmatrix}. \tag{107}$$

Hence

$$\lambda_{\mathrm{m}} = v_{\mathrm{g}} - K\alpha_{\mathrm{g}}\alpha_\ell \frac{\partial \rho_\ell}{\partial p}(v_{\mathrm{g}} - v_\ell)^3 + \mathcal{O}(\varepsilon^5). \tag{108}$$

Changes in the material composition are consequently propagated by the velocity of the gas bubbles, plus small correction terms representing volumetric changes due to compression. Note that $\lambda_{\mathrm{m}} = v_{\mathrm{g}}$ becomes an *exact* eigenvalue for $\eta = 0$, the limit of incompressible liquid [16].

4.5.4. Sonic waves. For the sonic waves, we obtain the following coefficients:

- *Downstream pressure wave*

(109)
$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sqrt{\varphi\psi} \\ \eta(\phi - \psi)/2 - \hat{\alpha}\varphi \\ \left(2\varphi(4\eta + 2\hat{\alpha}\eta\psi - \eta^2\psi - 4\hat{\alpha}) + (2\hat{\alpha} - \eta)^2\varphi^2 + \eta^2\psi^2\right)/(8\sqrt{\varphi\psi}) \\ \vdots \end{bmatrix},$$

- *Upstream pressure wave*

(110)
$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} -\sqrt{\varphi\psi} \\ \eta(\phi - \psi)/2 - \hat{\alpha}\varphi \\ -\left(2\varphi(4\eta + 2\hat{\alpha}\eta\psi - \eta^2\psi - 4\hat{\alpha}) + (2\hat{\alpha} - \eta)^2\varphi^2 + \eta^2\psi^2\right)/(8\sqrt{\varphi\psi}) \\ \vdots \end{bmatrix}.$$

Hence

$$\bar{v}^{\mathrm{p}} = \frac{m_{\mathrm{g}}v_{\mathrm{g}} + \zeta m_\ell v_\ell}{m_{\mathrm{g}} + \zeta m_\ell} + m_{\mathrm{g}}\alpha_\ell\kappa\frac{\partial\rho_\ell}{\partial p}(v_{\mathrm{g}} - v_\ell)\frac{\rho_\ell - \rho_{\mathrm{g}}}{2\hat{\rho}} + \mathcal{O}(\varepsilon^3), \tag{111}$$

and the sound velocity c may be written as

$$c = \sqrt{\frac{\kappa\rho_{\mathrm{g}}\rho_\ell}{K\alpha_{\mathrm{g}}(\rho_{\mathrm{g}} - \rho_\ell) + \rho_\ell}} + \mathcal{O}(\varepsilon)^2. \tag{112}$$

Remark 5. Note that $\rho_{\mathrm{g}} \ll \rho_\ell$ implies $c \ll \sigma$, and the requirement $\varepsilon \ll 1$ (87) has a significantly broader range of validity than the assumption of subsonic slip, $|v_{\mathrm{g}} - v_\ell| \ll c$.

Remark 6. The sonic eigenvalues may be written as

$$\lambda_{\mathrm{s}} = v_\ell \pm \sqrt{\frac{(\partial p/\partial\rho_{\mathrm{g}})\rho_{\mathrm{g}}}{(1 - K\alpha_{\mathrm{g}})\rho_\ell\alpha_{\mathrm{g}}}} + \mathcal{O}(\eta) + \mathcal{O}(\psi) + \mathcal{O}(\varepsilon^2), \tag{113}$$

which reduces to the result (67) when $p(\rho_{\mathrm{g}})$ satisfies the ideal gas law.

5. Two-fluid formulation. In this section, we perform the transformation required to write the *general* drift-flux model of section 3.1 in *canonical two-fluid form* as described in section 2.3. In other words, we replace the conservation equation (19), together with the slip relation (20), with equivalent evolution equations for the momentums of each phase.

5.1. Momentum evolution equations. We first derive an explicit gas momentum evolution equation for the general drift-flux model with slip relation (20). Our starting point is the previously derived differential (45), which becomes

$$\frac{\partial I_{\mathrm{g}}}{\partial t} = \frac{1}{\hat{\rho}}\left(m_{\mathrm{g}}\frac{\partial I}{\partial t} + \left(m_{\mathrm{g}}m_\ell\mu_{\mathrm{g}} + \zeta m_\ell v_{\mathrm{g}}\right)\frac{\partial m_{\mathrm{g}}}{\partial t} + \left(m_{\mathrm{g}}m_\ell\mu_\ell - m_{\mathrm{g}}v_\ell\right)\frac{\partial m_\ell}{\partial t}\right), \tag{114}$$

when written as a partial derivative with respect to t.

By using the conservation equations (27)–(29), we obtain the gas momentum evolution equation, written in terms of spatial derivatives

$$\begin{aligned}&\frac{\partial I_{\mathrm{g}}}{\partial t} + \frac{m_{\mathrm{g}}}{\hat{\rho}}\frac{\partial}{\partial x}\left(I_{\mathrm{g}}v_{\mathrm{g}} + I_\ell v_\ell + p\right) + \frac{\zeta m_\ell}{\hat{\rho}}v_{\mathrm{g}}\frac{\partial I_{\mathrm{g}}}{\partial x}\\ &- \frac{m_{\mathrm{g}}}{\hat{\rho}}v_\ell\frac{\partial I_\ell}{\partial x} + \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\left(\mu_{\mathrm{g}}\frac{\partial I_{\mathrm{g}}}{\partial x} + \mu_\ell\frac{\partial I_\ell}{\partial x}\right) = \frac{m_{\mathrm{g}}}{\hat{\rho}}Q.\end{aligned} \tag{115}$$

Further manipulation of derivatives yields

$$\begin{aligned}&\frac{\partial I_{\mathrm{g}}}{\partial t} + \frac{\partial}{\partial x}\left(I_{\mathrm{g}}v_{\mathrm{g}}\right) + \frac{m_{\mathrm{g}}}{\hat{\rho}}\frac{\partial p}{\partial x}\\ &+ \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\left(v_\ell\frac{\partial v_\ell}{\partial x} - \zeta v_{\mathrm{g}}\frac{\partial v_{\mathrm{g}}}{\partial x} + \mu_{\mathrm{g}}\frac{\partial I_{\mathrm{g}}}{\partial x} + \mu_\ell\frac{\partial I_\ell}{\partial x}\right) = \frac{m_{\mathrm{g}}}{\hat{\rho}}Q.\end{aligned} \tag{116}$$

5.1.1. Canonical form. Writing (116) under the canonical two-fluid form of section 2.3,

$$\frac{\partial I_{\mathrm{g}}}{\partial t} + \frac{\partial}{\partial x}\left(I_{\mathrm{g}}v_{\mathrm{g}}\right) + \alpha_{\mathrm{g}}\frac{\partial p}{\partial x} + \tau_{\mathrm{i}} = Q_{\mathrm{g}}, \tag{117}$$

where $Q = Q_{\mathrm{g}} + Q_\ell$, we find that the interface friction τ_{i} is given by

$$\begin{aligned}\tau_{\mathrm{i}} = {}&\alpha_{\mathrm{g}}\alpha_\ell\frac{\rho_{\mathrm{g}} - \zeta\rho_\ell}{\hat{\rho}}\frac{\partial p}{\partial x} + \frac{\zeta m_\ell}{\hat{\rho}}Q_{\mathrm{g}} - \frac{m_{\mathrm{g}}}{\hat{\rho}}Q_\ell\\ &+ \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\left(v_\ell\frac{\partial v_\ell}{\partial x} - \zeta v_{\mathrm{g}}\frac{\partial v_{\mathrm{g}}}{\partial x} + \mu_{\mathrm{g}}\frac{\partial I_{\mathrm{g}}}{\partial x} + \mu_\ell\frac{\partial I_\ell}{\partial x}\right).\end{aligned} \tag{118}$$

5.1.2. Liquid momentum evolution. By inserting (118) into the canonical liquid momentum equation (17), we obtain

$$\begin{aligned}&\frac{\partial I_\ell}{\partial t} + \frac{\partial}{\partial x}\left(I_\ell v_\ell\right) + \frac{\zeta m_\ell}{\hat{\rho}}\frac{\partial p}{\partial x}\\ &-\frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\left(v_\ell\frac{\partial v_\ell}{\partial x} - \zeta v_{\mathrm{g}}\frac{\partial v_{\mathrm{g}}}{\partial x} + \mu_{\mathrm{g}}\frac{\partial I_{\mathrm{g}}}{\partial x} + \mu_\ell\frac{\partial I_\ell}{\partial x}\right) = \frac{\zeta m_\ell}{\hat{\rho}}Q.\end{aligned} \tag{119}$$

5.2. Quasi-linear formulation. We may now express this rewritten drift-flux model in quasi-linear form:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = \mathbf{Q}, \tag{120}$$

similar to section 3.2. However, the matrix $\mathbf{A}$ is now 4×4 and $\mathbf{U}$ is given by

$$\mathbf{U} = \begin{bmatrix} \rho_{\mathrm{g}}\alpha_{\mathrm{g}} \\ \rho_\ell\alpha_\ell \\ \rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}} \\ \rho_\ell\alpha_\ell v_\ell \end{bmatrix}, \tag{121}$$

whereas the vector of sources is

$$\mathbf{Q} = \frac{1}{\hat{\rho}}\begin{bmatrix} 0 \\ 0 \\ m_{\mathrm{g}}Q \\ \zeta m_\ell Q \end{bmatrix}. \tag{122}$$

We now split (118) into four parts:

$$\tau_{\mathrm{i}} = \tau_p + \tau_v + \tau_\alpha + \tau_Q, \tag{123}$$

where

$$\tau_p = \alpha_{\mathrm{g}}\alpha_\ell\frac{\rho_{\mathrm{g}} - \zeta\rho_\ell}{\hat{\rho}}\frac{\partial p}{\partial x}, \tag{124}$$

$$\tau_v = \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\left(v_\ell\frac{\partial v_\ell}{\partial x} - \zeta v_{\mathrm{g}}\frac{\partial v_{\mathrm{g}}}{\partial x}\right), \tag{125}$$

$$\tau_\alpha = \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\left(\mu_{\mathrm{g}}\frac{\partial I_{\mathrm{g}}}{\partial x} + \mu_\ell\frac{\partial I_\ell}{\partial x}\right), \tag{126}$$

and

$$\tau_Q = \frac{\zeta m_\ell}{\hat{\rho}}Q_{\mathrm{g}} - \frac{m_{\mathrm{g}}}{\hat{\rho}}Q_\ell. \tag{127}$$

This defines a natural decomposition of the Jacobi matrix as follows:

$$\mathbf{A}(\mathbf{U}) = \mathbf{A}_0 + \mathbf{A}_p + \mathbf{A}_v + \mathbf{A}_\alpha, \tag{128}$$

i.e., one additional contribution for each differential term of the interface friction.

5.2.1. $\mathbf{A_0}$. The Jacobi matrix for the canonical two-fluid model with $\tau_{\mathrm{i}} = 0$ is [12]

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \kappa\rho_\ell\alpha_{\mathrm{g}} - v_{\mathrm{g}}^2 & \kappa\rho_{\mathrm{g}}\alpha_{\mathrm{g}} & 2v_{\mathrm{g}} & 0 \\ \kappa\rho_\ell\alpha_\ell & \kappa\rho_{\mathrm{g}}\alpha_\ell - v_\ell^2 & 0 & 2v_\ell \end{bmatrix}. \tag{129}$$

5.2.2. $\mathbf{A}_{\boldsymbol{p}}$. From (49) we obtain

$$\mathbf{A}_p(\mathbf{U}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \kappa\rho_\ell\alpha_{\mathrm{g}}\alpha_\ell\frac{\rho_{\mathrm{g}}-\zeta\rho_\ell}{\hat{\rho}} & \kappa\rho_{\mathrm{g}}\alpha_{\mathrm{g}}\alpha_\ell\frac{\rho_{\mathrm{g}}-\zeta\rho_\ell}{\hat{\rho}} & 0 & 0 \\ -\kappa\rho_\ell\alpha_{\mathrm{g}}\alpha_\ell\frac{\rho_{\mathrm{g}}-\zeta\rho_\ell}{\hat{\rho}} & -\kappa\rho_\ell\alpha_{\mathrm{g}}\alpha_\ell\frac{\rho_{\mathrm{g}}-\zeta\rho_\ell}{\hat{\rho}} & 0 & 0 \end{bmatrix}. \tag{130}$$

5.2.3. $\mathbf{A}_{\boldsymbol{v}}$. From

$$dv_{\mathrm{g}} = \frac{1}{m_{\mathrm{g}}}\,dI_{\mathrm{g}} - \frac{v_{\mathrm{g}}}{m_{\mathrm{g}}}\,dm_{\mathrm{g}} \tag{131}$$

and

$$dv_\ell = \frac{1}{m_\ell}\,dI_\ell - \frac{v_\ell}{m_\ell}\,dm_\ell \tag{132}$$

we obtain

$$\mathbf{A}_v(\mathbf{U}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\zeta m_\ell}{\hat{\rho}}v_{\mathrm{g}}^2 & -\frac{m_{\mathrm{g}}}{\hat{\rho}}v_\ell^2 & -\frac{\zeta m_\ell}{\hat{\rho}}v_{\mathrm{g}} & \frac{m_{\mathrm{g}}}{\hat{\rho}}v_\ell \\ -\frac{\zeta m_\ell}{\hat{\rho}}v_{\mathrm{g}}^2 & \frac{m_{\mathrm{g}}}{\hat{\rho}}v_\ell^2 & \frac{\zeta m_\ell}{\hat{\rho}}v_{\mathrm{g}} & -\frac{m_{\mathrm{g}}}{\hat{\rho}}v_\ell \end{bmatrix}. \tag{133}$$

5.2.4. $\mathbf{A}_{\boldsymbol{\alpha}}$. We directly obtain

$$\mathbf{A}_\alpha(\mathbf{U}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_{\mathrm{g}} & \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_\ell \\ 0 & 0 & -\frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_{\mathrm{g}} & -\frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_\ell \end{bmatrix}. \tag{134}$$

5.2.5. End result. Adding all contributions we obtain from (128)

(135)

$$\mathbf{A}(\mathbf{U}) =$$
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{m_{\mathrm{g}}}{\hat{\rho}}\left(\kappa\rho_\ell - v_{\mathrm{g}}^2\right) & \frac{m_{\mathrm{g}}}{\hat{\rho}}\left(\kappa\rho_{\mathrm{g}} - v_\ell^2\right) & \left(2-\frac{\zeta m_\ell}{\hat{\rho}}\right)v_{\mathrm{g}} + \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_{\mathrm{g}} & \frac{m_{\mathrm{g}}}{\hat{\rho}}v_\ell + \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_\ell \\ \frac{\zeta m_\ell}{\hat{\rho}}\left(\kappa\rho_\ell - v_{\mathrm{g}}^2\right) & \frac{\zeta m_\ell}{\hat{\rho}}\left(\kappa\rho_{\mathrm{g}} - v_\ell^2\right) & \frac{\zeta m_\ell}{\hat{\rho}}v_{\mathrm{g}} - \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_{\mathrm{g}} & \left(2-\frac{m_{\mathrm{g}}}{\hat{\rho}}\right)v_\ell - \frac{m_{\mathrm{g}}m_\ell}{\hat{\rho}}\mu_\ell \end{bmatrix}.$$

5.2.6. Eigenvalues. The eigenvalues of the matrix $\mathbf{A}$ are the roots of the polynomial equation

$$\begin{aligned} \lambda\Big[&(\lambda - v_{\mathrm{g}})(\lambda - v_\ell)(\hat{\rho}\lambda - m_{\mathrm{g}}v_{\mathrm{g}} - \zeta m_\ell v_\ell) \\ &+ m_{\mathrm{g}}m_\ell\left(\mu_\ell(\lambda - v_{\mathrm{g}})^2 - \mu_{\mathrm{g}}(\lambda - v_\ell)^2\right) \\ &+ \kappa\rho_{\mathrm{g}}\rho_\ell\left(\alpha_{\mathrm{g}}\alpha_\ell(\rho_{\mathrm{g}}\mu_{\mathrm{g}} - \rho_\ell\mu_\ell) - \alpha_{\mathrm{g}}(\lambda - v_\ell) - \zeta\alpha_\ell(\lambda - v_{\mathrm{g}})\right)\Big] = 0. \end{aligned} \tag{136}$$

By direct comparison with (60), we see that this may be written as

$$\lambda P(\lambda) = 0, \tag{137}$$

where $P(\lambda)$ is the eigenvalue polynomial for the original drift-flux model.

Remark 7. We have written the drift-flux model as a quasi-linear system of four equations by deriving two momentum equations which replace the mixed momentum equation and the slip law. As a consequence, the characteristic speeds of this system are given by (137) showing that a new characteristic speed $\lambda = 0$, representing the slip relation, has been added to the characteristic speeds already given by the drift-flux model.

This situation is similar to what is observed for a much simpler problem. Consider the scalar equation

$$u_t + f(u)_x = k'(x)g(u), \tag{138}$$

where k, f, and g are given functions. A common approach for deriving numerical schemes for the model (138) is to first write the model as a quasi-linear system of two equations, by adding the trivial equation $k_t = 0$, which gives

$$U_t + A(U)U_x = 0, \qquad U = \begin{pmatrix} u \\ k \end{pmatrix}, \qquad A(U) = \begin{pmatrix} f'(u) & -g(u) \\ 0 & 0 \end{pmatrix}. \tag{139}$$

The characteristic speeds of this system are given by $\lambda_1 = f'(u)$ and $\lambda_2 = 0$. If $f'(u) = 0$ for some u, then the eigenvalues coincide, and we have so-called resonance; see, for instance, [17] and the references therein for more on this. Note that this phenomenon might well also occur for our system (120)–(126), since one of the solutions of $P(\lambda) = 0$ corresponding to the slow material wave (see below for more details) can be zero. This happens when $v_{\mathrm{g}} = v_\ell = 0$.

It is interesting to note that the form (139) often is used as the starting point for designing numerical schemes for solving (138). In a similar manner we could imagine to use the above two-fluid form (120)–(126) as a starting point for developing a numerical scheme for the drift-flux model, e.g., by using the numerical schemes more recently proposed in [13, 14] for the two-fluid model.

6. Interface friction and wave velocities. In this section, we investigate how the wave structure of the two-fluid model gradually changes as it is transformed into a drift-flux model by addition of the various terms of (123). Our starting point is the *canonical model* with $\tau_i = 0$:

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}) + \frac{\partial}{\partial x}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}) = 0, \tag{140}$$

$$\frac{\partial}{\partial t}(\rho_\ell\alpha_\ell) + \frac{\partial}{\partial x}(\rho_\ell\alpha_\ell v_\ell) = 0, \tag{141}$$

$$\frac{\partial}{\partial t}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}) + \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}v_{\mathrm{g}}^2\right) + \alpha_{\mathrm{g}}\frac{\partial}{\partial x}(p) = Q_{\mathrm{g}}, \tag{142}$$

$$\frac{\partial}{\partial t}(\rho_\ell\alpha_\ell v_\ell) + \frac{\partial}{\partial x}\left(\rho_\ell\alpha_\ell v_\ell^2\right) + \alpha_\ell\frac{\partial}{\partial x}(p) = Q_\ell. \tag{143}$$

6.1. Wave structure of the canonical model. For different choices of τ_i, Toumi and coworkers [27, 28, 29] investigated the wave structure of the model with a perturbation technique. For $\tau_i = 0$, the wave velocities are precisely the eigenvalues of the matrix $\mathbf{A}_0$ given by (129). Now defining

$$\varepsilon = \frac{v_{\mathrm{g}} - v_\ell}{\hat{c}}, \tag{144}$$

where $\hat{c}$ is a *mixture* sonic velocity given by

$$\hat{c} = \sqrt{\frac{\rho_\ell \alpha_{\rm g} + \rho_{\rm g} \alpha_\ell}{(\partial \rho_{\rm g}/\partial p)\rho_\ell \alpha_{\rm g} + (\partial \rho_\ell/\partial p)\rho_{\rm g}\alpha_\ell}} = \sqrt{(\rho_\ell \alpha_{\rm g} + \rho_{\rm g}\alpha_\ell)\kappa}, \tag{145}$$

approximate eigenvalues for (140)–(143) were presented by Evje and Flåtten [12] as described below.

6.1.1. Material waves. Writing

$$\lambda_{\rm m} = \bar{v}^{\rm v} \pm \gamma, \tag{146}$$

we obtain

$$\bar{v}^{\rm v} = \frac{\rho_{\rm g}\alpha_\ell v_{\rm g} + \rho_\ell \alpha_{\rm g} v_\ell}{\rho_{\rm g}\alpha_\ell + \rho_\ell \alpha_{\rm g}} + \mathcal{O}(\varepsilon^3) \tag{147}$$

and

$$\gamma = i\frac{\sqrt{\rho_{\rm g}\rho_\ell \alpha_{\rm g}\alpha_\ell}(v_{\rm g} - v_\ell)}{\rho_{\rm g}\alpha_\ell + \rho_\ell \alpha_{\rm g}} + \mathcal{O}(\varepsilon^3). \tag{148}$$

Remark 8. Note that unless $v_{\rm g} = v_\ell$, γ is imaginary and the canonical two-fluid model with $\tau_i = 0$ loses hyperbolicity. Hence the inclusion of a differential interface friction τ_i is essential for obtaining a well-behaved mathematical solution.

Remark 9. Note that if $\rho_{\rm g} \ll \rho_\ell$, $\bar{v}^{\rm v} \approx v_\ell$ and the material waves travel with the velocity of the liquid phase. This is quite the opposite of the drift-flux model, where the velocity of the material wave corresponds to the gas velocity $v_{\rm g}$ (section 4.4).

6.1.2. Sonic waves. Writing

$$\lambda_{\rm s} = \bar{v}^{\rm p} \pm c, \tag{149}$$

we obtain

$$\bar{v}^{\rm p} = \frac{\rho_{\rm g}\alpha_\ell v_\ell + \rho_\ell \alpha_{\rm g} v_{\rm g}}{\rho_{\rm g}\alpha_\ell + \rho_\ell \alpha_{\rm g}} + \mathcal{O}(\varepsilon^3) \tag{150}$$

and

$$c = \hat{c}\left(1 + \mathcal{O}(\varepsilon^2)\right). \tag{151}$$

Remark 10. If $\rho_{\rm g} \ll \rho_\ell$, $\bar{v}^{\rm p} \approx v_{\rm g}$ and the part of the sonic wave that is transported along the flow travels with the velocity of the gas phase. Again this contrasts the drift-flux model, where the corresponding result of section 4.4 yields $\bar{v}^{\rm p} \approx v_\ell$.

6.2. Numerical investigations. In the framework of the canonical two-fluid model, the eigenvalues of the previous section correspond to $\tau_i = 0$, whereas the eigenvalues of section 4.4 correspond to the interface friction (123),

$$\tau_{\rm i} = \tau_p + \tau_v + \tau_\alpha + \tau_Q, \tag{152}$$

which was derived in section 5.1.1. We now study the relation between the interface friction and the wave velocities more closely, by looking at a specific example. More precisely, we consider a two-phase flow satisfying the Zuber–Findlay slip relation (63) with phasic properties roughly representing an air-water mixture.

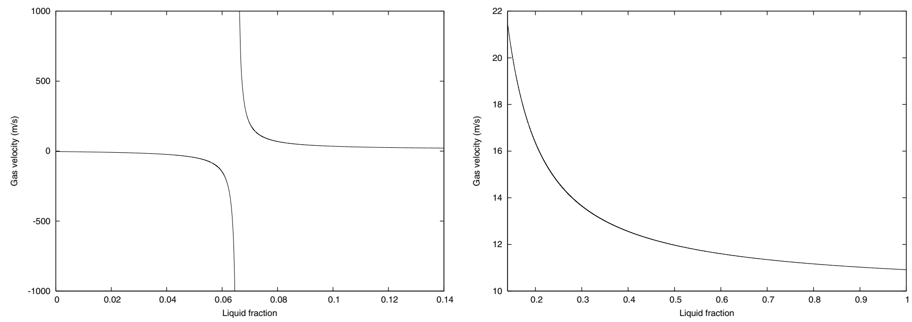

FIG. 1. *The Zuber–Findlay gas velocity as a function of liquid fraction. Left: Near singularity. Right: Physical region.*

6.2.1. Model parameters. In the following, we assume that the phasic velocities are related by the Zuber–Findlay slip relation

$$v_{\rm g} = K\left(\alpha_{\rm g} v_{\rm g} + \alpha_\ell v_\ell\right) + S, \tag{153}$$

where we choose

$$K = 1.07 \tag{154}$$

and

$$S = 0.216 \text{ m/s}. \tag{155}$$

Furthermore, we assume the flow conditions

$$v_\ell = 10 \text{ m/s}, \tag{156}$$
$$\rho_{\rm g} = 1.0 \text{ kg/m}^3, \tag{157}$$
$$\rho_\ell = 1000 \text{ kg/m}^3, \tag{158}$$
$$\partial p/\partial \rho_{\rm g} = 10^5 \text{ m}^2/\text{s}^2, \tag{159}$$
$$\partial p/\partial \rho_\ell = 10^6 \text{ m}^2/\text{s}^2. \tag{160}$$

6.2.2. Gas velocity. By inspecting the slip expression (153) we find there is a singularity corresponding to

$$\hat{\alpha} = \zeta \alpha_\ell = \frac{1 - K\alpha_{\rm g}}{K} = 0, \tag{161}$$

which with our choice of parameters occurs at

$$\alpha_\ell^{\rm crit} \approx 0.0654. \tag{162}$$

The gas velocity $v_{\rm g}$ changes sign from $-\infty$ to $+\infty$ across the singularity, as shown in Figure 1. However, $\alpha_\ell < \alpha_\ell^{\rm crit}$ implies large gas bubbles corresponding more or less to the *annular* flow regime, where the drift-flux model is not applicable [16]. Hence we discard the corresponding results as unphysical and base our further investigations on the assumption $\alpha_\ell > \alpha_\ell^{\rm crit}$.

6.2.3. Wave velocities. We now investigate the effect of the different terms of

$$\tau_{\mathrm{i}} = \tau_p + \tau_v + \tau_\alpha + \tau_Q \tag{163}$$

on the wave velocities of the canonical two-fluid model. Note that τ_Q, as given by (124), is purely nondifferential, and hence has no effect on the wave structure of the model. In the following plots we use the labels

- *two-fluid:* $\tau_i = 0$,
- *drift-flux:* $\tau_i = \tau_p + \tau_v + \tau_\alpha$

to denote the special choices of interface friction yielding the basic two-fluid and drift-flux wave structures, respectively, as described in sections 4 and 6.1.

Remark 11. Note that with our choice of slip relation (153), the expression (126) may by use of (101) and (102) be rewritten as

$$\tau_\alpha = (v_{\mathrm{g}} - v_\ell)\frac{m_{\mathrm{g}} m_\ell}{\hat{\rho}\alpha_\ell}\frac{\partial \alpha_\ell}{\partial t}. \tag{164}$$

In the following, wave velocities corresponding to different choices of τ_i (163) are calculated as the eigenvalues of the corresponding matrix $\mathbf{A}(\mathbf{U})$ as described in section 5.2. A numerical algorithm was used to calculate the eigenvalues, sorted in ascending order by their real parts as

$$\mathrm{Re}(\lambda_1) < \mathrm{Re}(\lambda_2) < \mathrm{Re}(\lambda_3) < \mathrm{Re}(\lambda_4). \tag{165}$$

Here λ_1 and λ_4 are sonic waves, whereas λ_2 and λ_3 represent slow waves.

6.2.4. Slip wave. As noted in Remark 7, the slip relation manifests itself as a stationary wave for the *drift-flux* interface friction ($\tau_i = \tau_p + \tau_v + \tau_\alpha$). Hence one of the two material waves described in section 6.1.1, corresponding to $\tau_i = 0$, will gradually transform into this stationary "slip wave" as the terms (124)–(126) are added to the interface friction.

The effect of this is illustrated in Figure 2, where $|\lambda_2|$ is plotted as a function of liquid fraction. Already for $\tau_i = \tau_p + \tau_v$, we obtain $\lambda_2 = 0$, which is left unchanged by the addition of τ_α. Note that $\tau_\alpha = 0$ corresponds to a special case of the drift-flux model, where the slip relation satisfies $\mu_{\mathrm{g}} = \mu_\ell = 0$. Hence the "drift-flux" character of the system ($\lambda_2 \equiv 0$) is fully manifest in the τ_p and τ_v components of the interface friction.

6.2.5. Material wave. As seen by the analyses of section 4.5 and 6.1, one material wave is gradually transformed from (146) ($\lambda_{\mathrm{m}} \approx v_\ell$) into (108) ($\lambda_{\mathrm{m}} \approx v_{\mathrm{g}}$).

This is illustrated in Figure 3, where $\mathrm{Re}(\lambda_3)$ is plotted as a function of liquid fraction. Note that without the inclusion of τ_α, the wave velocity is constant. This demonstrates the fact that $\tau_\alpha = 0$ implies that the slip is independent of volume fraction.

6.2.6. Sound velocity. Following sections 4.4.2 and 6.1.2, we write the sonic waves as a combination of two components as follows:

$$\lambda_{\mathrm{s}} = \bar{v}^{\mathrm{p}} \pm c, \tag{166}$$

where $\bar{v}^{\mathrm{p}}$ represents the part of the sonic wave that is transported with the flow, whereas c is the sonic velocity with respect to $\bar{v}^{\mathrm{p}}$. Hence we get

$$c = \frac{\lambda_4 - \lambda_1}{2} \tag{167}$$

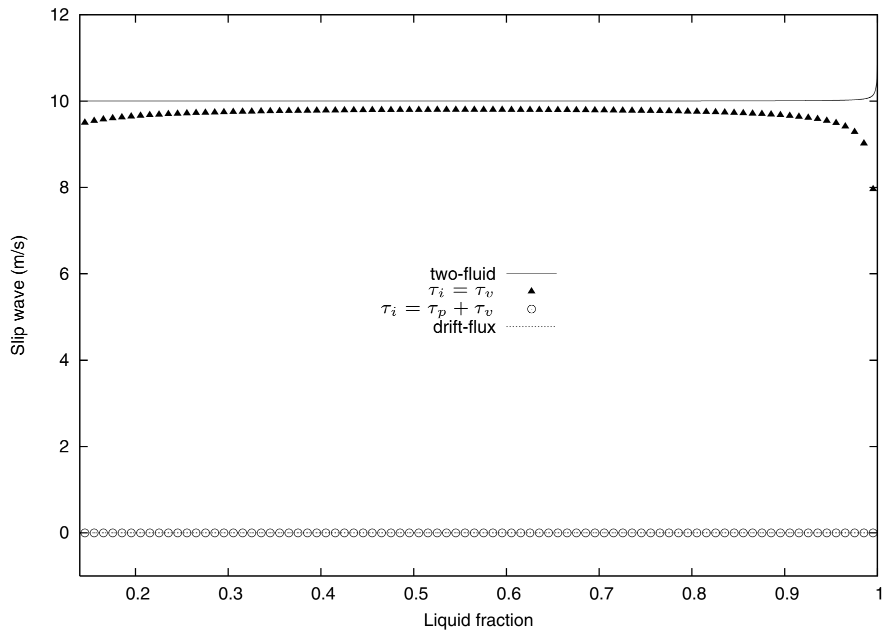

FIG. 2. *Slip wave velocity as a function of liquid fraction.*

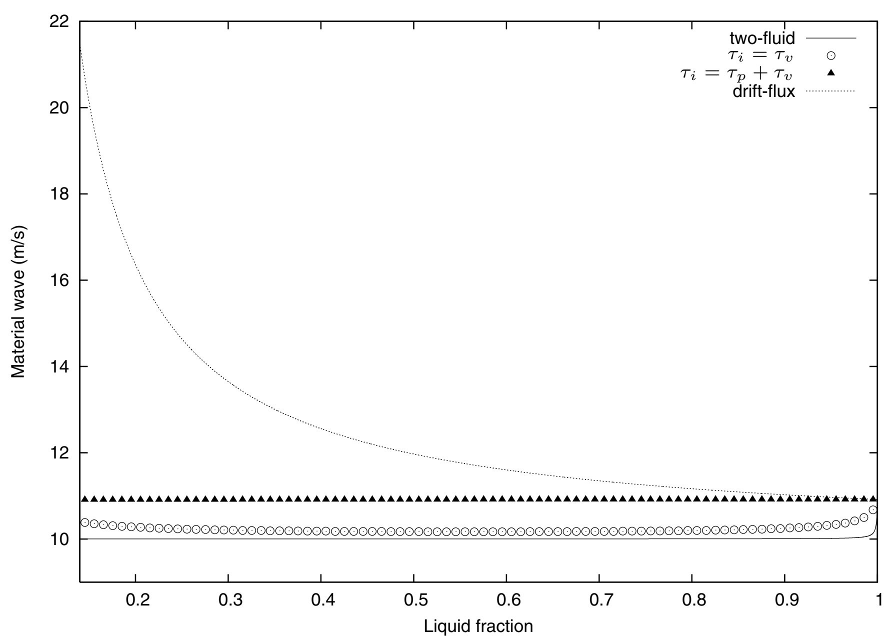

FIG. 3. *Material wave velocity as a function of liquid fraction.*

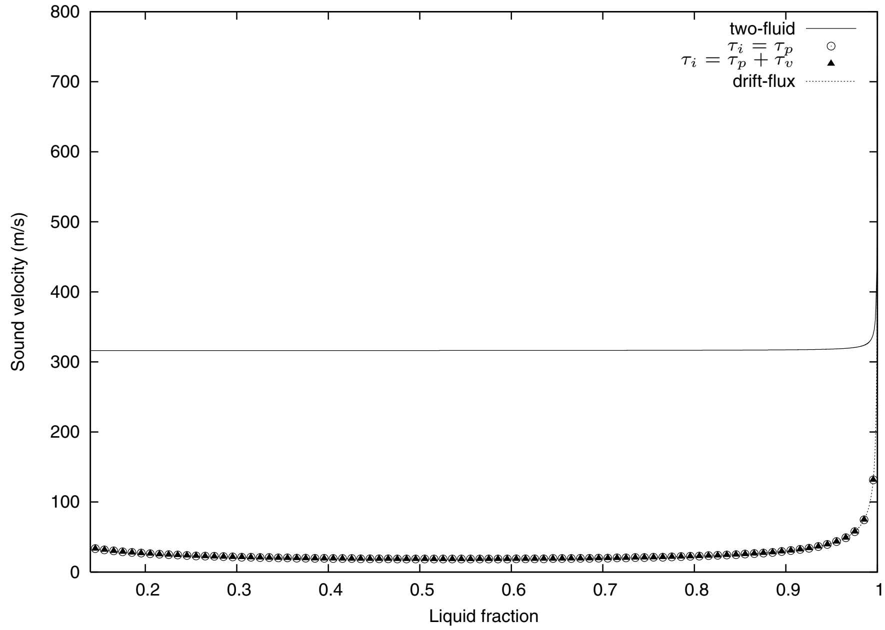

FIG. 4. *Mixture sound velocity as a function of liquid fraction.*

and

$$\bar{v}^{\mathrm{p}} = \frac{\lambda_1 + \lambda_4}{2}. \tag{168}$$

In Figure 4, the sound velocity c is plotted as a function of liquid fraction. We observe that c is transformed from the two-fluid sound velocity (145) into the drift-flux sound velocity (112) by the action of τ_p alone; the terms τ_v and τ_α have no additional effect.

Remark 12. This plot illustrates the fact that whereas for the two-fluid model

$$c_{\mathrm{tf}} \approx c_{\mathrm{g}}, \tag{169}$$

the drift-flux sonic velocity satisfies

$$c_{\mathrm{df}} \ll \min(c_{\mathrm{g}}, c_\ell). \tag{170}$$

A similar parabolic-like shape for $c_{\mathrm{mix}}(\alpha_\ell)$ was also derived by Nguyen, Winther, and Greiner [21] by considering the interface as an elastic wall. They also pointed out that this shape is consistent with experimental data for *mixed* flows.

6.2.7. Sonic transport velocity. The sonic transport velocity $\bar{v}^{\mathrm{p}}$ is plotted in Figure 5. We get more or less the inverse of Figure 3; now $\bar{v}^{\mathrm{p}} \approx v_{\mathrm{g}}$ (two-fluid model) is transformed into $\bar{v}^{\mathrm{p}} \approx v_\ell$ (drift-flux model) by the action of the interface friction (118).

7. Summary. A quasi-linear form of the *drift-flux* two-phase flow model has been derived. The wave structure of this model has been investigated by a perturbation technique, extending previous results of Théron [26] and Benzoni-Gavage [5].

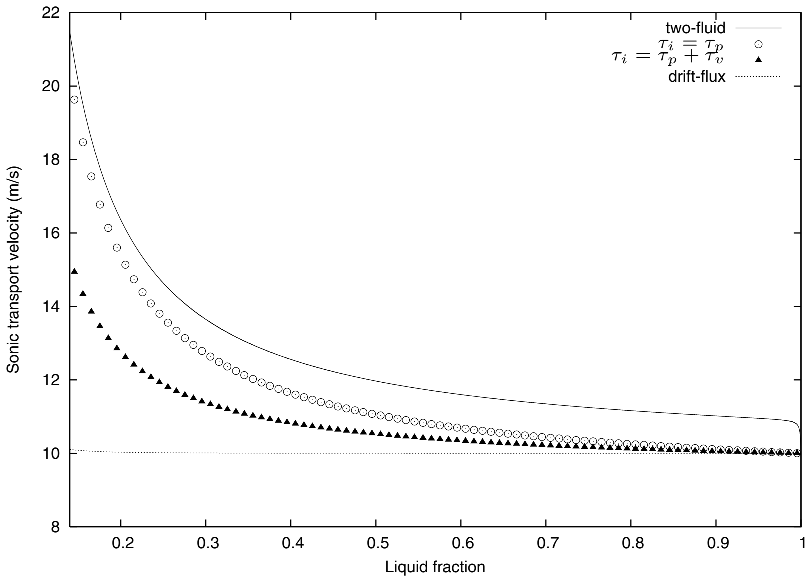

FIG. 5. *Sonic transport velocity* $\bar{v}^{\mathrm{p}}$ *as a function of liquid fraction.*

The drift-flux model has further been rewritten within the framework of a more general *two-fluid* model, by derivation of the proper form of the interface friction τ_i. Here the slip relation is represented as a stationary wave.

The interface friction τ_i may be split into four parts

$$\tau_i = \tau_p + \tau_v + \tau_\alpha + \tau_Q, \tag{171}$$

where the following hold:

- The terms τ_p and τ_v make up the *drift-flux* nature of the system (stationary slip wave).
- The term τ_p is almost exclusively associated with the mixture sound velocity c.
- The term τ_α is associated with the slow waves, imposing a dependency of volume fraction on the material wave.

The drift-flux and two-fluid formulations are often considered to be different modeling strategies with different domains of applicability. The unification presented in this paper may facilitate the implementation of both models within a single computer code. Furthermore, the link presented between the *observable* slip velocity and the underlying interface friction may serve as an aid for developing better physical models for two-phase flows.

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