Determination of the Maximum Water-Free Production Rate of a Horizontal Well With Water/Oil/Interface Cresting

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ABSTRACT

The number of horizontal oil wells has been increasing rapidly in the past few years because of several advantages of their application. However, further improvement of oil production rate of horizontal wells is limited by the encroachment of the water crest when bottom water exists. This paper focuses on the theoretical and numerical analysis of water-oil-interface cresting behavior in horizontal wells. The objectives of this investigation include: (1) to find a simple approach to determination of the maximum water-free production rate of horizontal wells with water-oil-interface cresting, and (2) to determine the location of the water-oil-interface under critical condition.

The maximum water-free oil production rate (critical rate), which is defined as the oil producing rate at which the unstable water crest forms, is determined analytically for horizontal oil wells using conformal mapping theory. The resulting solution shows that the critical oil production rate is proportional to the conductivity and thickness of the oil reservoir, and water-oil density contrast. This critical oil rate is also a function of the critical crest height and wellbore location in the oil bearing formation.

The lateral extension of the water crest under critical condition is also determined analytically. This solution indicates that the water-oil-interface cresting is a local phenomena in an oil reservoir. The water crest extends laterally within one thickness of the oil reservoir.

Example calculations are confirmed by a numerical reservoir simulator which has matched field data. This solution provides a new approach to evaluating performance of horizontal wells.

INTRODUCTION

The production of oil through a well that horizontally penetrates an oil layer underlain by a water zone causes the oil-water interface to deform into a crest shape. As the production rate is increased, the height of the water crest above the original oil-water contact also increases until at a certain production rate the water crest becomes unstable and water is produced into the well. The maximum water-free production rate is called the critical oil rate.

One of the reasons for drilling horizontal wells is to enhance the oil production rate in some areas where the water coning is a severe problem in vertical wells. However, the encroachment of the water crest is still a limitation for further improving oil production rate in horizontal wells, especially for some cases where the thickness of the oil reservoir is small. Determination of the maximum water-free oil production rate (critical rate) is a very practical problem in horizontal well evaluation. It is also of vital importance to know the location of the water crest for evaluating the oil recovery from horizontal wells. Since horizontal wells have been drilled commercially in the oil industry only in recent years, very few studies have been reported about water-oil cresting behavior.

Assuming the horizontal well is situated near the roof of the reservoir for the lateral edge drive and the bottom water drive cases, Giger presented an analytical 2-D model of water cresting before breakthrough for horizontal wells. Since he used the free surface boundary condition and assumed that the free surface is at a large distance, the oil height in
the model is difficult to choose. In fact, he showed that the interface is tilted with asymptotic parabolic shape, which means that nowhere does it tend to its initial shape, but instead, at some distance from the well, it drops lower than initial water-oil-contact, as low as infinity. Although the mathematical solution was modified by the author, he still suggested that these solutions should not be used for small values of dimensionless drainage radius. In fact, an ideal solution similar to Giger's solution was also given by Efros3 in the early sixties. But again, it is doubtful to be used in practice.

Chaperon4 studied the behavior of cresting toward horizontal wells in an anisotropic formation assuming constant interface elevation at a finite distance. Her approach is identical to that used by Muskat4: Equilibrium conditions are stated first. This approach requires expression of viscous flow potential pertinent to the geometry of the flow created by a horizontal well. However, the same as in Muskat's approach, the presence of the immobile water crest which reduces the cross section of flow was not taken into consideration in Chaperon's solution. Since she neglected the flow restriction due to the immobile water in the crest, her theory might give an optimistic evaluation of critical rate.

Kuchuk et al6 analyzed the pressure transient behavior of horizontal wells with and without a gas cap or aquifer. They considered the irregular two-phase boundary (crest-shaped interface) as a constant pressure boundary and assumed that the gravity effect is negligible and the viscosity of fluid is constant throughout the medium. The existence of the gas or water crest does not affect the pressure response according to their solution. Ozkan and Raghavan7 investigated the time-dependent performance of horizontal wells subject to bottom water drive. They assumed the reservoir boundary at the top of the formation and the boundaries at the lateral extent of the formation to be impermeable, and that an active aquifer at the bottom of the reservoir would yield an effect identical to that of a constant pressure boundary located at the original water-oil contact. The most questionable assumption they made is that the mobility of water in the flooded portion of the oil zone (water in water-crest) is the same as the mobility of the oil. Furthermore, they assumed the density difference between the oil and water to be negligible. These two assumptions are equivalent to an assumption that the moving water-oil-interface does not exist.

Joshi8 made an augmentation of well productivity with slant and horizontal wells using Giger's theory. Joshi8 also compared the results given by the above theories and found out that they are conflicting and different by a factor of up to 20. Papatzacos et al9,11 solved the water breakthrough time for horizontal wells using the moving boundary method with gravity equilibrium assumed in the crests. The results in their papers are based on semi-analytical solutions for time development of a gas or water crest and simultaneous gas and water crests in an infinite reservoir with a horizontal well placed in the oil column. The essential assumption they introduced concerning gas and water is that they are, at each time, in static equilibrium. In other words they participate in the movement of the interface boundaries by expanding when pressure falls, but their flow is neglected. This assumption is in contradiction with another assumption they made: incompressible flow. Their solution maybe valid in the infinite-acting period at low rates. In their semi-analytical solution the total value of numerical errors, which are introduced by the time-step procedure, convergence acceleration and roundoff, was estimated at 5% - 10%. Recently, Yang and Wattenbarger12 presented two correlation methods for water cresting calculations based on numerical simulation.

Guo and Lee13,14 investigated the mechanism of water cresting process graphically. Their studies show that the unstable water crest does not exist for some cases where the pressure gradient beneath the wellbore is lower than the hydrostatic pressure gradient of water. They employed the conformal mapping theory to solve the critical rate. Their results of critical rate computations using the analytical solution matched the results given by a numerical simulator. However, their theory can only be used when the wellbore is located at the roof of an oil reservoir in water-oil-interface cresting systems, or at the bottom of an oil reservoir in gas-oil-interface dipping systems.

In this study, the water cresting behavior in an oil reservoir penetrated by a horizontal well below its roof (at any level) is investigated. The water cresting system is analyzed using both the conformal mapping theory and numerical simulation method. The maximum water-free oil rate and location of the water crest under the critical condition are analytically determined. The height of the water crest under the critical condition is determined based on the numerical simulator runs. Their results of critical rate computations using the analytical solution matched the results given by the numerical simulator. The solution presented in this paper is also applicable to gas-oil-interface dipping systems in which an oil reservoir is penetrated by a horizontal well above its bottom (at any level).

STATEMENT OF THE PROBLEM

Physics of Water Cresting

Since a steady state flow condition prevails during most of the life of oil producing wells, the water cresting problem is usually dealt with in this condition. In the steady state condition, a constant production rate causes a constant pressure drawdown at every point within the constant potential boundaries, which results in a stable potential distribution in the reservoir. When oil flow rate and pressure at the outer boundary are fixed, the spatial variation (distribution) of pressure in the reservoir is a function of reservoir conductivity, which is defined as the permeability of reservoir rock divided by the viscosity of flowing fluid in the reservoir.

In water cresting systems, the upward dynamic force due to wellbore drawdown causes water at the bottom of the oil zone to rise to a certain height at which the dynamic force is balanced by the weight of water beneath this point. Since as the lateral distance from the wellbore increases, the pressure drawdown and upward dynamic force caused by it decrease, the height of the balance-point decreases along the lateral direction. Therefore, the locus of the balance-point is a stable crest-shaped water-oil-interface when the production rate is less than a certain rate. Oil flows above the water-oil-interface, while water remains stationary below the interface.
As the production rate is increased, the height of the water crest above the original oil-water contact also increases until, at a certain production rate, the water crest becomes unstable and water is produced into the well. The reason the water crest becomes unstable at a certain point is that the upward dynamic force caused by wellbore drawdown near the wellbore is so high that it cannot be balanced by the weight of water below the point. In other words, the dynamic pressure gradient above the critical point (beneath wellbore) is everywhere greater than the hydrostatic pressure gradient of water. Therefore, the water in the crest above this critical point can no longer remain stationary, and it flows upward to "try" to build up another balance until water breaks through into the wellbore. Determination of the maximum water-free oil production rate (critical rate) and location of the water crest is the problem to be solved in this investigation.

Assumptions

1. Oil reservoir is homogeneous and isotropic,
2. Oil is an incompressible fluid,
3. Steady state flow condition prevails,
4. Capillary pressure is negligible, abrupt two-phase interface exists,
5. Wellbore is horizontal and straight.

It will be shown in discussions section of this paper that the assumption of the isotropic reservoir can be removed and the solution for an anisotropic reservoir remains in the same form as an isotropic reservoir.

Governing Equation

Steady flow of incompressible fluids through homogeneous porous media is governed by Laplace Equation. The Laplace equation is expressed as

$$\nabla^2 \phi = 0$$  \hspace{1cm} (1)

where the Laplace operator, $\nabla^2$, in cartesian system is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$  \hspace{1cm} (2)

and the velocity potential, $\phi$, is defined as

$$\phi = -\frac{k}{\mu} (\rho_o g y + p)$$  \hspace{1cm} (3)

Boundary Conditions

Figure 1 illustrates a 2-D flow domain in an oil reservoir underlain by a water zone. The outer boundary A-A is a constant influx boundary for a constant production rate. At this boundary, the Darcy's velocities are $v_y = 0$ and

$$v_x = \frac{q}{2d}. $$

The lower boundary A-B-C, the inner boundary C-D-E and the upper boundary E-A are all streamlines. The boundary condition for wellbore (point D) may be treated as a line sink at which the velocity is infinite. The difficulty in treating the boundary B-C is that its shape is flow rate dependent and unknown. However, the velocity potential along this boundary can be expressed as

$$\phi_{woi} = -\frac{k}{\mu} (\rho_o g y_{woi} + p_{woi})$$

or

$$\phi_{woi} = -\frac{k\rho_o g}{\mu} \left( y_{woi} + \frac{p_{woi}}{\rho_o g} \right).$$  \hspace{1cm} (4)

Since $p_{woi} = -\rho_w g y_{woi}$, Eq. (4) becomes

$$\phi_{woi} = \frac{k g \Delta \rho}{\mu} y_{woi}.$$  \hspace{1cm} (5)

where $\Delta \rho = \rho_w - \rho_o$ and $\rho_w$ is the density of water.

ANALYTICAL SOLUTION

The difficulty in obtaining a direct analytical solution to a flow problem involving such boundaries as two-phase interfaces lies in the fact that the locations of these boundaries are not known a priori and are parts of the required solution. However, if the boundaries of the flow region are mapped on a new plane in which the velocities act as coordinates, the boundaries of the flow region in this plane are always known. This plane is called the hodograph plane. The mapping of a flow region in a physical plane on to a hodograph plane is actually isogonal rather than conformal. Therefore, in order to use conformal mapping theory to solve the flow problem, one or several intermediate planes are required.

The step-by-step derivation of the solution is shown in Appendices A and B. The resulting solution is summarized as below.

Critical Oil Rate:

The critical oil production rate, which is defined as the oil producing rate at which the unstable water crest forms is determined for horizontal oil wells using conformal mapping theory. The resulting solution is given by the following three equations:

$$q = \frac{2\pi H_D}{\mu \ln \left[ \frac{t_D (t_C + 1)}{(t_C - t_D)} \right]},$$  \hspace{1cm} (6)

$$\frac{2\pi dk g \Delta \rho}{\mu q} = 2\sqrt{t_C (1 + t_C)} - \ln \left( \frac{\frac{1 + t_C}{t_C} - 1}{\frac{1 + t_C}{t_C} + 1} \right),$$  \hspace{1cm} (7)

and

$$\frac{\mu}{k g \Delta \rho} \left[ \frac{2\sqrt{t_C}}{\pi} \left( \sqrt{t_D-t_C} + \frac{1}{\sqrt{t_C}} \tan^{-1} \sqrt{t_D-t_C} \right) - 1 \right] = 0.$$  \hspace{1cm} (8)

The critical rate, $q$, can be solved by a numerical method from these three equations.
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OF A HORIZONTAL WELL WITH WATER-OIL-INTERFACE CRESTING

For convenience of field application, the above solution is presented graphically in Fig. 2 in terms of dimensionless variables. The dimensionless variables are defined as following:

Dimensionless Critical Rate, \( q_D \):

\[
q_D = \frac{\mu q}{2\pi kd \Delta \rho} \tag{9}
\]

in consistent units, or

\[
q_D = 20231 \frac{\mu q}{kd \Delta \rho} \tag{10}
\]
in field units.

Dimensionless Critical Crest Height at the Cusp Point, \( H_D \):

\[
H_D = \frac{H_c}{d}. \tag{11}
\]

Wellbore Location Index, \( L_w \):

\[
L_w = \frac{c}{d}. \tag{12}
\]

Lateral Extension of Water Crest:

The height of the water crest in critical condition is given by

\[
h_c = \frac{\mu q}{2\pi kd \Delta \rho} \ln \left[\frac{-t_D (t + 1)}{(t - t_D)}\right]. \tag{13}
\]

The lateral distance can be expressed by

\[
x = -\int_{t_c}^{t} \frac{\mu q}{\pi^2 kd \Delta \rho} \left[-\sqrt{t_c (t_c - t)}
+ \frac{1}{2} \ln \left(\frac{1 - \sqrt{\frac{t - t_D}{t_c}}}{1 + \sqrt{\frac{t - t_D}{t_c}}}\right) \right] \left(\frac{1}{t + 1} - \frac{1}{t - t_D}\right) dt. \tag{14}
\]

which can be integrated numerically. The height of water crest, \( h_c \), given by Eq. (13) can be plotted versus lateral distance, \( x \), given by Eq. (14) with parameter \( t \). This plot gives the solution of the lateral extension of the water crest. This solution is presented graphically in Fig. 3 in terms of dimensionless variables. The dimensionless variables are defined as following:

Dimensionless Critical Crest Height, \( h_D \):

\[
h_D = \frac{h_c}{d}. \tag{15}
\]

Dimensionless Lateral Distance, \( x_D \):

\[
x_D = \frac{x}{d}. \tag{16}
\]

It can be seen from Fig. 3 that the water crest extends laterally within one thickness of the oil zone. This solution indicates that the water cresting is a local phenomena in the oil reservoir.

**NUMERICAL SIMULATION**

To confirm the analytical solution given by Eqs. (6), (7) and (8), a 2-D two-phase numerical simulator was built. This numerical model consists of variable number of grid blocks and simulates the flow condition around a horizontal wellbore penetrating an oil reservoir underlain by a water zone. Normalized relative permeability data utilized in the simulator are shown in Table 1.

**Model Validation:**

For validation of the numerical model in this study, field data were taken from horizontal well A5(RD) in Helder oil field on the Dutch Continental Shelf. Field data are listed in Table 2. Since an oil rate less than the critical rate was not economical, the well was produced at a supercritical rate of 2000 BOPD. Data from this well were chosen for comparison with our simulator output because this well has a long horizontal wellbore. In this case a 2-D flow pattern dominates the flow domain, which is consistent with 2-D flow pattern in our simulator. Using these field data as input for our simulator, the simulator predicts 7 days of water crest breakthrough time, which is, fortunately, the same number as observed in field operation.

**Sensitivity Runs:**

Sensitivity analysis was made by varying only one parameter for each run, while keeping others at their base values. Simulation input data and results are shown in Table 3. By examining the data in Table 3, one can draw the following conclusions for the simulation:

1. Critical rate is almost directly proportional to the horizontal permeability.
2. Critical rate is insensitive to vertical permeability.
3. Critical rate is sensitive to drainage radius when drainage area is small.
4. Critical rate is proportional to the thickness of oil zone.
5. Critical rate is inversely proportional to the viscosity of oil.
6. Critical rate is proportional to the density contrast between water and oil.
7. Critical rate is insensitive to water viscosity.
8. Critical rate is inversely proportional to the wellbore height in the oil column.

Conclusions (1), (2), and (4) are reasonable because horizontal flow dominates and contributes to the total oil production rate in thin oil zones where water crests exist. Conclusion (2) is consistent with the analysis given by Chaperon. Conclusion (3) is easy to understand from the material balance point of view. Conclusions (4) and (5) are obvious according to Darcy's law. Conclusions (6) and (8) reflect the mechanism of water cresting discussed in the second section of this paper. Conclusion (7) results from the fact that the move-
ment of the water crest encroaching into the wellbore is very slow.

**COMPARISON OF THE ANALYTICAL SOLUTION WITH THE SIMULATION**

It is noticed that conclusions (1), (4), (5), (6), (7) and (8) in the above section are qualitatively consistent with our analytical solution given by Eqs. (6), (7) and (8). Quantitative comparison between the critical rates calculated using these equations and computed by the simulator is also shown in Table 3. It can be seen that for most cases, the analytical solution matches the numerical simulation.

**DISCUSSIONS**

**Anisotropy**

The governing equation of the problem for an anisotropic reservoir is

\[
k_x \frac{\partial^2 \Phi}{\partial x^2} + k_y \frac{\partial^2 \Phi}{\partial y^2} = 0,
\]

where the potential, \( \Phi \), is defined as

\[
\Phi = \frac{1}{\mu} (\rho_c g y + p).
\]

The change in variables to be used to convert an anisotropic case to isotropic form affects \( x \) and \( y \) coordinates only. The transformation is such that:

\[
k_x' \frac{\partial^2 \Phi}{\partial x'^2} + k_y' \frac{\partial^2 \Phi}{\partial y'^2} = k' \left( \frac{\partial^2 \Phi}{\partial x'^2} + \frac{\partial^2 \Phi}{\partial y'^2} \right),
\]

where the effective parameters are

\[
k' = \sqrt{k_x k_y}
\]

\[
x' = x \sqrt{k'/k_x},
\]

\[
y' = y \sqrt{k'}/k_y,
\]

and \( x \) and \( y \) should be principal permeability axes.

Since

\[
k' \left( \frac{\partial^2 \Phi}{\partial x'^2} + \frac{\partial^2 \Phi}{\partial y'^2} \right) = \left( \frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial y'^2} \right),
\]

where the potential, \( \phi' \), is

\[
\phi' = \frac{k'}{\mu} (\rho_c g y' + p),
\]

Eq. (17) can be expressed as below:

\[
\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial y'^2} = 0.
\]

Eq. (25) has the same form as Eq. (1) except the effective parameters are used. Therefore, the solution to Eq. (1) is also the solution to Eq. (17) if effective parameters are used. Hence, for anisotropic reservoirs, Eqs. (6), (7) and (8) become

\[
q = \frac{2\pi H_d c' k' g \Delta \rho}{\mu \ln \left[ -t_D (t + 1) \right]},
\]

\[
\frac{2\pi d' k' g \Delta \rho}{\mu q} = 2\sqrt{t_C(1 + t_C)} - \ln \left( \frac{\sqrt{1 + t_C} - 1}{\sqrt{1 + t_C} + 1} \right),
\]

and

\[
\frac{\mu}{k' g \Delta \rho} \left[ \frac{2\sqrt{t_C}}{\pi} \left( \sqrt{t_D - t_C} + \frac{1}{\sqrt{t_C}} \tan^{-1} \frac{\sqrt{t_D - t_C}}{t_C} \right) - 1 \right] = 0.
\]

respectively. Therefore, Fig. 2 is still correct for an anisotropic reservoir if one uses the following dimensionless variables:

\[
q_D = \frac{\mu q}{2\pi k' g d' \Delta \rho},
\]

in consistent units, or

\[
q_D = 20231 \frac{\mu q}{k' d' \Delta \rho},
\]

in field units, where

\[
d' = d \sqrt{k'/k_y}.
\]

Similarly, Eqs. (13) and (14) become

\[
h'_c = \frac{\mu q}{2\pi k' g \Delta \rho} \ln \left[ -t_D (t + 1) \right],
\]

and

\[
x' = - \int_{t_c}^{t_D} \frac{\mu q}{\pi^2 k' g \Delta \rho} \left[ -\sqrt{t_C} (t_C - t) 
\right.
\]

\[
+ \frac{1}{2} \ln \left( \frac{1 - \sqrt{t_C - t_c}}{1 + \sqrt{t_C - t_c}} \right) \left( \frac{1}{t + 1} - \frac{1}{t - t_D} \right) dt.
\]

respectively, where

\[
h'_c = h_c \sqrt{k'/k_y}.
\]

Again, Fig. 3 is still correct for an anisotropic reservoir if one uses the effective parameters.

**Critical Crest Height**

The critical crest height, \( H_D \), has not exactly been known. The results of experiments conducted by Muskat and Wyckoff indicate that this critical crest height, for the
particular system represented by their electrical conduction models, is in the vicinity of 76% of the total distance between the bottom of the well and the normal water level. Muskat also concluded in his later work that the critical crest height is about 60% to 75% of that total distance. Dagan and Bear concluded that the critical crest height for seawater fresh-water interface cresting systems is 30% to 50% of that total distance. Since, as demonstrated by Guo and Lee, the unstable water crest does not exist for cases where the pressure gradient beneath the wellbore is lower than the hydrostatic pressure gradient of water, we believe that the critical crest height is a function of pressure gradient beneath the wellbore.

To make extensive use of Fig. 2, the value of $H_D$ should be determined. In fact, applying curve fitting procedure to the $H_D$ data given by the simulator, an empirical formula is found as follow:

$$H_D = 0.033 (1.18 - 0.00246 d')(2.286 \Delta \gamma + 0.77) \left(100 - 67 L_w\right) (\log k' + 8.14) \log X_s. \quad (35)$$

**Calculation Example**

The resulting analytical solution represented by Fig. 2 and Fig. 3 is easy to use in practice. An example of using Fig. 2 is shown below.

Length of horizontal wellbore, $L = 1000 \text{ feet}$,
Thickness of oil reservoir underlain by a water zone, $d = 120 \text{ feet}$,
Horizontal permeability of the oil reservoir, $k_z = 50 \text{ md}$,
Vertical permeability of the oil reservoir, $k_y = 25 \text{ md}$,
Oil viscosity in reservoir conditions, $\mu = 0.8 \text{ cp}$,
Oil formation volume factor, $B_o = 1.2 \text{ rb/stb}$,
Density of water, $\rho_w = 64 \text{ lb/ft}^3$,
Density of oil, $\rho_o = 49 \text{ lb/ft}^3$,
Drainage length, $2X_s = 2640 \text{ ft}$,
Wellbore location, $L_w = 0.7$.

Since $\Delta \rho = 15 \text{ lb/ft}^3$, we know $\Delta \gamma = 0.1 \text{ psi/ft}$. The effective permeability and thickness of the reservoir are

$$k' = \sqrt{(50)(25)} = 35 \text{ md},$$
and

$$d' = (120)\sqrt{35/25} = 142.7 \text{ ft}.$$  

The dimensionless height of water crest at cusp point, $H_D$, can be estimated using Eq. (35):

$$H_D = (0.033)(1.18 - 0.00246(142.7))(2.286)(0.1) + 0.77 \left[100 - (67)(0.7)\right] \log(35) + 8.14 \log(1320)$$

$$= 44 \%$$

The value of $q_D$ corresponding to $H_D = 0.44$ and $c/d = 0.7$ in Fig. 2 is 0.28.

Substituting $q_D$, $k'$, $d'$ and other given data into Eq. (30) gives

$$0.28 = (20231) \frac{(0.8)q}{(35)(142.7)(64 - 49)},$$

which yields

$$q = 1.31 \text{ rb/D ft}.$$  

The total critical rate over the producing wellbore is estimated as

$$Q_c = qL/B_o = (1.31)(1000) / (1.2) = 1092 \text{ STB/D}.$$  

**Conclusions**

1. An analytical solution is provided to estimate the critical oil production rate of a horizontal well in an anisotropic reservoir. The critical rate per unit length of wellbore is proportional to the conductivity and thickness of the oil reservoir, water-oil density contrast, and is related to the critical crest height and wellbore location.

2. The location of the water crest under the critical condition is also determined analytically. This solution indicates that the water crest extends laterally within one thickness of the oil zone. Therefore, the water cresting is a local phenomena in the oil reservoir.

3. A numerical reservoir simulator has been built to check the analytical solution. The output of the simulator matched field data. This numerical model confirmed the analytical solution.

4. An empirical formula for determining the critical crest height is presented based on the numerical simulation, which aids the extensive application of the analytical solution in performance evaluation of horizontal wells.

**Nomenclature**

$A$ = constant of integration in transformation of Schwarz and Christoffel.
$B$ = constant of integration in transformation of Schwarz and Christoffel.
$B_o$ = formation volume factor of oil, rb/stb.
$c$ = distance of wellbore from bottom of oil zone, ft[M].
$d$ = thickness of oil zone, ft[M].
$g$ = $980 \text{ cm/s} = 32 ft/s$, constant of gravitation.
$h$ = hodograph complex plane.
$h_c$ = height of water crest, ft[M].
$h_D$ = dimensionless height of water crest, ft[M].
$H$ = height of water crest at cusp point, ft[M].
$H_D$ = dimensionless height of water crest at cusp point.
$i$ = $\sqrt{-1}$.
$k$ = isotropic permeability of the oil reservoir, md.
$k'$ = effective permeability of the oil reservoir, md.
$k_z$ = horizontal permeability of the oil reservoir, md.
$k_y$ = vertical permeability of the oil reservoir, md.
$L$ = length of perforated horizontal wellbore, ft[M].
$L_w$ = $c/d$, wellbore location index, dimensionless.
$p$ = pressure, psi[Pa].
$p_wi$ = pressure on water-oil interface, psi[Pa].
$q$ = critical oil flow rate per unit length of wellbore in payzone, bbl/D ft[M^3/s/M].
METHOD OF SOLVING THE PROBLEM

The conformal mapping method was adopted to solve the 2-D flow problem. Two important concepts are to be reviewed first.

1. Complex Potential

A potential, \( \phi \), defined as

\[
\phi = -\frac{k}{\mu}(\rho_ogy + p)
\]

is called velocity potential because Darcy's velocities, \( v_x \) and \( v_y \), in the oil reservoir can be determined based on this potential. That is

\[
v_x = \frac{\partial \phi}{\partial x}, \quad v_y = \frac{\partial \phi}{\partial y}.
\]

For incompressible fluids one has the equation of continuity:

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.
\]

From Eqs. (A.2) and Eq. (A.3) it is seen that the velocity potential \( \phi \) is harmonic, i.e. satisfies Laplace's equation

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.
\]
It follows that there must exist a conjugate harmonic function, $\psi$, such that

$$\Omega(z) = \phi(x, y) + i\psi(x, y) \quad (A.5)$$

is analytic. The function $\Omega$ is, as well known, called the complex potential.

Differentiating Eq. (A.5) gives

$$\frac{d\Omega}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial y} - i\frac{\partial \phi}{\partial y} = v_x - iv_y. \quad (A.6)$$

Eq. (A.6) is a very important equation relating physical plane $z = x + iy$, hodograph plane $h = v_x + iv_y$ and complex potential plane $\Omega = \phi + i\psi$.

2. Hodograph Mapping

Mapping of straight boundaries of a flow region in hodograph plane is not a difficult task because the directions, and sometimes the magnitudes, of velocities on the boundaries are known. Therefore, the hodograph mapping of straight boundaries of our flow region will not be discussed here. However, the mapping of water-oil interface on the hodograph plane is not so obvious and needs to be discussed in detail.

It has been shown that the velocity potential along the water-oil-interface can be expressed as

$$\phi_{woi} = \frac{k g \Delta \rho}{\mu} y_{woi}. \quad (A.7)$$

where $\Delta \rho = \rho_w - \rho_o$ and $\rho_w$ is the density of water. The velocity along the water-oil-interface, $v_l$, may be obtained by differentiating Eq. (A.7) with respect to the curve linear length, $l$, of the interface:

$$v_l = \frac{k g \Delta \rho}{\mu} \frac{\partial y_{woi}}{\partial l}. \quad (A.8)$$

Multiplying Eq. (A.8) by $v_l$ gives

$$v_l^2 = \frac{k g \Delta \rho}{\mu} v_l \frac{\partial y_{woi}}{\partial l}. \quad (A.9)$$

Since

$$v_y = v_l \frac{\partial y}{\partial l}$$

and

$$v_l^2 = v_x^2 + v_y^2,$$

Eq. (A.9) may be rewritten as

$$v_x^2 + \left( v_y - \frac{k g \Delta \rho}{2\mu} \right)^2 = \left( \frac{k g \Delta \rho}{2\mu} \right)^2 \quad (A.10)$$

Eq. (A.10) shows that the water-oil interface in the hodograph plane is part of a circle with center at point $(0, kg \Delta \rho/2\mu)$ and radius of $(kg \Delta \rho/2\mu)$.

The general procedure for solving a flow problem by the conformal mapping method is summarized in the following steps:

1. Given the boundary of a flow domain in physical plane $z = x + iy$, map it on to the hodograph plane $h = v_x + iv_y$.

2. Find a transformation with which the flow domain in the hodograph plane can be mapped on to a modified hodograph plane $\varepsilon = \alpha + i\beta$ and the mapping between the physical plane and the modified hodograph plane is conformal.

3. Map the boundary of the flow domain in the hodograph plane on to the modified hodograph plane.

4. Map the boundary of the flow domain in the complex potential plane $\Omega = \phi + i\psi$.

5. Determine a conformal mapping relationship $\varepsilon = \varepsilon(\Omega)$ that maps the $\varepsilon-$plane on the $\Omega-$plane. Sometimes this cannot be derived directly and an intermediate plane, or several such planes, are required. For example, we may map the flow domain in the $\varepsilon-$plane on to the upper half of some $\varepsilon = t + is$ plane by $\varepsilon = \varepsilon(\Omega)$; we also map the flow domain represented in the $\Omega-$plane on the same upper half of the $\varepsilon-$plane by $\Omega = \Omega(\varepsilon)$. Finally, we establish the required relationship $\varepsilon = \varepsilon(\Omega)$.

6. Solve differential equation

$$\frac{d\Omega}{dz} = v_x - iv_y$$

for $\Omega = \Omega(z)$ to obtain velocity potential function $\phi$ and stream function $\psi$.

In solving our water-oil coning problem, we focused on the determination of critical oil rate. Therefore, instead of solving for velocity potential function and stream function, we solve the location of two-phase interface. From the resulting solution we determine the critical production rate.

APPENDIX B

DETERMINATION OF THE CRITICAL OIL RATE AND LATERAL EXTENSION OF THE WATER CREST

Since the flow region in physical plane is symmetric, only half of the flow region is studied. The flow domain in physical plane $z = x + iy$ is mapped in Fig. 5(a) under critical condition.

Mapping of the flow domain in hodograph plane is shown in Fig. 5(b), based on the mapping of the domain in physical plane and Eq. (A.10).

Because of the inconformity of mapping from physical plane to hodograph plane, we need to modify the hodograph plane to use conformal mapping theory. Among many transformations, the following one may be utilized to obtain the conformity:

$$\varepsilon = \alpha + i\beta = \frac{ih}{v_x^2 + v_y^2} = \frac{i}{v_x - iv_y} \quad (B.1)$$

This transformation is chosen because it simply relates to the flow domains represented in $z-$plane and in $\Omega-$plane.
through Eq. (A.6). In fact, substituting it into Eq. (A.6) gives

\[ \frac{d\Omega}{dz} = \frac{i}{\varepsilon}, \]

from which one obtains

\[ dz = -i\varepsilon d\Omega. \tag{B.2} \]

Eq. (B.2) is the fundamental equation of solving our problem.

Using Eq. (B.1), the flow domain in hodograph plane is mapped on to the modified hodograph plane ($\varepsilon$–plane) as shown in Fig. 5(c).

Since the value of velocity potential $\phi$ is a relative one, it does not matter where we map each point of flow boundary on the complex potential plane, as long as these points are arranged in such an order as to follow one direction on a stream line. Choosing the stream function as zero along the lower boundary of the flow domain in physical domain and assuming the velocity potential at point $B$ is zero, we map the flow domain in complex potential plane as shown in Fig. 5(d).

Since it is difficult to directly relate the flow domain represented in the modified hodograph plane and that in the complex potential plane, we may indirectly relate them through a third plane. We can first map the flow domains represented in both of the two planes to the upper half of an intermediate plane by two transformations, and then combine the two transformations. The flow domain represented by the upper half of the intermediate plane, $\Omega = t + is$, is shown in Fig. 5(e). In the figure, $t_A = -1$, $t_B = 0$ and $t_E = \infty$ are chosen.

The transformation between $\varepsilon$–plane and $\Omega$–plane may be obtained by employing the Transformation of Schwarz and Christoffel:

\[ \varepsilon = A \int (\Omega - t_A)^{2\pi/\pi - 1}(\Omega - t_B)^{\pi/\pi - 1} d\Omega + B. \tag{B.3} \]

Since we have chosen $t_A = -1$, $t_B = 0$ and $t_E = \infty$, Eq. (B.3) may be simplified as

\[ \varepsilon = A \int \frac{\Omega + 1}{\Omega - t_C} d\Omega + B \tag{B.4} \]

which can be integrated along the real axis as

\[ \varepsilon = 2A \left( \sqrt{t - t_C} + \frac{1}{\sqrt{t_C}} \tan^{-1} \sqrt{\frac{t - t_C}{t_C}} \right) + B. \tag{B.5} \]

By examining the change in $\varepsilon$ at point $B$, one can determine the constant $A$:

\[ A = \frac{\mu\sqrt{t_C}}{\pi kg \Delta\rho}. \tag{B.6} \]

Substitution of Eq. (B.6) into Eq. (B.5) gives

\[ \varepsilon = \frac{2\mu\sqrt{t_C}}{\pi kg \Delta\rho} \left( \sqrt{t - t_C} + \frac{1}{\sqrt{t_C}} \tan^{-1} \sqrt{\frac{t - t_C}{t_C}} \right) + B. \tag{B.7} \]

By applying Eq. (B.7) at point $C$ we can determine constant $B$:

\[ B = -\frac{\mu}{kg \Delta\rho}. \tag{B.8} \]

Substituting Eq. (B.8) into Eq. (B.7) gives

\[ \varepsilon = \frac{\mu}{kg \Delta\rho} \left[ \frac{2\sqrt{t_C}}{\pi} \left( \sqrt{t - t_C} + \frac{1}{\sqrt{t_C}} \tan^{-1} \sqrt{\frac{t - t_C}{t_C}} \right) - 1 \right]. \tag{B.9} \]

The transformation between $\Omega$–plane and $\mathfrak{R}$–plane may be again obtained by employing the Transformation of Schwarz and Christoffel:

\[ \Omega = A \int (\mathfrak{R} - t_A)^{0/\pi - 1}(\mathfrak{R} - t_D)^{0/\pi - 1} d\mathfrak{R} + B. \tag{B.10} \]

Since we have chosen $t_A = -1$, $t_B = 0$ and $t_E = \infty$, Eq. (B.10) may be simplified as

\[ \Omega = A \int \frac{d\mathfrak{R}}{(\mathfrak{R} + 1)(\mathfrak{R} - t_D)} + B \tag{B.11} \]

which can be integrated along the real axis as

\[ \Omega = -\frac{A}{1 + t_D} \ln \left( \frac{t + 1}{t - t_D} \right) + B. \tag{B.12} \]

By examining the change in $\Omega$ at point $A$, one can determine the constant $A$:

\[ \frac{A}{1 + t_D} = \frac{q}{2\pi}. \tag{B.13} \]

Substitution of Eq. (B.13) into Eq. (B.12) gives

\[ \Omega = -\frac{q}{2\pi} \ln \left( \frac{t + 1}{t - t_D} \right) + B. \tag{B.14} \]

By applying Eq. (B.14) at point $B$, we can determine constant $B$:

\[ B = \frac{q}{2\pi} \ln \left( \frac{1}{1 - t_D} \right). \tag{B.15} \]

Substituting Eq. (B.15) into Eq. (B.14) gives

\[ \Omega = -\frac{q}{2\pi} \ln \left( \frac{-t_D(t + 1)}{t - t_D} \right). \tag{B.16} \]

The solution of water-oil interface can be obtained solving Eq. (B.2). Differentiating Eq. (B.16) gives

\[ d\Omega = -\frac{q}{2\pi} \left( \frac{1}{t + 1} - \frac{1}{t - t_D} \right) dt. \tag{B.17} \]

Substitutions of Eq. (B.9) and Eq. (B.17) into Eq. (B.2) gives

\[ dz = -\frac{2i\mu}{\pi^2 kg \Delta\rho} \left[ \sqrt{t_C(t - t_C)} + \frac{1}{\sqrt{t_C}} \tan^{-1} \sqrt{\frac{t - t_C}{t_C}} - \frac{\pi}{2} \right] \left( \frac{1}{t + 1} - \frac{1}{t - t_D} \right) dt. \tag{B.18} \]
The imaginary part of Eq. (B.18) is
\[ dy = \frac{\mu q}{2\pi kg \Delta \rho} \left( \frac{1}{t+1} - \frac{1}{1-t_D} \right) dt. \] (B.19)

The height of water cone \((h_c)\) can be obtained by integration of Eq. (B.19):
\[ h_c = y - y_B = \int_y^{y_B} dy = \int_t^{t_D} \frac{\mu q}{2\pi kg \Delta \rho} \left( \frac{1}{t+1} - \frac{1}{1-t_D} \right) dt = \frac{\mu q}{2\pi kg \Delta \rho} \ln \left[ \frac{-t_D(t+1)}{(t-t_D)} \right]. \] (B.20)

At the cusp point \(C\), \(t = t_C\). The height of the water cone \((H_c)\) is
\[ H_c = y_C - y_B = \frac{\mu q}{2\pi kg \Delta \rho} \ln \left[ \frac{-t_D(t_C+1)}{(t_C-t_D)} \right]. \] (B.21)

To determine the constant \(t_D\), one can apply Eq. (B.9) at point \(D\):
\[ \frac{i2d}{q} = \frac{\mu}{kg \Delta \rho} \left[ \frac{2\sqrt{t_D}}{\pi} \left( \sqrt{1-t_C} \right) + \frac{1}{\sqrt{t_C}} \tan^{-1} \left( \frac{t_D-t_C}{t_C} \right) - 1 \right]. \] (B.22)

The constant \(t_D\) can be solved from Eq. (B.22) if constant \(t_C\) is known. To determine the constant \(t_C\), one can apply Eq. (B.9) at point \(A\):
\[ \frac{i2d}{q} = \frac{\mu}{kg \Delta \rho} \left[ \frac{2\sqrt{t_C}}{\pi} \left( \sqrt{-1-t_C} \right) + \frac{1}{\sqrt{t_C}} \tan^{-1} \left( \frac{-1-t_C}{t_C} \right) - 1 \right]. \] (B.23)

i.e.,
\[ \frac{2\pi kg \Delta \rho}{\mu q} = 2\sqrt{t_C(1+t_C)} - \ln \left( \frac{\sqrt{1+t_C} - 1}{\sqrt{1+t_C} + 1} \right) \] (B.24)

from which \(t_C\) can be solved numerically.

Assuming the dimensionless critical-cone-height is \(H_D\), the real critical-cone-height \(H_c\) is then \(H_Dc\). Therefore, the critical oil production rate for unit length of wellbore, according to Eq. (B.21), is
\[ q_c = \frac{2\pi H_Dckg \Delta \rho}{\mu \ln \left[ \frac{-t_D(t_C+1)}{(t_C-t_D)} \right]}. \] (B.25)

The critical rate, \(q_c\), can be solved by a numerical method from Eqs. (B.22), (B.24) and (B.25).
### TABLE 1
Relative Permeabilities, Normalized

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### TABLE 2
Helder Field Parameters, Well A5(RD)

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### TABLE 3
Critical Rate Given by the Analytical Solution and Simulation

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<th>$k_y$ (md)</th>
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DETERMINATION OF THE MAXIMUM WATER-FREE PRODUCTION RATE OF A HORIZONTAL WELL WITH WATER-OIL INTERFACE CRESTING

Fig. 1 -- A schematic diagram to illustrate water cresting

Fig. 2 -- Solution of dimensionless critical rate

Fig. 3 -- Lateral extension of the critical water crest

Fig. 4 -- Lateral extension of the critical water crest
Fig. 5 -- Mappings of flow domain for a water coning system