UNIVERSITETET I STAVANGER
INSTITUTT FOR MATEMATIKK OG NATURVITENSKAP

FYS 610 Many-particle quantum mechanics

Suggested solutions, exercises for 17 March 2017

PROBLEM 17:

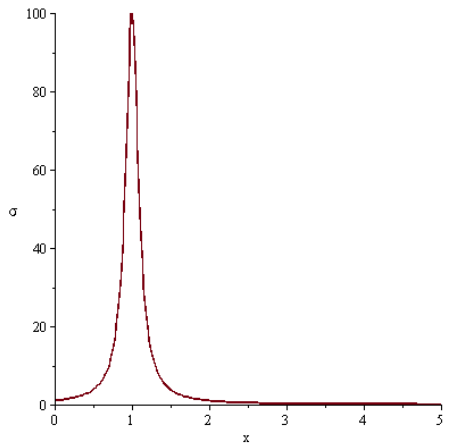
a) Replacing the propagator in S (7.93), one finds, absorbing ϵ in $\mathit{\Gamma}$:

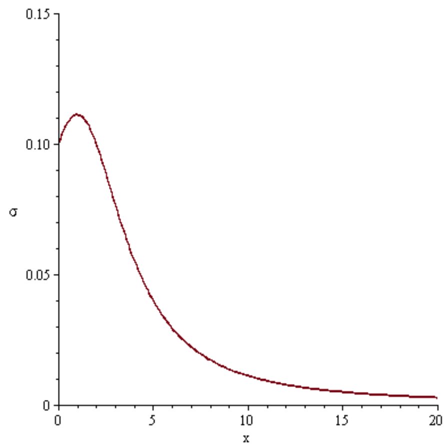
$$\mathrm{i}\mathcal{M}_s = \frac{-\mathrm{i}g^2}{s - m^2 + \mathrm{i}m\mathit{\Gamma}}\,.$$

The cross section follows as in S (7.96) for the s-channel alone ($x = s/m^2$):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^4}{64\pi^2 E_{\mathrm{CM}}^2}\frac{1}{(s-m^2)^2 + m^2\mathit{\Gamma}^2} = \frac{g^4}{64\pi^2 E_{\mathrm{CM}}^2}\frac{1}{(x-1)^2 + \left(\frac{\mathit{\Gamma}}{m}\right)^2}\,.$$

b)





Plot of the cross section (arbitrary units) for $\mathit{\Gamma}/m = 0.1$ (left) and $\mathit{\Gamma}/m = 3$ (right).

c) With $f(x) = \frac{1}{x+\mathrm{i}\epsilon}$ we find:

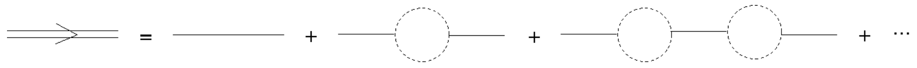
$$\lim_{\epsilon\to 0}\mathrm{Im}f(x) = \lim_{\epsilon\to 0}\mathrm{Im}\,\frac{x-\mathrm{i}\epsilon}{x^2+\epsilon^2} = \lim_{\epsilon\to 0} -\frac{\epsilon}{x^2+\epsilon^2} = 0 \qquad (x\neq 0)\,.$$

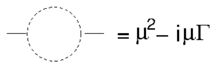
Furthermore for $\epsilon > 0$:

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{Im}f(x) = -\epsilon\int_{-\infty}^{\infty}\frac{\mathrm{d}\,x}{x^2+\epsilon^2} = -\epsilon\frac{1}{\epsilon}\arctan\left(\frac{x}{\epsilon}\right)\Big|_{-\infty}^{\infty} = -\pi \quad \underset{\epsilon\to 0}{\longrightarrow} \quad -\pi\,.$$

Thus we must have $\lim_{\epsilon\to 0} f(x) = -\pi\delta(x)$, and the result follows by setting $x = p^2 - m^2$. [The integral could also have been done by contour integration.]

d) With the ϕ propagators as fully drawn lines and the ψ propagators as dashed lines, we have:





where μ and Γ are both proportional to g^2. One can sum this series just as we did in Problem 15 (S problem 7.4) and in lecture note 10. Even the loop contains an imaginary part, we can sum the diagrams as before, obtaining:

$$\widetilde{D}_F(p) = \frac{\widetilde{D}_F^0(p)}{1 + \mathrm{i}(\mu^2 - \mathrm{i}\mu\Gamma)\widetilde{D}_F^0(p)} = \frac{1}{\left(\widetilde{D}_F^0(p)\right)^{-1} + \mathrm{i}(\mu^2 - \mathrm{i}\mu\Gamma)}$$

$$= \frac{\mathrm{i}}{p^2 - (m^2 + \mu)^2 + \mathrm{i}\mu\Gamma} \, .$$

Here $\widetilde{D}_F^0(p)$ is the free propagator, and we have dropped the irrelevant $\mathrm{i}\epsilon$. We see that the propagator has acquired a width, in addition to the renormalization of the mass, $m^2 \to m_R^2 + \mu^2$.

e) If $\Gamma \to 0$, the imaginary part of the ϕ-loop will just contribute a factor $-\pi\delta(p^2 - \mu^2)$, which means that it behaves precisely like an on-shell, stable, particle of mass μ. From the energy-time uncertainty relation, if the energy is sharp, the life-time is infinite. Increasing Γ means that the energy uncertainty of the virtual state increases, hence that the lifetime decreases. [This hand-waving argument can be made mathematically precise, showing that $\Gamma = \frac{1}{\tau}$, where τ is the expected life-time of the state.]

Problem 18:

a) In lecture note 9, eq. (9.2), we found the retarded Green's function as:

$$D_R(x-y) = \theta(x^0 - y^0)\langle 0|[\phi_0(x), \phi_0(y)]|\,0\rangle = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2 - m^2}\, e^{-\mathrm{i}k(x-y)} \, .$$

In the following we shall only need the case $x^0 > y^0$, in which case the k^0 integration can be easily carried out by contour integration. Since for the retarded propagator both poles are in the *lower* k^0 plane, at $k^0 = \pm(\omega_k - \mathrm{i}\epsilon)$, which is were we have to close the integration contour for $x^0 > y^0$, we find:

$$D_R(x-y) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_k}\left(e^{-\mathrm{i}k(x-y)} - e^{\mathrm{i}k(x-y)}\right) \qquad (x^0 > y^0) \, ,$$

where $k^\mu = [\omega_k, \mathbf{k}]$, and we have changed integration variable $\mathbf{k} \to -\mathbf{k}$ in the second integral.

Since $\mathrm{i}D_F(y-x)$ is Green's function for the Klein–Gordon operator, we can write the solution of:

$$(\,\Box + m^2)\phi(x) = j(x)$$

as:

$$\phi(x) = \phi_0(x) + \mathrm{i}\int \mathrm{d}^4y\, D_R(x-y) j(y)\,.$$

Furthermore, since $j(y) = 0$ for $y^0 > T$, we can use the formula we have found for $D_R(x-y)$ for all times x^0 later than T. Hence:

$$\phi(x) = \phi_0(x) + \mathrm{i}\int \mathrm{d}^4y \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_k}\left(e^{-\mathrm{i}k(x-y)} - e^{\mathrm{i}k(x-y)}\right) j(y)\,.$$

But the y-integral here then only yields $\tilde{\jmath}(k)$ and $\tilde{\jmath}^*(k)$ in the two terms, with $k^0 = \omega_k$. Collecting terms, we recover the expression given. We see that the field is a coherent sum of a quantum fields, involving $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$, and a classical field, involving $\tilde{\jmath}(k)$. [In the samme manner, one can calculate the *coherent* response of a system containting electrons, *i.e.* an atom, to an external electromagnetic field, like a light beam. This describes the interplay between spontaneous decay, which is a quantum process, and excitation/ionization induced by the classical field. This is essential for the theory of lasers.]

b) We can simply observe that $\phi(x)$ can be expressed in terms of the operators:

$$b_\mathbf{k} = a_\mathbf{k} + \frac{\mathrm{i}\,\tilde{\jmath}(k)}{\sqrt{2\omega_k}}\,, \qquad b^\dagger_\mathbf{k} = a^\dagger_\mathbf{k} - \frac{\mathrm{i}\,\tilde{\jmath}^*(k)}{\sqrt{2\omega_k}}\,,$$

which satisfy the same commutation relations as $a_\mathbf{k}, a^\dagger_\mathbf{k}$. Hence one find the Hamiltonian density expressed in terms of the $b_\mathbf{k}$'s in precisely the same manner as one find it for the free theory expressed in terms of the $a_\mathbf{k}$'s, the result being the formulas stated in the problem. It is straightforward to verify that $\phi(x)$ satisfy the Heisenberg equation of motion.

c) The vacuum of the free theory is a solution of $a_\mathbf{k}|\,0\,\rangle = 0$, We therefore find:

$$\langle 0|H|\,0\rangle = \int \mathrm{d}^3\mathbf{k}\frac{1}{2}|\tilde{\jmath}(k)|^2$$

$$\langle 0|N|\,0\rangle = \int \mathrm{d}^3\mathbf{k}\frac{1}{2\omega_k}|\tilde{\jmath}(k)|^2\,.$$

Thus we can interpret $|\tilde{\jmath}(k)|^2/2\omega_k$ as the momentum space density of created particles. If $j(x)$ is time independent for $-T \le x^0 < T$, we find:

$$\tilde{\jmath}(k) = \int \mathrm{d}^3\mathbf{x}\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} j(\mathbf{x}) \int_{-T}^{T} \mathrm{d}\,x^0\, e^{\mathrm{i}\omega_k x^0} = \frac{2}{\omega_k}\sin(\omega_k T)\,\tilde{\jmath}(\mathbf{k})\,.$$

We see that for short $\omega_k T$ we have $2\frac{\sin\omega_k T}{\omega_k} \to 2T$, so initially the field strenght increases proportionally with time, but for larger $\omega_k T$ the created modes start to interfere.