UNIVERSITETET I STAVANGER

INSTITUTT FOR MATEMATIKK OG NATURVITENSKAP

FYS 610 Many-particle quantum mechanics

Suggested solutions, exercises for 24 February 2017

PROBLEM 12:

a) The decay rate for a muon at rest is given by eq. (5.24):

$$\mathrm{d}\Gamma = \frac{1}{2m}|\mathcal{M}_{fi}|^2 \mathrm{d}\Pi_{\mathrm{LIPS}}\,,$$

where m is the muon mass. Before we continue, we observe that the matrix element:

$$|\mathcal{M}_{fi}|^2 = 32G_F^2(m^2 - 2mE)mE\,,$$

obviously makes sense only for $E \leq m/2$, which is actuallythe maximal energy allowed for the electron. From energy-momentum conservation it can be seen that the three massless particles all have a maximum possible energy of $m/2$.

If we neglect the masses of the decay products, the differential Lorentz-invariant phase space for three-body final state is ($E = E_e$ is the electron energy) is, from eq. (5.21):

$$\mathrm{d}\Pi_{\mathrm{LIPS}} = (2\pi)^4\delta^4(p_\mu - p_e - p_\nu - p_{\bar\nu})\frac{1}{(2\pi)^9}\frac{\mathrm{d}^3\,\mathbf{p}_e}{2E}\,\frac{\mathrm{d}^3\mathbf{p}_\nu}{2E_\nu}\,\frac{\mathrm{d}^3\mathbf{p}_{\bar\nu}}{2E_{\bar\nu}}\,.$$

Here the subscripts ν and $\bar\nu$ refers to the muon neutrino and the electron antineutrino, respectively. We use momentum delta-function to integrate over $\mathbf{p}_{\bar\nu}$, so $\mathbf{p}_{\bar\nu} = -\mathbf{p}_e - \mathbf{p}_\nu$ (remember $\mathbf{p}_\mu = 0$). We then have $E_{\bar\nu} = |E\hat{\mathbf{p}}_e + E_\nu\hat{\mathbf{p}}_\nu|$ since $\mathbf{p}_\nu = E_\nu\hat{\mathbf{p}}_\nu$ etc for massless particles. Thus:

$$\Gamma = \frac{G_F^2}{16\pi^5}\int E\,\mathrm{d}E\,\mathrm{d}^2\hat{\mathbf{p}}_e\,E_\nu\,\mathrm{d}E_\nu\,\mathrm{d}^2\hat{\mathbf{p}}_\nu\,\frac{1}{E_{\bar\nu}}\,\delta(m - E - E_\nu - E_{\bar\nu})E(m^2 - 2mE)\,,$$

If we measure the direction of the muon neutrino from that of the electron, we have $E_{\bar\nu}^2 = E^2 + E_\nu^2 + 2EE_\nu\cos\theta$, where θ is the angle between $\mathbf{p}_e$ and $\mathbf{p}_\nu$. We can then perform the angular integrations as follows:

$$\int \mathrm{d}^2\hat{\mathbf{p}}_e\,\mathrm{d}^2\hat{\mathbf{p}}_\nu\frac{\delta(m - E - E_\nu - E_{\bar\nu})}{E_{\bar\nu}} = \int \mathrm{d}^2\hat{\mathbf{p}}_e\,\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi\frac{\delta(m - E - E_\nu - E_{\bar\nu})}{E_{\bar\nu}}$$

$$= 8\pi^2\int_0^{\frac{1}{2}m}\mathrm{d}E_{\bar\nu}\,\frac{\mathrm{d}\cos\theta}{\mathrm{d}E_{\bar\nu}}\frac{\delta(m - E - E_\nu - E_{\bar\nu})}{E_{\bar\nu}}$$

$$= 8\pi^2\int_0^{\frac{1}{2}m}\mathrm{d}E_{\bar\nu}\frac{\delta(m - E - E_\nu - E_{\bar\nu})}{EE_\nu} = \frac{8\pi^2}{EE_\nu}\theta(E + E_\nu - \tfrac{1}{2}m)\,,$$

since we must have $0 \leq E_{\bar{\nu}} = m - E - E_\nu \leq \frac{1}{2}m$. The remaining integrals can now easily be done, yielding:

$$\begin{aligned} \Gamma &= \frac{G_F^2}{2\pi^3} \int_0^{\frac{1}{2}m} E\,\mathrm{d}E(m^2 - 2mE) \int_0^{\frac{1}{2}m-E} \mathrm{d}E_\nu \\ &= \frac{G_F^2 m}{2\pi^3} \int_0^{\frac{1}{2}m} \mathrm{d}E\,E(m^2 - 2mE)(\tfrac{1}{2}m - E) = \frac{G_F^2 m^5}{192\pi^3} \end{aligned}$$

b) We see that $\tau = 1/\Gamma$ has energy-dimension $4 - 5 = -1$, which is correct for a time. If we insert the constants given, we find:

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m^5} = 3.26 \cdot 10^{15}\,\mathrm{MeV}^{-1}\,.$$

To convert this to a more useful form, we multiply by $\hbar = 6.58\,\mathrm{MeV{\cdot}s}$, yielding $\tau = 2.14\,\mu\mathrm{s}$. The discrepancy, 2.7 %, is too large to be a measurement error, and is probably mostly due to the neglection of the electron mass and deficiencies in the Fermi theory for weak interactions.

Problem 13:

a) The electromagnetic Lagrangian density is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu \partial^\mu\right) \\ &= \frac{1}{2}\left[\left(\partial_\mu A_\nu\right)^2 - \left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right)\right] = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)\,. \end{aligned}$$

From *Schwartz*, eq. (3.35), we then find the energy-momentum tensor, using the results of sect. 3.4:

$$\begin{aligned} \mathcal{T}^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\lambda}\partial^\nu A^\lambda - g^{\mu\nu}\mathcal{L} = -\left(\partial^\mu A^\lambda - \partial^\lambda A^\mu\right)\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L} \\ &- F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\,. \end{aligned}$$

We evidently have $\mathcal{T}^{\mu\nu} - \mathcal{T}^{\nu\mu} \neq 0$.

b) With

$$E^i = -\partial_t A^i - \partial^i A^0 \qquad B^i = \epsilon^{ijk}\partial^j A^k\,,$$

we find by direct substitution:

$$\begin{aligned} \mathcal{T}^{00} &= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \boldsymbol{\nabla}\cdot\left(A^0\mathbf{E}\right)\,, \\ \mathcal{T}^{0i} &= \left(\mathbf{E}\times\mathbf{B}\right)^i + \boldsymbol{\nabla}\cdot\left(A^i\mathbf{E}\right)\,. \end{aligned}$$

c) From the antisymmetry of $K^{\lambda\mu\nu}$ we find:

$$\partial_\mu\partial_\lambda K^{\lambda\mu\nu} = -\partial_\lambda\partial_\mu K^{\mu\lambda\nu} = -\partial_\mu\partial_\lambda K^{\lambda\mu\nu} = 0\,,$$

so

$$\partial_\mu \widetilde{\mathcal{T}}^{\mu\nu} = \partial_\mu \mathcal{T}^{\mu\nu} + \partial_\mu \partial_\lambda K^{\lambda\mu\nu} = \partial_\mu \mathcal{T}^{\mu\nu} = 0\,.$$

d) With the suggested $K^{\lambda\mu\nu}$ we have, using the Maxwell-equations $\partial_\mu F^{\mu\nu} = 0$:

$$\widetilde{\mathcal{T}}^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L} - \partial_\lambda \left(F^{\lambda\mu} A^\nu\right) = F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\,.$$

This expression is manifestly symmetric in μ and ν. The total energy and momentum are unchanged:

$$\widehat{P}^\mu = \int \mathrm{d}^3x\, \widehat{\mathcal{T}}^{0\mu} = \int \mathrm{d}^3x\, \mathcal{T}^{0\mu} + \int \mathrm{d}^3x\, \partial_\nu K^{\nu 0\mu} = P^\mu + \int \mathrm{d}^3x\, \partial_i K^{i0\mu} = P^\mu\,.$$

where we first have used that $K^{00\mu} = 0$ by the antisymmetry in the first pair of indices, and the divergence theorem to convert the last integral into a surface integral at infinity, which vanishes by the standard assumption about the asymptotic behavior of the fields.

e) By direct substitution, we find:

$$\widetilde{\mathcal{T}}^{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\,, \qquad \widetilde{\mathcal{T}}^{0i} = \left(\mathbf{E} \times \mathbf{B}\right)^i\,.$$

These are the standard expression for the electromagnetic energy density and the Poynting-vector for the momentum density.

Problem 6: See suggested solutions for 10.02 2017.