UNIVERSITETET I STAVANGER

Institutt for matematikk og naturvitenskap

FYS 610 Many-particle quantum mechanics

Exercises for 31 March 2017

Problem 19: Show that the boost operator for a Dirac spinor in the Weyl representation takes the form

$$D(\boldsymbol{\beta}) = \cosh\frac{\beta}{2}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} - \sinh\frac{\beta}{2}\begin{pmatrix} \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} \end{pmatrix}.$$

[Hint: See Lecture notes 15.]

Problem 20: The gamma matrices in the Weyl representation are:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

a) Show that the traditional Bjorken-Drell representation, $\gamma'^0 = \mathrm{Diag}(\mathbb{1}_2, -\mathbb{1}_2)$, $\gamma'^i = \gamma^i$, is obtained by the unitary (actually orthogonal) transformation $\gamma'^\mu = V^\dagger\gamma^\mu V$, with:

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}.$$

b) Use V to show that the solution in this representation can be written:

$$u'(p) = \begin{pmatrix} \sqrt{\omega_p + m}\,\mathbb{1}_2\,\xi \\ \sqrt{\omega_p - m}\,(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma})\,\xi \end{pmatrix}.$$

c) Discuss the non-relativistic of the solution in the previous part.

Problem 21:

a) Verify by direct substitution that the positive energy solutions of the Dirac equation in the Weyl representation:

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix},$$

indeed solve the equation. [Hint: Remember $(p\cdot\sigma)(p\cdot\bar{\sigma}) = m^2$.]

b) Show by the same method that if we also write the negative energy solutions in the form:

$$\psi(x) = v'(p)e^{-\mathrm{i}p\cdot x},$$

the spinors $v'(p)$ take the form:

$$v'(p) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi \\ -\sqrt{p\cdot\sigma}\,\xi \end{pmatrix},$$

c) Show that the matrix V constructed from the solutions in parts a and b above, which we can write on block matrix form as:

$$V = \frac{1}{\sqrt{2p^0}} \begin{pmatrix} \sqrt{p \cdot \sigma} & \sqrt{p \cdot \bar{\sigma}} \\ \sqrt{p \cdot \bar{\sigma}} & -\sqrt{p \cdot \sigma} \end{pmatrix} .$$

is indeed unitary.

d) Write the Hamiltonian,

$$H = P^0 = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m = \gamma^0 (\boldsymbol{\gamma} \cdot \mathbf{p} + m) \, .$$

in matrix form in the Weyl representation.

e) Use V to diagonalize H. [V is called a Foldy–Wouthuysen transformations.]

f) Find the eigenspinors of H in the diagonal form.