UNIVERSITETET I STAVANGER

INSTITUTT FOR MATEMATIKK OG NATURVITENSKAP

FYS 610 Many-particle quantum mechanics

Suggested solutions, exercises for 17 February 2017

PROBLEM 6: See Suggested solutions for 10.02 2017.

PROBLEM 8: See Suggested solutions for 10.02 2017.

PROBLEM 10:

a) Making a partial integration, the lagrangian density can be rewritten:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,,$$

which leads to the Euler-lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = \Box\phi - m^2\phi + \frac{\lambda}{6}\phi^3 = 0\,.$$

The constant solutions, $\phi = c$, follow as:

$$\frac{\lambda}{6}c^3 - m^2 c = 0 \qquad \Longrightarrow \qquad c_0 = 0\,, \quad c_+ = \sqrt{\frac{6m^2}{\lambda}}\,, \quad c_- = -\sqrt{\frac{6m^2}{\lambda}}\,.$$

The energy density is equal to the Hamiltonian density:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\,,$$

which for the three constant solutions found yields the energy densities:

$$\mathcal{E} = \mathcal{T}^{00} = \mathcal{H} = -\frac{m^2c^2}{2} + \frac{\lambda c^4}{24} = 0, -\frac{3}{2}\frac{m^4}{\lambda}, -\frac{3}{2}\frac{m^4}{\lambda}\,.$$

Thus we see that $c_\pm = \pm\sqrt{6m^2/\lambda}$ both yields minima, which are even *global minima*, *i.e.* they both yield the ground state energy density of the system, since the non-constant terms in $\mathcal{H}$ are positive. On the other hand, $c_0 = 0$ is a *local point of inflection* of the full $\mathcal{H}$, because we can decrease $\mathcal{H}$ by adding small constant values to $\phi = c_0$, but increase it by adding space- or time-dependent values.

b) Under the symmetry transformation $\phi \to -\phi$ we see that $c_+ \leftrightarrow c_-$, so neither state is invariant.

c) Let $c = c_+$ for definiteness. Since it is constant, we have $\partial_\mu \pi = \partial_\mu \phi$, so the Lagrangian density expressed in terms of π reads,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\dot{\pi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\pi)^2 + \frac{1}{2}m^2(\pi + c)^2 - \frac{\lambda}{4!}(\pi + c)^4 \\ &= \frac{1}{2}\dot{\pi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\pi)^2 + \frac{1}{2}m^2\pi^2 - \frac{\lambda c}{4}\pi^3 - \frac{\lambda}{4!}\pi^4 + \frac{3}{2}\frac{m^4}{\lambda}\,.\end{aligned}$$

where we have substituted the value of c^2. The equation of motions now become:

$$= \Box\pi + 2m^2\pi + \frac{3\lambda c}{4}\pi^2 + \frac{\lambda}{6}\pi^3 = 0\,,$$

which indeed has a solution $\pi = 0$. The transformation $\phi \to -\phi$ corresponds to the combined transformation $(\pi, c) \to (-\pi, -c)$, which is a symmetry of $\mathcal{L}$. [We can also read off from $\mathcal{L}$ that after quantization π gives rise to a particle of *positive* rest mass $\sqrt{2}m$.]

Problem 11:

a) We find the Euler-Lagrange equations as in *Schwartz* sec. 3.4, with an additional term in $\mathcal{L}$:

$$\partial_\mu F^{\mu\nu} + m^2 A^\mu = J^\nu\,.$$

Since $F^{\mu\nu} = -F^{\nu\mu}$, we have $\partial_\mu\partial_\nu F^{\mu\nu} = -\partial_\nu\partial_\mu F^{\nu\mu} = 0$ (this is also seen by an explicit calculation), we find if J^μ is conserved ($\partial_\mu J^\mu = 0$):

$$0 = \partial_\mu\partial_\nu F^{\mu\nu} = \partial_\nu J^\nu - m^2\partial_\nu A^\nu = m^2\partial_\nu A^\nu \qquad \Longrightarrow \qquad \partial_\nu A^\nu = 0\,,$$

so A^μ satisfies the Lorentz condition.

b) Like in *Schwartz* sec. 3.4, the Lorentz condition implies:

$$\partial_\mu F^{\mu\nu} = \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = \Box A^\nu$$

so the equations of motion can be written as four linear partial differential equations:

$$\Box A^\mu + m^2 A^\mu = J^\mu\,.$$

In the present case we shall find the static potential from a point charge at the origin, like in *Schwartz* sec. 3.4.2, so $A^\mu = (\rho, \mathbf{0}$, with $\rho = e\delta^3(\mathbf{x})$. Hence $\mathbf{A} = \mathbf{0}$, while A^0 is a solution of:

$$\left(-\boldsymbol{\nabla}^2 + m^2\right) A^0(\mathbf{x}) = e\delta^3(\mathbf{x})$$

Taking Fourier transforms, this becomes:

$$(\mathbf{k}^2 + m^2)\tilde{A}^0(\mathbf{k}) = e \qquad \Longrightarrow \qquad A^0(\mathbf{k}) = \frac{e}{k^2 + m^2}\,,$$

where $k = |\mathbf{k}|$. Transforming back, we find by introducing spherical coordinates, like in *Schwartz* eq. (3.62):

$$\begin{aligned}A^0(\mathbf{x}) &= e\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{e^{\mathrm{i}\mathbf{k}\mathbf{x}}}{\mathbf{k}^2 + m^2} = \frac{e}{4\pi^2}2\pi\int_0^\infty \frac{k^2\,\mathrm{d}k}{k^2 + m^2}\frac{e^{\mathrm{i}kr} - e^{-\mathrm{i}kr}}{\mathrm{i}kr} \\ &= \frac{e}{4\pi^2 r\mathrm{i}}\int_{-\infty}^\infty \frac{k\,\mathrm{d}k}{k^2 + m^2}e^{\mathrm{i}kr}\,.\end{aligned}$$

c) This integral can be evaluated by contour integration (or be looked up in an integral table). Since $r > 0$, we must close the contour in the upper half plane. Since

$$\frac{k}{k^2+m^2} = \frac{k}{(k+\mathrm{i}m)(k-\mathrm{i}m)}\,,$$

we see that there is only one pole, at $k = \mathrm{i}m$, in this plane, with residue:

$$\mathrm{Res}_{k=\mathrm{i}m} = \frac{\mathrm{i}m}{2\mathrm{i}m} e^{-mr} = \frac{1}{2} e^{-mr}\,.$$

(this is the same as in *Schwartz*, except $\delta \to m$, which remains finite). Hence we have:

$$A^0(\mathbf{x}) = \frac{e}{4\pi^2 r\mathrm{i}} 2\pi\mathrm{i}\,\mathrm{Res}_{k=\mathrm{i}m} = \frac{e}{4\pi}\frac{e^{-mr}}{r}$$

d) This follows trivially, since $\lim_{m\to 0} e^{-mr} = 1$.

e) Introducing standard units, the Yukawa potential is:

$$A^0(r) = \frac{e}{4\pi\epsilon_0}\frac{e^{-cmr/\hbar}}{r} = \frac{e}{4\pi\epsilon_0}\frac{e^{-r/r_s}}{r}\,.$$

Here $r_s = \hbar/mc$ is a screening length, while $2\pi r_s$ is called the *Compton wavelength.* For distances $r \ll r_s$, the Yukawa and the Coulomb potentials are the same, but for $r \gg r_s$ the former vanishes, so it only has a finite range. It was introduced to describe the short range of nuclear forces, so r_s was thought to be the range of the strong nuclear force, around 10^{-15}m. Yukawa predicted that there should exist particles, now identified with the pion, with a corresponding rest energy $mc^2 = \hbar c/r_s \approx 200\,\mathrm{MeV}$. Indeed, the pion mass is $139.5\,\mathrm{MeV}$. This screened potential had actually been introduced in physics earlier by Peter Debye, to describe electric screening in plasmas and electrolytes, for which it is still in wide use.

e) We can rewrite the free, massless part of $\mathcal{L}$, interchanging indices $\mu \leftrightarrow \nu$ and making further index gymnastics necessary to identify equal terms:

$$\begin{aligned}
\mathcal{L}_0 = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} &= \frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= \frac{1}{2}\Big((\partial^\mu A^\nu)^2 - (\partial^\nu A^\mu)(\partial_\mu A_\nu)\Big) \\
&= \frac{1}{2}\Big((\partial^\mu A^\nu)^2 - \partial^\nu[A^\mu(\partial_\mu A_\nu)] + A^\mu\partial_\mu\partial^\nu A_\nu\Big) \\
&= \frac{1}{2}\Big((\partial^\mu A^\nu)^2 - \partial^\nu[A^\mu(\partial_\mu A_\nu)] + \partial_\mu[A^\mu\partial^\nu A_\nu] - (\partial_\mu A^\mu)(\partial^\nu A_\nu)\Big) \\
&= \mathcal{L}_0' + \partial_\mu X^\mu\,.
\end{aligned}$$

Here

$$\mathcal{L}_0' = \frac{1}{2}\Big((\partial^\mu A^\nu)^2 - (\partial_\mu A^\mu)(\partial_\nu A^\nu)\Big)$$

only differs from $\mathcal{L}_0$ by a the total derivative $\partial_\mu X^\mu$, with $X^\mu = A^\mu\partial^\nu A_\nu - A^\nu\partial^\mu A_\nu$, and so gives rise to the same equations of motion. If we plug $\partial_\mu A^\mu = 0$ into $\mathcal{L}_0'$, we simply have $\mathcal{L}_0' = \frac{1}{2}(\partial^\mu A^\nu)^2$, which also has Euler-Lagrange equation $\Box A^\mu = 0$. Thus the equations of motion are unchanged when the constraint is included from the start. It is the term $\frac{1}{2}m^2 A_\mu^2$ in the original Lagrangian which enforces it, but, of course, without it we have a theory that differs in more respect than just the absence of the constraint.