UNIVERSITETET I STAVANGER

INSTITUTT FOR MATEMATIKK OG NATURVITENSKAP

FYS 610 Many-particle quantum mechanics

Exercises for 10 February 2017

PROBLEM 5: Consider an infinite one-dimensional chain of point particles, all of mass m, connected by identical elastic springs of length a and spring constant k. Assume that the motion takes place along the string, so you can disregard the transverse directions. Let $\eta_i(t)$ be the displacement of particle i from its equilibrium position at time t. [See *Goldstein* ch. 13.1 for hints].

a) Show that the Lagrangian of the system can be written:

$$L = \frac{1}{2}\sum_i \left[m\dot{\eta}_i^2 - k(\eta_{i+1}-\eta_i)^2\right] .$$

b) Find the Euler-Lagrange equations, and show that they can be written:

$$\frac{m}{a}\ddot{\eta}_i - ka\left[\frac{\eta_{i+1}-\eta_i}{a^2} - \frac{\eta_i-\eta_{i-1}}{a^2}\right] = 0 .$$

We shall now take the continuum limit of this model, letting $a \to 0$ in such a manner that the mass density, $\mu = m/a$, and Young's module $Y = ka$ remain finite. [The correctness of the expression for Y may be checked by dimensional analysis].

c) Let $\eta(x,t)$ be the displacement of the piece of string which is at position x in equilibrium. Show that the equation of motion for the string becomes the wave equation:

$$\mu\frac{\partial^2\eta}{\partial t^2} - Y\frac{\partial^2\eta}{\partial x^2} = 0 ,$$

Verify that this equation has longitudinal wave solutions of the form $\eta(x,t) = \eta_0\cos(x \pm vt + \delta)$, with $v^2 = Y/\mu$.

d) Show that the continuum limit of the Lagrangian in a) can be written:

$$L = \int \mathrm{d}x\,\mathcal{L} \qquad \mathcal{L} = \frac{1}{2}\left[\mu\left(\frac{\partial\eta}{\partial t}\right)^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right] .$$

Further, show that the resulting Euler-Lagrange equation is the one found in the previous part.

PROBLEM 6: Find the components of the (two-dimensional) energy-momentum tensor of the Lagrangian in problem 5d) above. Also find the corresponding conserved quantities.

For later reference, we give the general formula for the conserved Noether current. If $\phi^n(x) \to \phi^n(x) + \delta\phi^n$ and $x^\mu \to x^\mu + \delta x^\mu$ is an infinitesimal symmetry transformation for the action $S = \int \mathrm{d}x \mathcal{L}(\{\phi^n, \partial_\mu \phi^n\}, t)$, then the conserved current is:

$$J^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^n} \delta\phi^n - \mathcal{T}^\mu_\nu \delta x^\nu \, ,$$

where the energy momentum tensor, $\mathcal{T}^{\mu\nu}$ is defined by:

$$\mathcal{T}^{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^n} \partial^\nu \phi^n - g^{\mu\nu} \mathcal{L} \, .$$

Problem 7:

a) Show that the actions of free massless scalar and vector fields, including electromagnetism, are invariant under the scale transformation $x^\mu \to a x^\mu$, $\phi^n \to \phi/a$.

b) Find the conserved current following from this transformation, expressed in terms of $\mathcal{T}^{\mu\nu}$.

Problem 8. An infinitesimal Lorentz transformation may be written $y^\mu = x^\mu + \delta x^\mu$ where $\delta x^\mu = \omega^{\mu\nu} x_\nu$, with an antisymmetric tensor, $\omega^{\mu\nu} = -\omega^{\nu\mu}$. Show that this ensures $y_\mu y^\mu = x_\mu x^\mu$, and use it to solve problem 3.2 in *Schwartz*.

[**Problem 9**: *Schwartz*, problem 3.3. We drop this. There is something fishy here]