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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 9, 28.2 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

The retarded Greens-function

In *Schwartz*, sec. 6.2, the Feynman propagator is found as the vacuum expectation value of the time-ordered product of two fields, which turns out to be a Greeen's function for the Klein–Gordon equations. It is, however, a Green's function with rather non-classical boundary conditions, as revealed by the positions of the poles in the complex ω plane (see *Schwartz*, fig. 6.1). This boundary condition comes from the scattering boundary condition, via the LSZ formula. There are, however, also other useful Green's functions, corresponding to different boundary conditions, which can be found by the same technique.

In classical field theory we are mostly interested in the **retarded Green's function**, which describes the future evolution of a field when we know it at all points in space at some fixed time. Such problems can, of course, also be formulated for quantum fields. The starting point is still eq. 2.51 in *Schwartz*:

$$D(x_1 - x_2) = \langle 0|\phi_0(x_1)\phi_0(x_2)|\,0\rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_k}e^{\mathrm{i}k(x_2-x_1)}\,. \qquad (S\ 6.25)$$

Assuming $t_1 > t_2$, we can then write the vacuum expectation value of the *commutator* of the fields at:

$$\begin{aligned}
&\langle 0|[\phi_0(x_1),\phi_0(x_2)]|\,0\rangle = D(x_1-x_2) - D(x_2-x_1)\\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_k}\left[e^{\mathrm{i}k(x_2-x_1)} - e^{-\mathrm{i}k(x_2-x_1)}\right]\,.\\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}\int\frac{\mathrm{d}\,\omega}{2\pi\mathrm{i}}\left[e^{\mathrm{i}\omega(t_2-t_1)}\frac{1}{\omega-(\omega_k-\mathrm{i}\epsilon)} - e^{\mathrm{i}\omega(t_2-t_1)}\frac{1}{\omega+(\omega_k-\mathrm{i}\epsilon)}\right]\\
&= \mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\,e^{\mathrm{i}k(x_2-x_1)} \qquad (t_1-t_2>0)\,.
\end{aligned} \qquad (9.1)$$

In the next to the last line we have used the invariance of $\mathrm{d}\mathbf{k}$ to make the variable change $\mathbf{k}\to-\mathbf{k}$ in the second term and performed the ω-integral by exploiting that $t_1-t_2>0$ to close the contour in the lower complex plane, picking up the residues at the two poles $\omega=\pm(\omega_k-\mathrm{i}\epsilon)$.

If instead $t_1-t_1<0$, we can instead close the ω-countour in the *upper* half plane, where there are no poles, so the result is zero. We thus have shown that:

$$D_R(x_1-x_2) = \theta(x_1^0-x_2^0)\langle 0|[\phi_0(x_1),\phi_0(x_2)]|\,0\rangle = \mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\,e^{\mathrm{i}k(x_2-x_1)}\,, \quad (9.2)$$

where $\theta(t)$ is the Heaviside function. This is easily confirmed to be a Green's function for the Klein–Gordon equation, up to a trivial factor:

$$(\Box + m^2) D_R(x_1 - x_2) = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, e^{\mathrm{i}k(x_2 - x_1)} = -\mathrm{i}\, \delta^4(x_1 - x_2)\,. \tag{9.3}$$

This retarded Green's function can be used to find the solution of the Klein–Gordon equation, classical or quantum, with a classical source, $j(x)$, which acts only for a finite time, say $j(x) = 0$ for $x^0 < t_0$. If the equation of motion is:

$$(\Box + m^2)\phi(x) = j(x)\,, \tag{9.4}$$

the solution can be written:

$$\phi(x) = \phi_0(x) + i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} D_R(x - y) j(y)\,, \tag{9.5}$$

valid for both classical and quantal fields $\phi(x)$. We see that by construction D_R makes $\phi(x) = \phi_0(x)$ for $x^0 < t_0$. The *advanced* Green's function, used for solving problems where we want to reconstruct the field at earlier times from that of a later time, can be found in the same manner as:

$$D_A(x_1 - x_2) = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2} \, e^{-\mathrm{i}k(x_2 - x_1)}\,. \tag{9.6}$$