UNIVERSITETET I STAVANGER
INSTITUTT FOR MATEMATIKK OG NATURVITENSKAP

FYS 610 Many-particle quantum mechanics

Suggested solutions, exercises for 7 April 2017

PROBLEM 28:

a) The LSZ formula, eq. (19.1), reads in this:

$$\langle p', s'|S|\, p, s\rangle = \left[\mathrm{i}\,\bar{u}^{s'}(p')\int \mathrm{d}^4x' e^{-\mathrm{i}p'x'} S_F^{-1}(x')\right]\left[\mathrm{i}\,u^s(p)\int \mathrm{d}^4x\, e^{-\mathrm{i}px} S_F^{-1}(x)\right]$$
$$\times \langle \Omega|T\left\{\psi^s(x)\bar{\psi}^{s'}(x')\right\}|\,\Omega\rangle$$

To the lowest order in perturbation theory *Schwartz* eq. (8.64) reduces to:

$$\langle \Omega|T\left\{\psi^s(x)\bar{\psi}^{s'}(x')\right\}|\,\Omega\rangle = \frac{\langle 0|T\left\{\psi^s(x)\bar{\psi}^{s'}(x')e^{\mathrm{i}\int \mathrm{d}^4y\mathcal{L}_I[\psi]}\right\}|\,0\rangle}{\langle 0|T\left\{e^{\mathrm{i}\int \mathrm{d}^4y\mathcal{L}_I[\psi]}\right\}|\,0\rangle}$$
$$\approx \frac{\langle 0|T\left\{\psi^s(x)\bar{\psi}^{s'}(x')(1+\mathrm{i}q\int \mathrm{d}^4y\,\bar{\psi}(y)\gamma^\mu\psi(y)A_\mu(y))\right\}|\,0\rangle}{\langle 0|T\left\{(1+\mathrm{i}q\int \mathrm{d}^4y\,\bar{\psi}(y)\gamma^\mu\psi(y)A_\mu(y))\right\}|\,0\rangle}$$
$$= S_F^{ss'}(x-x') + \mathrm{i}q\int \mathrm{d}^4y \sum_{rr'} S_F^{sr}(x-y)(\gamma^\mu)^{rr'} S_F^{r's'}(y-x')A_\mu(y)\,,$$

where we have used Wick's theorem, inserted spinor indices, and cancelled the bubble diagram generated by $\langle 0|T\left\{\psi(y)\bar{\psi}(y)\right\}|\,0\rangle$. If we insert this result in the LSZ formula, shifting the integration variable $x \to x-y$, $x' \to x'-y$, we see that the first term vanishes because of momentum conservation if $p \neq p'$, while the propagators cancel yielding two delta functions which takes care of the two of the integrals, while the remaining integral yields the Fourier transform of A_μ. Also, since A_μ does not couple to the spins, $s' = s$ is conserved, and will be dropped. Thus the end result is [possibly up to a sign!]:

$$\langle p', s'|S|\, p, s\rangle = \mathrm{i}q\bar{u}^{s'}(p')\gamma^\mu u^s(p)\tilde{A}_\mu(p'-p)\,,$$

b) This follows immediately from the previous result, if we retain all the terms in the expansion of $e^{\mathrm{i}\int \mathrm{d}^4y\mathcal{L}_I}$.

c) With $A^\mu(x) = A^\mu(\mathbf{x})$, we have:

$$\tilde{A}^\mu(p) = \int \mathrm{d}^4x\, e^{\mathrm{i}p\cdot x}A^\mu(x)\int \mathrm{d}^3\mathbf{x}\, e^{-\mathrm{i}\mathbf{x}\cdot\mathbf{p}}A^\mu(\mathbf{x})\int_{-\infty}^{\infty}\mathrm{d}\,x^0 e^{\mathrm{i}x^0p^0} = 2\pi\delta(p^0)\tilde{A}^\mu(\mathbf{p})\,,$$

from which the result follows.

d) This follows from a above, with

$$\mathcal{M} = q\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(\mathbf{p}' - \mathbf{p})\,.$$

e) The result follows precisely as in *Schwartz* eq. (5.19-22), except that there is only one incoming (and one outgoing) particle. Using $\delta(0) = \frac{T}{2\pi}$, we have instead of eq. (S 5.18):

$$|\langle p', s'|S|\, p, s\rangle|^2 = \delta(p^0 - p'^0)T(2\pi)|\mathcal{M}|^2\,.$$

Instead of eq. (S 5.20) we have ($E = p^0$)

$$\begin{aligned}
\mathrm{d}P &= \frac{|\langle p', s'|S|\, p, s\rangle|^2}{\langle p', s'|p', s'\rangle\langle P, s|p, s\rangle}\mathrm{d}\Pi = \frac{\delta(p^0 - p'^0)T(2\pi)}{(2p^0V)(2p'^0V)}|\mathcal{M}|^2\frac{V\mathrm{d}^3\mathbf{p}'}{(2\pi)^3} \\
&= \frac{T}{V}\frac{2\pi\delta(p^0 - p'^0)}{(2p^0)(2p'^0)}|\mathcal{M}|^2\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3}\,.
\end{aligned}$$

Thus:

$$\mathrm{d}\sigma = \frac{V}{T}\frac{1}{v_i}\mathrm{d}P = \frac{1}{v_i}\frac{1}{2p^0}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2p'^0}|\mathcal{M}(p, s \to p', s')|^2(2\pi)\delta(p'^0 - p^0)\,.$$

Furthermore, from part d) above:

$$\mathcal{M} = 4\pi Ze^2\bar{u}(p')\gamma^0 u(p)\frac{1}{(\mathbf{p} - \mathbf{p}')^2}\,.$$

f) Since energy is conserved, so is the magnitude of the momentum, $|\mathbf{p}| = |\mathbf{p}'|$, so we have:

$$(\mathbf{p} - \mathbf{p}')^2 = \mathbf{p}^2 + \mathbf{p}'^2 - 2\mathbf{p}\cdot\mathbf{p}' = 2\mathbf{p}^2(1 - \cos\theta) = 4\mathbf{p}^2\sin^2\frac{\theta}{2}\,.$$

Since $\mathcal{M}$ is the same for both spins and spin is conserved, averaging over initial spin directions and summing over the final ones yields just a factor bye. Since, as we have shown before, $\delta(p^0 - p'^0) = \frac{p^0}{|\mathbf{p}|}\delta(|\mathbf{p}| - |\mathbf{p}'|)$. Furthermore $v_i\, p^0 = |\mathbf{p}|$, so after integrating over $|\mathbf{p}'$, using the delta function, we find:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathbf{p}^2}{16\pi^2 v_i\, p^0|\mathbf{p}|}\frac{16\pi^2Z^2e^4}{16\mathbf{p}^4\sin^4\frac{\theta}{2}}\left|\bar{u}^\dagger(p')u(p)\right|^2 = \frac{Z^2e^4}{16\mathbf{p}^4\sin^4\frac{\theta}{2}}\left|\bar{u}^\dagger(p')u(p)\right|^2\,.$$

Unfortunately, we have not shown how to evaluate

$$\left|\bar{u}^\dagger(p')u(p)\right|^2 = 4m^2\left(1 - \frac{|\mathbf{p}|^2}{m^2}\sin^2\frac{\theta}{2}\right)\,.$$

g) Straightforward.