UNIVERSITETET I STAVANGER

Institutt for matematikk og naturvitenskap

FYS 610 Many-particle quantum mechanics

Exercises for 21 April 2017

Problem 28: [Hard] If we treat the electromagnetic potential A^μ as a classical external field, the interaction Lagrangian is

$$\mathcal{L}_I = q\bar{\psi}\gamma^\mu\psi A_\mu,$$

where $A^\mu(x)$ is a given function pf x and q is the charge of the quantized Dirac particle.

a) Show that the S-matrix element for inelastic scattering, $p' \neq p$, for a particle in this potential to the lowest order in the interaction is given by:

$$\langle p', s'|S|\, p, s\rangle = \mathrm{i}q\bar{u}^{s'}(p')\gamma^\mu u^s(p)\tilde{A}_\mu(p'-p)\,,$$

where $\tilde{A}(p)$ is the Fourier transform of $A_\mu(x)$.

b) Show that $\langle p', s'|S|\, p, s\rangle$ can be evaluated by Feynman rules, where the interaction with the external field is simply represented by:

$$p \;\longrightarrow\!\!\bullet\!\!\longrightarrow\; p' \quad = \quad \mathrm{i}q\gamma^\mu\tilde{A}_\mu(p-p')\,.$$

No additional 4-momentum conservation is implied.

c) Show that for a static (time-independent) 4-vector potential $A^\nu(x)$, the above vertex takes the form:

$$p \;\longrightarrow\!\!\bullet\!\!\longrightarrow\; p' \quad = \quad \mathrm{i}q\gamma^\mu\tilde{A}_\mu(\mathbf{p}-\mathbf{p}')\,,$$

where energy is now conserved at the vertex.

d) Show that for a static potential the S-matrix element can be written:

$$\langle p', s'|S|\, p, s\rangle = \mathrm{i}\mathcal{M}(2\pi)\delta(p'^0 - p^0)\,.$$

e) Show that the cross section for a static $A^\mu(x)$ cam be written:

$$\mathrm{d}\sigma = \frac{1}{v_i}\frac{1}{2p^0}\frac{\mathrm{d}^3p'}{(2\pi)^3 2p'^0}\left|\mathcal{M}(p,s\to p',s')\right|^2(2\pi)\delta(p'^0-p^0)\,,$$

where v_i is the initial velocity in the coordinate system where $A_\mu(x)$ is static. Calculate $\mathcal{M}$ for the scattering of an electron of charge $q=-e$ $(e>0)$ from the Coulomb potential of a nucleus, $A^\mu = \delta_{\mu,0}Ze/r$. $[\widetilde{1/r} = 4\pi/\mathbf{q}^2]$.

f) Show that the unpolarized cross section, averaged over initial and summed over final spin states, is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2e^4m^2}{16|\mathbf{p}|^4\sin^4\frac{\theta}{2}}\left(1 - \frac{|\mathbf{p}|^2}{m^2}\sin^2\frac{\theta}{2}\right)\,.$$

This is called the *Mott cross section*.

g) Show that the above result reduces to the Rutherford cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2e^4}{4m^2v^4\sin^4\frac{\theta}{2}}\,.$$

in the non-relativistic limit, $|\mathbf{p}| \to mv$, $v \ll 1$.