UNIVERSITETET I STAVANGER
Institutt for matematikk og naturvitenskap

FYS 610 Many-particle quantum mechanics

Suggested solutions, exercises for 27 January 2017

Problem 1:

The unitary matrix connecting the two bases has matrix elements defined by:

$$| e_i' \rangle = \sum_j U_{ji} | e_j \rangle ,$$

(note the order of the indices on U!). But from the expansion of the unit operator in the unprimed basis we also have:

$$| e_i' \rangle = \mathbb{I} | e_i' \rangle = \sum_j | e_j \rangle \langle e_j | e_i' \rangle .$$

Since the expansion coefficients of a vector in any basis are unique, we read off $U_{ji} = \langle e_j | e_i' \rangle = \langle e_i' | e_j \rangle^*$. Similarly, by interchanging the two bases we have

$$| e_i \rangle = \sum_j U_{ji}^{-1} | e_j' \rangle = \sum_j | e_j' \rangle \langle e_j' | e_i \rangle ,$$

so $U_{ji}^{-1} = \langle e_j' | e_i \rangle = \langle e_i | e_j' \rangle^* = U_{ij}^* = U_{ji}^\dagger$. For any matrix A we then have

$$A'_{ij} = \langle e_i' | A | e_j' \rangle = \sum_{kl} \langle e_i' | e_k \rangle \langle e_k | A | e_l \rangle \langle e_l | e_j' \rangle = \sum_{kl} U_{ik}^\dagger A_{kl} U_{lj} ,$$

which expresses the matrix equation $A' = U^\dagger A U$. [Note that this relates just the matrix elements of A in two different bases (a passive transformation). Whether or not the corresponding active transformation is interesting is another issue.]

Problem 2:

a) With $f(k^0) = {k^0}^2 - \omega_k^2$ we find from the formula given:

$$\int_\infty^\infty \mathrm{d}k^0 \, \delta({k^0}^2 - \omega_k^2)\theta(k^0) = \int_\infty^\infty \mathrm{d}k^0 \left(\frac{1}{2\omega_k}\delta(k^0 - \omega_k) + \frac{1}{2\omega_k}\delta(k^0 + \omega_k) \right) \theta(k^0)$$
$$= \frac{1}{2\omega_k} ,$$

since the factor $\theta(k^0)$ ensures that the second term with $k^0 = -\omega_k$ vanishes.

b) We know that for any Lorentz transformation, $k'^\mu = \Lambda^\mu_\nu k^\nu$, proper or improper, we have $\det \Lambda = \pm 1$, as is also evident from eqs. (S 2.13-14). Thus the Jacobi determinant of the transformation $k \to k'$ is 1, so:

$$\mathrm{d}^4 k = \left| \det \left(\frac{\partial k^\mu}{\partial k'^\nu} \right) \right| \mathrm{d}^4 k' = \left| \det \Lambda^{-1} \right| \mathrm{d}^4 k' = \frac{\mathrm{d}^4 k'}{|\det \Lambda|} = \mathrm{d}^4 k'.$$

c) Combining the two previous results, and assuming that we only consider Lorentz transformations which preserve time ordering, so both k^0 and k'^0 have the same sign, we find from the previous two parts:

$$\int \frac{\mathrm{d}^3 \mathbf{k}}{2\omega_k} = \int \mathrm{d}^4 k \, \delta({k^0}^2 - \omega_k^2) \theta(k^0) = \int \mathrm{d}^4 k \, \delta(k^2 - m^2) \theta(k^0) \, .$$

The last integrand is clearly Lorentz invariant, so the first must also be.

Problem 2

a) With $f(k^0) = {k^0}^2 - \omega_k^2$ we find from the formula given:

$$\int_{\infty}^{\infty} \mathrm{d}k^0 \, \delta({k^0}^2 - \omega_k^2) \theta(k^0) = \int_{\infty}^{\infty} \mathrm{d}k^0 \left(\frac{1}{2\omega_k} \delta(k^0 - \omega_k) + \frac{1}{2\omega_k} \delta(k^0 + \omega_k) \right) \theta(k^0)$$
$$= \frac{1}{2\omega_k} \, ,$$

since the factor $\theta(k^0)$ ensures that the second term with $k^0 = -\omega_k$ vanishes.

b) We know that for any Lorentz transformation, $k'^\mu = \Lambda^\mu_\nu k^\nu$, proper or improper, we have $\det \Lambda = \pm 1$, as is also evident from eqs. (S 2.13-14). Thus the Jacobi determinant of the transformation $k \to k'$ is 1, so:

$$\mathrm{d}^4 k = \left| \det \left(\frac{\partial k^\mu}{\partial k'^\nu} \right) \right| \mathrm{d}^4 k' = \left| \det \Lambda^{-1} \right| \mathrm{d}^4 k' = \frac{\mathrm{d}^4 k'}{|\det \Lambda|} = \mathrm{d}^4 k'.$$

c) Combining the two previous results, and assuming that we only consider Lorentz transformations which preserve time ordering, so both k^0 and k'^0 have the same sign, we find from the previous two parts:

$$\int \frac{\mathrm{d}^3 \mathbf{k}}{2\omega_k} = \int \mathrm{d}^4 k \, \delta({k^0}^2 - \omega_k^2) \theta(k^0) = \int \mathrm{d}^4 k \, \delta(k^2 - m^2) \theta(k^0) \, .$$

The last integrand is clearly Lorentz invariant, so the first must also be.

Problem 3: In this problem we use the fact that the rules of differentiations also apply to operator expressions, as long as we handle commutation relations properly. Note that $[a, f(a)] = 0$ for any operator a and function f not involving an operator not commuting with a.

a) We find, using $[a, a^\dagger] = 1$:

$$\partial_z \left(e^{-za^\dagger} a e^{za^\dagger} \right) = e^{-za^\dagger}(-a^\dagger a + aa^\dagger)e^{za^\dagger} = e^{-za^\dagger}[a, a^\dagger]e^{za^\dagger} = 1.$$

b) From the previous part, we find by integration:

$$e^{-za^\dagger} a e^{za^\dagger} = z + c\,,$$

where c is a constant of integration. Inserting $z = 0$ we see that $c = a$. From this and $a|\,0\,\rangle = 0$ we find:

$$a\,|\,z\,\rangle = ae^{za^\dagger}|\,0\,\rangle = e^{za^\dagger}(z + a)|\,0\,\rangle = ze^{za^\dagger}|\,0\,\rangle = z|\,z\,\rangle\,.$$

Thus $|\,z\,\rangle$ is an eigenstate of a with eigenvalue z.

c) From:

$$|\,n\,\rangle = \frac{1}{\sqrt{n!}} {a^\dagger}^n |\,0\,\rangle$$

we have:

$$|\,z\,\rangle = \sum_k \frac{z^k}{k!} {a^\dagger}^k |\,0\,\rangle = \sum_k \frac{z^k}{\sqrt{k!}} |\,k\,\rangle\,.$$

Thus:

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\,.$$

d) The expectation values of p and q in the coherent state z follows from the expressions for p and q in Lecture notes 4, with $\hbar = 1$, by exploiting that $|\,z\,\rangle$ is an eigenvector for a, and letting the creation operators operate to the left:

$$\langle z|q|\,z\rangle = \sqrt{\frac{1}{2m\omega}} \langle z|(a + a^\dagger)|\,z\rangle = \sqrt{\frac{1}{2m\omega}} (z + z^*)\langle z|z\rangle\,,$$
$$\langle z|p|\,z\rangle = \frac{1}{\mathrm{i}}\sqrt{\frac{m\omega}{2}} \langle z|(a - a^\dagger)|\,z\rangle = \frac{1}{\mathrm{i}}\sqrt{\frac{m\omega}{2}} (z - z^*)\langle z|z\rangle\,.$$

Similarly, from $[a, a^\dagger] = 1$:

$$\begin{aligned}
\langle z|q^2|\,z\rangle &= \frac{1}{2m\omega}\langle z|a^2 + aa^\dagger + a^\dagger a + {a^\dagger}^2|\,z\rangle = \frac{1}{2m\omega}\langle z|a^2 + 2a^\dagger a + {a^\dagger}^2 + 1|\,z\rangle \\
&= \frac{1}{2m\omega}\left[(z + z^*)^2 + 1\right]\langle z|z\rangle\,, \\
\langle z|p^2|\,z\rangle &= -\frac{m\omega}{2}\langle z|a^2 - aa^\dagger - a^\dagger a - {a^\dagger}^2|\,z\rangle = -\frac{2m\omega}{2}\langle z|a^2 - 2a^\dagger a + {a^\dagger}^2 - 1|\,z\rangle \\
&= -\frac{1}{2m\omega}\left[(z - z^*)^2 - 1\right]\langle z|z\rangle\,.
\end{aligned}$$

Thus

$$\Delta q^2 = \frac{1}{2m\omega}\left[(z+z^*)^2 + 1 - (z+z^*)^2\right] = \frac{1}{2m\omega}\,,$$
$$\Delta p^2 = \frac{m\omega}{2}\left[-(z-z^*)^2 + 1 + (z-z^*)^2\right] = \frac{m\omega}{2}\,.$$

Hence:

$$\Delta q \Delta p = \frac{1}{2}\,,$$

which is indeed the lowest value allowed by Heisenberg's uncertainty relation.

e) Assume $a^\dagger$ has an eigenvector $|\,w\,\rangle$ with eigenvalue w:

$$a^\dagger |\,w\,\rangle = w|\,w\,\rangle\,.$$

We can expand this state in the complete set $\{|\,n\,\rangle\}$:

$$|\,w\,\rangle = \sum_k c_k |\,k\,\rangle\,.$$

where the c_k's are the expansion coefficients. Inserting this in the eigenvalue equation, and using the defining properties of $a^\dagger$, we find:

$$a^\dagger |\,w\,\rangle = \sum_{k=0} c_k a^\dagger |\,k\,\rangle = \sum_{k=0} c_k \sqrt{k+1}|\,k+1\,\rangle$$
$$= w|\,w\,\rangle = w\sum_{k=0} c_k|\,k\,\rangle = wc_0|\,0\,\rangle + w\sum_{k=1} c_k|\,k\,\rangle = wc_0|\,0\,\rangle + w\sum_{k=0} c_{k+1}|\,k+1\,\rangle\,.$$

Now a fundamental property of linear vector spaces is that the components of a vector are unique in any basis. Therefore, if $w = 0$, this immediately yields $c_k = 0$ for all k, so $|\,w\,\rangle$ vanishes. If $w \neq 0$, we still must have $c_0 = 0$. The other coefficients must satisfy a recursion relation which is easily solved:

$$c_{k+1} = \frac{\sqrt{k+1}}{w} c_k\,, \qquad \Longleftrightarrow \qquad c_k = \frac{\sqrt{k!}}{w^k} c_0 = 0\,.$$

Hence $|\,w\,\rangle$ does not exist.

Problem 4:

a) Assume that the resolution of the identity has the form:

$$\mathbb{I} = \int \frac{\mathrm{d}^3\mathbf{k}}{M(k)} |\,\mathbf{k}\,\rangle\langle \mathbf{k}|\,.$$

Applying this to the orthonormality condition for the basis vectors themselves, we find:

$$\langle \mathbf{p}|\mathbf{p}'\rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{M(k)} \langle \mathbf{p}|\mathbf{k}\rangle\langle \mathbf{k}|\mathbf{p}'\rangle = N(p)N(p')\int \frac{\mathrm{d}^3\mathbf{k}}{M(k)} \delta(\mathbf{p}-\mathbf{k})\delta(\mathbf{k}-\mathbf{p}')$$
$$= \frac{N(p)^2}{M(p)}\delta(\mathbf{p}-\mathbf{p}')\,.$$

Hence $M(k) = N(k)$.

b) Since a rescaling of the operators will not change the fact that the commutator is a just a number, we can evaluate it in the vacuum state:

$$\begin{aligned}\langle 0|[a_{\mathbf{p}'}, a_{\mathbf{p}}^\dagger]|\,0\rangle &= \int \frac{\mathrm{d}^3\mathbf{k}}{N(k)}\left(\langle 0|a_{\mathbf{p}'}|\,\mathbf{k}\rangle\langle\mathbf{k}|a_{\mathbf{p}}^\dagger|\,0\rangle - \langle 0|a_{\mathbf{p}}^\dagger|\,\mathbf{k}\rangle\langle\mathbf{k}|a_{\mathbf{p}}|\,0\rangle\right)\\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{N(k)}\left(f(p')^* f(p)\langle\mathbf{p}'|\mathbf{k}\rangle\langle\mathbf{k}|\mathbf{p}\rangle - 0\right)\\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{N(k)} f(p')^* f(p) N(p') N(p)\delta(\mathbf{p}'-\mathbf{k})\delta(\mathbf{k}-\mathbf{p})\\ &= |f(p)|^2 N(p)\delta(\mathbf{p}'-\mathbf{p})\,,\end{aligned}$$

so $[a_{\mathbf{p}'}, a_{\mathbf{p}}^\dagger] = G(p)\delta(\mathbf{p}'-\mathbf{p})$ with $G(p) = |f(p)|^2 N(p)$.