Hysteretic Θ(S) Curve Prediction: Comparison of Two Models

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(Received: 8 July 2003; accepted in final form: 14 June 2004)

Abstract. The prediction of the first-order wetting and drying scanning curves is attempted by two different methods. These are: Mualem model II and Poulovassilis and Kargas (P–K). Model II by Mualem was chosen deliberately as the most appropriate. Experimental (S) data obtained in the laboratory for a sand mixture and a real soil were used for the comparison. Moreover, data presented originally by Poulovassilis (1970) were also used for the same purpose. It is shown that the P–K method gives better results than Mualem’s model II. Some remarks on Mualem’s model are also included.

Key words: soil water retention curves, scanning curves, capillary hysteresis, pore-water distribution function, independent domain models.

1. Introduction

The phenomenon of hysteresis exhibited by certain hydraulic properties of soil and of porous media, in general, is a complex and quite interesting one.

Haines (1930) was the first to show, by performing experiments in stable porous media, that the relationship between soil water content (Θ) and suction (S) under which soil water content is retained is not unique, but follows different curves depending upon the direction in which they are traversed and the reversal suction at which wetting gives place to drying and vice versa. For the interpretation of the relationship between suction and pore water content Haines considered the geometry of the pore space responsible, as it contains relatively large pores connected with narrower ones. The significance of hysteresis in the study of soil water movement and in the development of the soil moisture profiles, together with the need to further explore the refinements of the phenomenon, was later on, investigated in a greater extent. The independent domain concept first developed for the study of hysteresis in magnetism (Preisach, 1935; Neel, 1942; Everett and Whitton, 1952; Everett and Smith, 1954; Everett, 1954, 1955; Enderby, 1955, 1956) was also applied for the study of hysteresis in porous materials (Poulovassilis, 1962). This theory was proved to be quite satisfactory for the description of the hysteretic relationship Θ–S for some porous bodies (Poulovassilis, 1962, 1970; Talsma, 1970).

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Some years later though, it was shown (Topp and Miller, 1966; Topp, 1969; Poulavassilis and Childs, 1971) that the theory of the independent domains could not in general suffice for the prediction of the hysteretic relationship $\Theta - S$ for all porous bodies. This reason led Poulavassilis and Childs (1971) and Poulavassilis and El-Ghamry (1978) to develop the dependent domain theory, where the hypothesis that for some porous bodies, the water accumulation or withdrawal depends on the values of the variables at reversal was introduced. Philip (1964) was the first to present a computational scheme, resting upon the similarity hypothesis, which was formally equivalent to the independent domain theory. In this scheme one could calculate hysteretic drying and wetting scanning curves using data solely from the main hysteretic branches of wetting and drying. Later on, a number of researchers attempted to simplify the algorithms upon which the calculation of the hysteretic scanning curves was based, and this was reflected in a number of papers (Mualem, 1973, 1974; Mualem and Dagan, 1975; Parlange, 1976; Mualem, 1977; Mualem and Miller, 1979; Mualem, 1984).

In this paper, a comparison between model II of Mualem (Mualem, 1974) and the recent method of Poulavassilis and Kargas (2000) for calculating hysteretic scanning curves in the $\Theta - S$ hysteretic loop is presented. Model II of Mualem was deliberately chosen for compatibility reasons first, and second because this model is more convenient than model I. It is also noted that all later models of Mualem are based on II, trying, with some correction functions to mitigate the weaknesses of model II, for the cases where the independent domain theory is not satisfactory and in others to reduce the needs for extra experimental data.

2. Theory

2.1 MODEL II (MUALEM, 1974)

It is assumed that pores behave independently in the process of emptying or refilling porous bodies. A similarity hypothesis, quite similar to that of Philip (1964), although less restrictive, is employed. This, permits the distribution function of the water elements, entering or leaving the porous medium, to be expressed in a simple way as the product of two functions one having as independent variable the $f_S$ (the suction value of filling) and the other $e_S$ (the suction value of emptying). The distribution function $f$, which is a two-dimensional function of the independent variables $f_S$ and $e_S$ may thus be described as

$$f(f_S, e_S) = h(e_S) l(f_S),$$

(2.1)

e.g., by the product of two marginal distribution functions $h(e_S)$ and $l(f_S)$ indicating the independency of the $h$ and $l$ functions. One should note that the expression
represents the relative volume of pore space which fills when the suction, during the process of imbibition is relaxed from the value \( f_S \) to the value \( f_S - d \) \( f_S \) and empties when the suction, during the process of water withdrawal is increased from the value \( e_S \) to the value \( e_S + d \) \( e_S \). In order to include the reversible amount of water in the drying–wetting processes, Mualem, in his model II replaces the triangular space, upon which the independent variables \( f_S \) and \( e_S \) are defined, with a square (see Figure 1). Moreover, instead of using as independent variables the suctions \( f_S \) and \( e_S \), he introduces a normalized pair of variables \((\bar{p}, \bar{r})\) through the transformation

\[
(\epsilon S, \delta S) \rightarrow (\bar{p}, \bar{r}),
\]

\(\bar{p}\) and \(\bar{r}\) are the dimensionless radii of the pores and their openings, respectively, and are appropriately defined in the region of the square \( R \).

\((\bar{p}, \bar{r}) \in R = (0 \leq \bar{p} \leq 1, 0 \leq \bar{r} \leq 1)\), as is shown in the figure below through the expressions

\[
\bar{p} = \frac{p - R_{\text{min}}}{R_{\text{max}} - R_{\text{min}}}
\]

and

\[
\bar{r} = \frac{r - R_{\text{min}}}{R_{\text{max}} - R_{\text{min}}},
\]

\[2.2\]

\[2.3\]

Figure 1. The filled pore diagrams in the \( \bar{r}, \bar{p} \) plane (the shadowed domain) for (a) the main wetting process and (b) the main drying process (Mualem, 1974).
where
\[ p = \frac{2\gamma}{\rho g \cos \alpha} \]  
(2.4)

and
\[ r = \frac{2\gamma}{\rho g cS}, \]  
(2.5)

\( \gamma \) is the surface tension coefficient of water (\( L^0 M T^{-2} \)), \( \rho \) the density of water (\( L^{-3} M \)) and \( R_{\text{max}}, R_{\text{min}} \) are the maximum and minimum pores’ radii.

In Figure 1 (borrowed from Mualem (1974) the filled pore diagrams in the \( (\tilde{\rho}, \tilde{r}) \) plane (the shadowed domain) are shown (a) the main wetting process and (b) the main drying process. It is obvious that the region OBC extending to the half-left triangle represents the reversible water and that the half-right triangle OAB represents the region where the hysteric water elements are distributed.

In the transformed plane \( (\tilde{\rho}, \tilde{r}) \), the distribution function (2.1) may be written as
\[ f(\tilde{\rho}, \tilde{r}) = l(\tilde{\rho}) h(\tilde{r}). \]  
(2.6)

If we further use the relations
\[ L(\tilde{R}) = \int_{0}^{\tilde{R}} l(\tilde{\rho}) d\tilde{\rho} \]  
(2.7)

and
\[ H(\tilde{R}) = \int_{0}^{\tilde{R}} h(\tilde{r}) d\tilde{r}, \]  
(2.8)

where \( \tilde{R} \) is a specific value that the independent variables \( \tilde{\rho} \) or \( \tilde{r} \) take and it shows the upper limit of integration (see Figure 1 and for more details see Mualem, 1974). Then, it is obvious that
\[ L(0) = 0 \text{ and } H(0) = 0 \]

If one takes, conventionally that
\[ H(1) = 1, \]
then the effective moisture at imbibition \( \Theta(R) = \Theta - \Theta_{\text{min}} \) may easily be obtained by integration as follows:
\[ f\Theta(\tilde{R}) = \int_{0}^{\tilde{R}} l(\tilde{\rho}) d\tilde{\rho} \int_{0}^{1} h(\tilde{r}) d\tilde{r} = L(\tilde{R}) H(1) = L(\tilde{R}). \]  
(2.9)
Another expression of Equation (2.9) is
\[ L(S) = f \Theta(S), \]  
which represents a convenient way of defining \( L \). The respective expression for \( H \) is
\[ H(S) = \frac{e \Theta(S) - f \Theta(S)}{\Theta_u - f \Theta(S)}, \]
where \( \Theta_u \) is the effective moisture at saturation. From the definitions of the functions \( L \) and \( H \) one can easily calculate the scanning curves, either wetting or drying of any order. In this respect the first-order scanning drying curve is obtained by the expression
\[ \Theta(S_{\text{min}}^S) = f \Theta(S) + \left[ \frac{|f \Theta(S) - f \Theta(S_1)|}{\Theta_u - f \Theta(S_1)} \right] \left[ e \Theta(S) - f \Theta(S) \right] \]
and the respective first-order scanning wetting curve is obtained by the expression
\[ \Theta(S_{\text{max}}^S) = f \Theta(S) + \left[ \frac{|\Theta_u - f \Theta(S)|}{\Theta_u - f \Theta(S_1)} \right] \left[ e \Theta(S_1) - f \Theta(S_1) \right]. \]
The expression \( \Theta(S_{\text{min}}^S) \) (and other similar ones) is simply a notation which was suggested by Enderby (1955) and adopted by Everett (1955) and has been used by Mualem in a series of papers (1973, 1974, 1982, 1984) to designate a path in the hysteretic domain. Thus, \( \Theta(S) = \Theta(S_{\text{min}}^S) \) means that the process is: \( S \) growing first from \( S_{\text{min}} \) to \( S_1 \) (wetting) and from \( S_1 \) decreases to \( S \) (drying). In other words the above expression represents a primary drying scanning curve.

For more details one may consult Mualem (1974).

2.2. THE PROPOSED METHODOLOGY

A distribution function \( \bar{F} \) is introduced which retains the features of the distribution function \( F \) of the independent domain concept (see Poulosvassiliis, 1962). Thus, the slope of the wetting boundary curve of a reproducible hysteresis loop at a suction value \( S_i = -P_i \) \( (P \) being the soil water pressure) is given by (see Figures 2 and 3)
\[ - \int_{S_i}^{S_{\text{max}}} \bar{F}(eS_{\text{w}}, S) d eS = f \Theta_i \]
and the slope of the drying boundary curve at the value \( S_n \) by
\[ - \int_{S_{\text{max}}}^{S_n} \bar{F}(eS_{\text{d}}, S) d eS = e \Theta_n \]
Figure 2. Boundary curves of a hypothetical reproducible hysteresis loop describing the water content-suction relationship of a porous body.

Figure 3. Distribution diagram explaining the processes of wetting and drying.
as in the case of the independent domain model where, however, the distribution function $F$ replaces $\bar{F}$. $f_i \Theta_i$ denotes the slope of the wetting boundary curve at the suction value $S_i$ (point A). To this suction a pore water content $f_i \Theta_i$ corresponds. The slope of the drying boundary curve (at the same value $S_i$, e.g. at point B) is denoted by $e_i \Theta_i$. At this point a new value of pore water content $e_i \Theta_i$ corresponds. $e_i \Theta_i$ is always larger than $f_i \Theta_i$ except at the two end points of the hysteresis loop where these two variables have the same value i.e when $S = S_{\text{min}}$ and $S = S_{\text{max}}$, the former being the suction value the two boundary curves meet at saturation and the latter the maximum suction value applied (see Figure 2). For obtaining the values of the distribution function $\bar{F}$ over the area OAB of the distribution diagram shown in Figure 3 according to the independent domain concept a set of experimental primary wetting or drying scanning curves must be available apart from the boundary curves of a reproducible hysteresis loop (see Poulosavissilis, 1962). However, the values of the distribution function $\bar{F}$ can be obtained by partitioning the slopes of the wetting boundary curve of an available reproducible hysteresis loop proportionately to the slopes on the drying boundary curve of the same loop or vice versa, i.e., by partitioning the slopes of the drying boundary curve proportionately to the slopes on the wetting boundary curve. Water elements which enter the pore space when the suction relaxes from $S_i$ to $S_i - \delta S$ following the wetting boundary curve (see Figure 2) have to leave the pore space in the suction range from $S_i$ to $S_{\text{max}}$, after a reversal of the trend of suction from wetting to drying at a $S_i$, contributing thus to the slope of the drying boundary curve in the suction range from $S_i$ to $S_{\text{max}}$. It is supposed that the contribution of these elements to the slope of drying boundary curve is proportional to the magnitude of that slope in the suction range from $S_i$ to $S_{\text{max}}$. Similarly, water elements which leave the pore space when the suction increases from $S_i$ to $S_i + \delta S$ following the drying boundary curve have to re-enter the pore space in the suction range from $S_i + \delta S$ to $S_{\text{min}}$, after a reversal of the trend of suction from drying to wetting at $S_i + \delta S$, contributing thus to the slope of the wetting boundary curve in the suction range from $S_i + \delta S$ to $S_{\text{min}}$. It is again supposed that the contribution of these elements to the slope of the wetting boundary curve is proportional to the magnitude of that slope in the suction range from $S_i + \delta S$ to $S_{\text{min}}$. The procedure for obtaining the values $\bar{F}$ is as follows: The difference $e_i \Theta_i - f_i \Theta_i$ for all suction values applied is recorded. This difference for the suction value $S_i$ is $e_i \Theta_i - f_i \Theta_i$ (see Figure 2, linear segment AB) and it is defined that

$$
\int_{S_{\text{min}}}^{S_{\text{max}}} \int_{S_i}^{S_f} \bar{F}(f_i S_{\text{w}} S) d f_i S d e_i S = e_i \Theta_i - f_i \Theta_i, \quad (2.16)
$$
the quantity under the integral being the pore water standing over the area CDEBC of the distribution diagram shown in Figure 3. Similarly for the suction value \( S_n \)

\[
\int_{S_{\min}}^{S_{\max}} \int_{eS_n}^{eS} F_{i}(iS, eS) dS d eS = e \Theta_n - e \Theta_n, \tag{2.17}
\]

this quantity being the pore water standing over the area HQRBH. The value of the distribution function \( F \) at the point \( G \) \((iS, eS_n)\) is now given by

\[
\bar{F}_{i,n} = \frac{e \Theta_n'}{e \Theta_n - e \Theta_n} \left[ e \Theta_n' - \int_{eS_n}^{eS} F_{i}(iS, eS) dS \right], \tag{2.18}
\]

when the slope of the wetting boundary at \( S_i \) is partitioned proportionately to the slopes of the drying boundary for \( eS > eS_i \) or by

\[
\bar{F}_{i,n} = \frac{e \Theta_n'}{e \Theta_n - e \Theta_n} \left[ e \Theta_n' - \int_{eS_n}^{eS} F_{i}(iS, eS) dS \right], \tag{2.19}
\]

when the slope of the drying boundary at \( S_n \) is partitioned proportionately to the slopes of the wetting boundary for \( eS < eS_n \). The two \( \bar{F}_{i,n} \) values given by Equations (2.18) and (2.19) are equal since, as it can be shown, the right-hand side of both equations is equal to

\[
\left[ e \Theta_n' - \int_{eS_n}^{eS} F_{i}(iS, eS) dS \right] \left[ e \Theta_n' - \int_{eS_n}^{eS} F_{i}(iS, eS) dS \right], \tag{2.20}
\]

the denominator representing the pore water standing over the area HGEBH which is common to the area CDEBC and HQRBH. Thus the partitioning of the slopes in both cases results in giving the same distribution of \( F \) which satisfies the slope Equations (2.14) and (2.15) for all values of \( iS \) and \( eS \). The quantities under the integrals in Equations (2.18) and (2.19) must be determined beforehand by computing the \( \bar{F}_{i,n} \) values for \( iS > iS_i \) in the case of Equation (2.18) or the \( \bar{F}_{i,n} \) values for \( eS < eS_n \) in the case of Equation (2.19). These computations may start at the dry end of the hysteresis loop in the former case or at the wet end in the latter. It is supposed that the function \( F \) has zero values along the line OA of the distribution diagram (see also Philip, 1964). In the case the area OAB is subdivided by a grid of square mesh the above condition is approached by rendering the size of the mesh small
enough and assigning zero $\bar{F}$ values to the little triangles formed along the line OA (see Figure 4, where an example of computing $\bar{F}$ is given).

3. Materials and Methods

In order to study the phenomenon of hysteresis, two different apparatuses have been used in this work. The first one is a version of Haines’ apparatus (Haines, 1930) and the second one is a version of Richards’ pressure cell. The first was used for the determination of the hysteretic $\Theta$–$S$ relationship for a sand porous material, while the latter was used for the determination of the hysteretic $\Theta$–$S$ relationships for undisturbed soil samples.

3.1. DESCRIPTION

A schematic representation of the modified Haines’ apparatus, which has been used is shown in Figure 5. This device basically consists of a Buchner funnel with a porous plate at its base. An elastic, plastic pipe connects the Buchner funnel with a vertical glass pipe which is accurately volumetrically graded. This, enables the experimenter to detect with sufficient accuracy any

$\text{Figure 4. Giving an example for calculating } \bar{F} \text{ values. } \bar{F}_a = \frac{\Theta_i}{\Theta_{i+1}} \left( \Theta_{i+1} - \bar{F}_b \delta S \right), \bar{F}_b = \frac{\Theta_{i+1}}{\Theta_i} \left( \Theta_{i+1} - \bar{F}_a \delta S \right). \text{ The volume } \delta V \text{ of the water element standing on each square of the mesh is } \delta V = \bar{F} \delta S \delta \gamma S.$
changes in the water content of the sample, resulting from changes of the suction, imposed upon the water retained in the porous body (in this case, sand). The system was assumed to attain equilibrium, when no changes of the position of the meniscus in the glass pipe were detected. To reduce evaporation, during experimentation a PVC lid was applied on the upper open side of the funnel, as is shown in Figure 5. A similar device without the porous material, under study, was used, in the same environment in order to measure the evaporation losses.

The sand sample consisted of a 65% (by mass) fraction with particle diameters ranging between 0.5 and 0.71 mm and the rest 35% with particle diameters between 0.105 and 0.250 mm. The above sand mixture was subjected at a range of suctions from 0 to $-\frac{40}{100}$ cm (H$_2$O) in our attempt to determine its moisture retention curve (see Figure 8a,b).

3.2. RICHARDS CELLS

The Richards cells used in this work, have been constructed in the Laboratory of Agricultural Hydraulics. They operate according to the same principles as the original Richards cells (Figure 6). This device differs from the Haines’ apparatus in the fact that the sample is subjected to a greater than the atmospheric pressure (e.g. positive pressure) and the water is forced to leave the sample and drain in the free atmosphere.

In each device only one sample can be investigated. These cells are made from plexiglass due to the relative ease with which one could manipulate...
this material in comparison with other materials, and also due to its transparency and its capacity to sustain relatively moderate to higher pressures. Each cell consists of two square, and parallel plates with dimensions \((10 \times 10) \text{ cm}^2\) and in between them a plastic cylinder is placed. The soil sample is retained in this cylinder. The width of the upper plate is 2 cm and the width of the bottom plate is 3 cm. In the center of the upper plate an opening with a valve permits the connection or the disconnection of the cell to the pressure system (air compressor, pressure regulators etc.). In the bottom plate a shallow cylindrical space with height 3 cm and diameter 6.5 cm is used as a water reservoir. Just above this reservoir the porous plate is placed and on top of this we place the soil sample. The cylinder is tightly fixed with the plates with four screws and two o-rings, one at each plate. The water reservoir at the bottom plate has two exits, one for the air escape, resting at the side of the plate and the other, at the centre of the bottom giving access to the water, through a flexible plastic pipe. This pipe is connected to another cylindrical reservoir made of plexiglass with dimensions 3 cm height and 1.5 cm diameter. At the upper place of this cylindrical reservoir a special opening exists at the same level.

Figure 6. Richards cell. (1) a plexiglass plate, (2) cylinder, (3) a connecting valve, (4) elastic O-rings, (5) porous plate, (6) water reservoir, (7) exits, (9) water reservoir.
as the bottom of the soil sample to permit water entering or leaving the soil sample.

4. Results and Discussions

In this section the results for three different porous materials are presented. Table I shows the specific characteristics of these materials. In Figure 7a the comparison of estimated first-order drying scanning curves, according to the Mualem’s model II and P–K model, together with the experimental data are shown, for the granular porous body, as this has been described by Poulovassilis (1970). The first-order wetting scanning curves for the same porous body are shown in Figure 7b. In Figure 8 the first-order drying (Figure 8a) and first-order wetting (Figure 8b) scanning curves are shown for a sand mixture. Furthermore, the two models were tested against the experimental hysteretic $\Theta(S)$ data for the case of a real soil. These are shown in Figure 9. Calculated first-order drying (Figure 9a) and first-order wetting (Figure 9b) scanning curves, according to the models of Mualem and P–K are shown together with the appropriate data points.

From the comparison of the experimental points with those estimated by the two models (Mualem II and P–K) one may notice that P–K model gives results in closer agreement with the experimental data, than the model by Mualem. The larger deviations from the experimental data points observed in Mualem’s prediction may be attributed to the overestimation of the so called “reversible” water. This is shown to occur due to the high values, the gradient $(\partial \Theta / \partial S)$ assumes at the reversing points whether this is a reverse from wetting to drying or vice versa. A similar behaviour has been noticed by Mualem himself (Mualem, 1974) in his attempt to verify his model against the experimental results of Poulovassilis (1970).

Table I.

<table>
<thead>
<tr>
<th>Sand fractions (%)</th>
<th>Poulovassilis (1970):</th>
<th>Bulk density</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1.4–1 mm)</td>
<td>(0.7–0.6 mm)</td>
</tr>
<tr>
<td>Sand mixture:</td>
<td>15.9%</td>
<td>21.3%</td>
</tr>
<tr>
<td>Soil sample:</td>
<td>0.71–0.5 mm</td>
<td>0.25–0.105 mm</td>
</tr>
<tr>
<td>65%</td>
<td>35%</td>
<td></td>
</tr>
<tr>
<td>Sand</td>
<td>48.2%</td>
<td>28.8%</td>
</tr>
<tr>
<td>Clay</td>
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<tr>
<td>Loam</td>
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<tr>
<td>Bulk density</td>
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</table>
Figure 7. (a) Drying scanning curves predicted by model II (dot lines) and by (P–K) model (solid lines) and experimental points presented originally by Poulovassilis (1970). (b) Wetting scanning curves predicted by model II (dot lines) and by (P–K) model (solid lines) and experimental points presented originally by Poulovassilis (1970).
As an advantage of the P–K model one may mention the relative ease by which one could calculate the entries of the F-distribution and its closer connection to the geometry of the pore space. Moreover one could argue that P–K method could give satisfactory results even in cases where the
dependence of domains is present. This can be circumvented by using a drying scanning curve, besides the two main branches of the hysteretic loop, as this is shown by Poulavassilis and Kargas (2000). As to the safe application of Mualem’s model one should caution himself when using water
elements lying in the square region where the “reversible” water elements reside. In this case, one may be confronted to the “unrealistic” situation where the radius of the pore neck, \( r \), might be found to be larger than the radius of the pore itself. Another weakness of Mualem’s model is that for some porous bodies in the near saturation region, \( dH/dS = h(S) \) may attain negative values which means that the distribution function \( f \) attains negative values too, which is unacceptable. This has been noticed by Mualem in a previous paper (Mualem and Dagan, 1972) and this was the reason which led the author to introduce some corrections in the way \( H \) function is determined. A step forward would be the verification of the recently published “one-curve” hysteretic models (Haverkamp et al., 2002) against the P–K model. Of course these models are not directly compatible but a comparison could be attempted. Preliminary results appear to be encouraging. Another useful contribution in making hysteretic models tractable and easily handled is to develop algorithms by which one could produce scanning curves of any order and direction (wetting or drying), so that the hysteretic behaviour of soil water would be easily included in modeling soil water dynamics.

Acknowledgements
The authors would like to express sincere thanks to Prof. A. Poulovassilis, whose critical comments and scientific guidance for this work have always been constructive and helpful.

References


