UNIVERSITETET I STAVANGER
INSTITUTT FOR MATEMATIKK OG NATURVITENSKAP

FYS 610 Many-particle quantum mechanics

Suggested solutions, exercises for 31 March 2017

PROBLEM 19:

With the boost generator S^{0i} from eq. (15.15) we find:

$$D(\boldsymbol{\beta}) = e^{-\frac{\beta}{2}\begin{pmatrix} \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} \end{pmatrix}}$$

Now, for any 3-vector $\mathbf{v}$ we have, using eq. (13.16):

$$\begin{aligned} (\mathbf{v}\cdot\boldsymbol{\sigma})^2 &= v^i v^j(\delta_{ij}\mathbb{1}_2 + \mathrm{i}\epsilon_{ijk}\sigma^k) = \mathbf{v}^2\mathbb{1}_2\,, \\ (\mathbf{v}\cdot\boldsymbol{\sigma})^{2n} &= (\mathbf{v}^2)^n\,, \qquad (\mathbf{v}\cdot\boldsymbol{\sigma})^{2n+1} = (\mathbf{v}^2)^n(\mathbf{v}\cdot\boldsymbol{\sigma})\,. \end{aligned}$$

Thus, since $\hat{\boldsymbol{\beta}}^2 = \mathbb{1}_2$, we find:

$$\begin{aligned} D(\boldsymbol{\beta}) &= \sum_{n=0}^{\infty}\frac{(-\frac{\beta}{2})^n}{n!}\begin{pmatrix} \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} \end{pmatrix}^n \\ &= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}\sum_{n=0}^{\infty}\frac{(\frac{\beta}{2})^{2n}}{(2n)!} - \begin{pmatrix} \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} \end{pmatrix}\sum_{n=0}^{\infty}\frac{(\frac{\beta}{2})^{2n+1}}{(2n+1)!} \\ &= \cosh\frac{\beta}{2}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} - \sinh\frac{\beta}{2}\begin{pmatrix} \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} \end{pmatrix}\,. \end{aligned}$$

PROBLEM 20:

[There was an unfortunate printing error in V in the first version of this problem]

a) We find:

$$\begin{aligned} \gamma'^0 = V^\dagger\gamma^0 V &= \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \end{aligned}$$

and:

$$\begin{aligned} \gamma'^i = V^\dagger\gamma^i V &= \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \sigma^i & \sigma^i \\ -\sigma^i & \sigma^i \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \end{aligned}$$

b) We find from the solution for $u(p)$ in the Weyl representation, eq. (16.9):

$$u'(p) = V^\dagger\, u(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} [\sqrt{p\cdot\sigma} + \sqrt{p\cdot\bar{\sigma}}]\,\xi \\ [\sqrt{p\cdot\sigma} - \sqrt{p\cdot\bar{\sigma}}]\,\xi \end{pmatrix}$$

Here we can insert for the square roots involving the Pauli matrices from the expressions used to find $u(p)$ in lecture notes 16, with $\sigma^3 \to \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} = \sigma_\beta$:

$$\sqrt{\frac{p\cdot\sigma}{m}} = \cosh\frac{\beta}{2}\mathbb{1}_2 + \sinh\frac{\beta}{2}\sigma_\beta$$
$$\sqrt{\frac{p\cdot\bar{\sigma}}{m}} = \cosh\frac{\beta}{2}\mathbb{1}_2 - \sinh\frac{\beta}{2}\sigma_\beta\,.$$

Thus:

$$u'(p) = \sqrt{2m} \begin{pmatrix} \cosh\frac{\beta}{2}\,\xi \\ \sinh\frac{\beta}{2}\,\sigma_\beta \xi \end{pmatrix}$$

But, from $\cosh\beta = P^0/m = \omega_p/m$:

$$\cosh\frac{\beta}{2} = \sqrt{\frac{\cosh\beta+1}{2}} = \sqrt{\frac{\omega_p+m}{2m}} \qquad \sinh\frac{\beta}{2} = \sqrt{\frac{\cosh\beta-1}{2}} = \sqrt{\frac{\omega_p-m}{2m}}\,.$$

we find:

$$u'(p) = \begin{pmatrix} \sqrt{\omega_p+m}\,\xi \\ \sqrt{\omega_p-m}\,\sigma_\beta\,\xi \end{pmatrix}.$$

c) In the non-relativistic limit we have $\omega_p = \sqrt{m^2+p^2} \to m + p^2/2m$, so

$$u'(p) \to \begin{pmatrix} \sqrt{2m + \frac{p^2}{2m}}\,\xi \\ \sqrt{\frac{p^2}{2m}}\,\sigma_\beta\,\xi \end{pmatrix} \approx \sqrt{2m}\left[\begin{pmatrix} \xi \\ \frac{p}{2m}\sigma_\beta\,\xi \end{pmatrix} + \mathcal{O}\left(\frac{p^2}{m^2}\right)\right].$$

Thus to the lowest order in p/m the upper component is the dominant one, describing just the constant amplitude of a plane wave. The lower component is the first order relativistic correction, which is a pure spin-orbit coupling. [This is the reason the Bjorken–Drell representation is often used in situations where relativistic effects are relatively small].

Problem 21:

a) In the Weyl representation the Dirac equation can be written:

$$(p^\mu\gamma_\mu - m)u(p) = \left[\begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar\sigma & 0\end{pmatrix} - m\mathbb{1}_4\right] u(p) = 0\,.$$

Inserting the given expression for $u(p)$ and using the hint, we find:

$$\begin{aligned} p^\mu\gamma_\mu u(p) &= \begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar\sigma & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi\end{pmatrix} = \begin{pmatrix}(p\cdot\sigma)\sqrt{p\cdot\bar\sigma}\,\xi \\ (p\cdot\bar\sigma)\sqrt{p\cdot\sigma}\,\xi\end{pmatrix} \\ &= m\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi\end{pmatrix} = mu\,(p)\,. \end{aligned}$$

b) [Skipped, because we need to include an extra discussion of the signs of the components of p^μ.]

c) We se that $V^\dagger = V$.

$$V^\dagger V = \frac{1}{p^0}\begin{pmatrix}\sqrt{p\cdot\sigma} & \sqrt{p\cdot\bar\sigma} \\ \sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\sigma}\end{pmatrix}^2 = \frac{1}{2p^0}\begin{pmatrix} p\cdot\sigma + p\cdot\bar\sigma & 0 \\ 0 & p\cdot\sigma+p\cdot\bar\sigma\end{pmatrix} = \mathbb{1}_4\,.$$

since $p\cdot\sigma + p\cdot\bar\sigma = 2p^0\mathbb{1}_2$.

d) We find:

$$H = \gamma^0(\boldsymbol{\gamma}\cdot\mathbf{p} + m) = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0\end{pmatrix}\begin{pmatrix} m & \mathbf{p}\cdot\boldsymbol{\sigma} \\ -\mathbf{p}\cdot\boldsymbol{\sigma}, & m\end{pmatrix} = \begin{pmatrix} -\mathbf{p}\cdot\boldsymbol{\sigma} & m \\ m & \mathbf{p}\cdot\boldsymbol{\sigma}\end{pmatrix}\,.$$

e) Since both $w(p) = u(p)$ and $w(p) = v(p)$ are solutions of the Dirac equation with $p^0 = \pm\omega_p$, we have

$$\begin{aligned} (p^\mu\gamma_\mu - m)w(p) &= (\pm\gamma^0\omega_p - \mathbf{p}\cdot\boldsymbol{\gamma} - m)w(u) \\ \pm\omega_p w(p) &= \gamma^0(\boldsymbol{\gamma}\cdot\mathbf{p}+m)w(p) = Hw(p)\,. \end{aligned}$$

Thus the eigenvalues of H are $\pm\omega_p$, and H, as any Hermitean matrix, is diagonalized by the matrix of its eigenvectors, which is V. Hence:

$$H' = V^\dagger H V = \begin{pmatrix} \omega_p & 0 & 0 & 0 \\ 0 & \omega_p & 0 & 0 \\ 0 & 0 & -\omega_p & 0 \\ 0 & 0 & 0 & -\omega_p\end{pmatrix}\,.$$

f) These are obviously:

$$u(p) = \begin{pmatrix}\xi \\ 0\end{pmatrix}, \qquad v(p) = \begin{pmatrix} 0 \\ \xi\end{pmatrix},$$

for an arbitrary 2-spinor ξ, with eigenvalues ω_p for $u(p)$, $-\omega_p$ for $v(p)$, so $\psi(x)$ is just plane of fixed spin, described by ξ.