UNIVERSITETET I STAVANGER
Institutt for matematikk og naturvitenskap

FYS 610 Many-particle quantum mechanics

Suggested solutions, exercises for 7 April 2017

Problem 22:

a) From the fundamental anticommutation relation, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$ it follows:

$$(\gamma^0)^2 = \mathbb{1}_4\,; \qquad (\gamma^i)^2 = -\mathbb{1}_4\,; \qquad \gamma^\mu\gamma^j = -\gamma^j\gamma^\mu \quad (\mu \neq j)\,.$$

Thus from the definition of γ^5:

$$\begin{aligned}(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -(-1)^3\gamma^1\gamma^2\gamma^3(\gamma^0)^2\gamma^1\gamma^2\gamma^3 = (-1)^2\gamma^2\gamma^3(\gamma^1)^2\gamma^2\gamma^3 \\ &= (-1)^2\gamma^3(\gamma^2)^2\gamma^3 = \mathbb{1}_4\,.\end{aligned}$$

b) From $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ it follows that $\gamma_\mu\gamma^\mu = g_\mu{}^\mu = \delta_{\mu\mu} = 4$, so:

$$\begin{aligned}\not{p}\gamma^\mu &= p_\nu\gamma^\nu\gamma^\mu = p_\nu(2g^{\nu\mu} - \gamma^\mu\gamma^\nu) = 2p^\mu - \gamma^\mu\not{p} \quad \Longleftrightarrow \quad \{\not{p}, \gamma^\mu\} = 2p^\mu\,; \\ \gamma_\mu\not{p}\gamma^\mu &= \gamma_\mu(2p^\mu - \gamma^\mu\not{p}) = 2\not{p} - 4\not{p} = -2\not{p}\,.\end{aligned}$$

c) Using the previous result and $\not{p}^2 = p^2$, we have:

$$\begin{aligned}\not{p}\not{q} &= q_\mu\not{p}\gamma^\mu = 2p\cdot q - \not{q}\not{p} \quad \Longleftrightarrow \quad \{\not{p}, \not{q}\} = 2p\cdot q\,; \\ \gamma_\mu\not{p}\not{q}\not{p}\gamma^\mu &= \gamma_\mu\not{p}\not{q}(2p^\mu - \gamma^\mu\not{p}) = 2\not{p}\not{p}\not{q} - \gamma_\mu\not{p}(2q^\mu - \gamma^\mu\not{q})\not{p} \\ &= 2p^2\not{q} - 2\not{q}\,p^2 + \gamma_\mu\not{p}\gamma^\mu\not{q}\not{p} = -2\not{p}\not{q}\not{p}\,.\end{aligned}$$

d) From $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ it follows that γ^μ commutes with three of the gamma-matrix factors in the definition of γ^5 and commutes with the fourth, γ^μ, so it anticommutes with γ^5.

e) From the cyclical properties of the trace we have:

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = \frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] = g^{\mu\nu}\mathrm{Tr}\,\mathbb{1}_4 = 4g^{\mu\nu}\,.$$

Using this and the fundamental commutation relation we find:

$$\begin{aligned}\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] &= -\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\beta] + 2g^{\beta\nu}\mathrm{Tr}[\gamma^\mu\gamma^\alpha] \\ &= -\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\beta] + 8g^{\alpha\mu}g^{\beta\nu}\,.\end{aligned}$$

By repeated applications of this result we then have:

$$\begin{aligned}\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] &= -\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\beta] + 8g^{\alpha\mu}g^{\beta\nu}\,.\\ &= \mathrm{Tr}[\gamma^\alpha\gamma^\nu\gamma^\mu\gamma^\beta] + 8(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})\\ &= -\mathrm{Tr}[\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta] + (g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu})\,.\end{aligned}$$

But the matrices in the trace in the last expression are just a cyclical rearrangement of those on the left hand side, so by the invariance of the trace under cyclical permutations, we find:

$$\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu})\,.$$

Problem 23: Using the hint, $(\gamma^5)^2 = \mathbb{1}_4$, and the properties of the trace, we find

$$\mathrm{Tr}[\gamma^{\mu_1}\ldots\gamma^{\mu_n}] = \mathrm{Tr}[\gamma^5\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma^5] = (-1)^n\mathrm{Tr}[\gamma^{\mu_1}\ldots\gamma^{\mu_n}(\gamma^5)^2]\,,$$

so

$$\mathrm{Tr}[\gamma^{\mu_1}\ldots\gamma^{\mu_n}] = 0\,,$$

if n is odd.

Problem 24: We have:

$$\not{A}\not{B} = A_\mu B_\nu\frac{1}{2}\left(\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right) = A_\mu B_\nu\frac{1}{2}(2g^{\mu\nu} - 2\mathrm{i}\sigma^{\mu\nu}) = A\cdot B\,\mathbb{1}_4 - \mathrm{i}A_\mu B_\nu\,\sigma^{\mu\nu}\,.$$

Hence, if $[B_\nu,\gamma^\mu] = 0$, since $\mathrm{Tr}\sigma^{\mu\nu} = -2\mathrm{i}\,\mathrm{Tr}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] = 0$:

$$\mathrm{Tr}[\not{A}\not{B}] = 4A\cdot B\,.$$

Problem 25: We note the following relations:

$$\gamma^{0\dagger} = \gamma^0\,,\qquad \gamma^{i\dagger} = -\gamma^i$$

which, together with the fundamental anticommutation relation yields

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0\,.$$

Hence if $u(p)$ is a Dirac spinor, satisfying $(\not{p} - m\mathbb{1}_4)u(p) = 0$ and $\bar{u}'(p) = u'^\dagger(p)\gamma_0$ satisfies the conjugated Dirac equation:

$$\begin{aligned}\bar{u}'(p)(\not{p} - m\mathbb{1}_4) &= u'^\dagger(p)\gamma^0(p_\mu\gamma^\mu - m\mathbb{1}_4)\gamma^{0^2} = u'^\dagger(p)(p_\mu\gamma^{\mu\dagger}\gamma_0 - m\mathbb{1}_4)\gamma_0\\ &= [\gamma_0(p_\mu\gamma^\mu\gamma_0 - m\mathbb{1}_4)u'(p)]^\dagger = 0\end{aligned}$$

If a^μ is an arbitrary 4-vector, we then have, using the result in the previous problem:

$$\begin{aligned}0 &= \bar{u}'(q)(\not{q} - m\mathbb{1}_4)\not{a}u(p) + \bar{u}(q)\not{a}(\not{p} - m\mathbb{1}_4)u(p)\\ &= \bar{u}'(q)\left[-2m\,\gamma^\mu a_\mu + (p^\mu + q^\mu)a_\mu + \mathrm{i}(q_\nu - p_\nu)\sigma^{\mu\nu}a_\mu\right]u(p)\end{aligned}$$

But this equation is linear in a^μ, and if it is satisfied for all values of a^μ, the coefficient of a_μ must vanish. Equivalently, we may differentiate the equation with respect to a_μ. Hence we have proven the Gordon decomposition:

$$\bar{u'}(q)\gamma^\mu u(p) = \bar{u}'(q)\left[\frac{p^\mu + q^\mu}{2m} + \mathrm{i}\frac{q^\nu - p^\nu}{2m}\sigma^{\mu\nu}\right]u(p)\,.$$

Problem 26 [Not yet available]

Problem 27 [Not yet available]