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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 8, 1.2 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

The Schrödinger, Heisenberg and interaction pictures

In elementary non-relativistic quantum mechanics one is used to attack a problem by solving the time dependent Schrödinger equation, which in the bra-ket formalism can be written (we use a superscript S to distinguish the Schrödinger picture):

$$\mathrm{i}\frac{\partial}{\partial t}|\,\psi\,\rangle^S = H|\,\psi\,\rangle^S\,, \tag{8.1}$$

where H is the Hamiltonian. Any solution of this equation can be written:

$$|\,\psi(t)\,\rangle^S = U(t,t_0)|\,\psi_0\,\rangle^S\,, \tag{8.2}$$

where $U(t,t_o)$ is the *time evolution operator* and $|\,\psi(t_0)\,\rangle^S = |\,\psi_0\,\rangle^S$ is the wavefunction at $t = t_0$. Inserting this into the time dependent Schrödinger equation, one finds that $U(t,t_0)$ also satisfies it:

$$\mathrm{i}\frac{\partial}{\partial t}U(t,t_0) = HU(t,t_0)\,, \qquad U(t_0,t_0) = \mathbb{I}\,. \tag{8.3}$$

We also must have, for arbitrary t, t_0 and $|\,\psi(t)\,\rangle^S$:

$$|\,\psi_0\,\rangle^S = |\,\psi(t_0)\,\rangle^S = U(t_0,t)|\,\psi(t)\,\rangle^S = U(t_0,t)U(t,t_0)|\,\psi_0\,\rangle^S$$

for any $|\,\psi_0\,\rangle$, so

$$U(t_0,t)U(t,t_0) = \mathbb{I} \qquad \Longleftrightarrow \qquad U(t_0,t) = U^{-1}(t,t_0)\,. \tag{8.4}$$

Indeed, U is a unitary operator. This follows, because since $H = H^\dagger$, the Hermitean conjugate of eq. (8.3) is:

$$-\mathrm{i}\frac{\partial}{\partial t}U^\dagger(t,t_0) = U^\dagger(t,t_0)H\,, \qquad U(t_0,t_0) = \mathbb{I}\,. \tag{8.5}$$

Hence, with $U = U(t,t_0)$:

$$\frac{\partial}{\partial t}\left(U^\dagger U\right) = \left(\frac{\partial}{\partial t}U^\dagger\right)U + U^\dagger\left(\frac{\partial}{\partial t}U\right) = \mathrm{i}(U^\dagger H)U - \mathrm{i}U^\dagger(HU) = 0\,.$$

Thus $C = U^\dagger U$ is independent of t. Evaluating it at $t = t_0$ one finds $C = \mathbb{I}$. In the same manner one finds $UU^\dagger = \mathbb{I}$, so U is unitary or $U^{-1} = U^\dagger$. If the Hamiltonian does not depend explicitly on time, $\partial H/\partial t = 0$, we find the unique solution of eq. (8.3) by inspection:

$$U(t, t_0) = U(t - t_0) = e^{-\mathrm{i}H(t-t_0)} \,.$$

We shall give an explicit formula for $U(t, t_o)$ in the case of a time dependent H when we have introduced the time ordering operator in ch. 6.1 of *Schwartz*.

In the Schrödinger formulation of quantum mechanics, we have that the important operators, like x, or q, and p are time independent. But in classical mechanics, these operators are time dependent. Indeed, solving a problem of classical mechanics normally involves finding this dependence. Indeed, from the canonical quantization prescription (see Note 1), we found that the quantum observable associated with a classical dynamical variable A, including q and p, satisfies Heisenberg's equation of motion:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{\mathrm{i}}[A, H] + \frac{\partial A}{\partial t} \,. \tag{1.11}$$

Thus, A is generally time dependent. But H only changes with time if it contains an explicit time dependence.

This issue is resolved by realizing that the Heisenberg equation of motion actually applies to the operator in a different representation of vectors in the Hilbert space, called the *Heisenberg picture*, which we shall designate with a superscript H. The two pictures must, of course, be unitarily equivalent (see Problem 1 in the exercises). To avoid confusion, we now introduce superscripts also for the operators. The time evolution of any physical observable in the two systems will be the same, provided

$${}^{S}\langle\psi(t)|A^S|\,\phi(t)\rangle^S = {}^{H}\langle\psi|A^H(t)|\,\phi\rangle^H \tag{8.6}$$

for any states $|\,\psi\,\rangle, |\,\phi\,\rangle$ and operator A. Since the two pictures describe the same physical system, at some fixed but arbitrary time, t_0, we can adjust the phases of the wavefunctions to make the two pictures agree:

$$|\,\psi\,\rangle^H = |\,\psi(t_0)\,\rangle^S = U^\dagger(t, t_0)|\,\psi(t)\,\rangle^S \,, \tag{8.7}$$

for any $|\,\psi\,\rangle$. Afterwards, and even before, $|\,\psi(t_0)\,\rangle^S$ evolves according to eq. (8.2), while $|\,\psi\,\rangle^H$ remains unchanged, $\partial|\,\psi\,\rangle^H/\partial t = 0$.

The physical interpretation requires that eq. (8.6) remains satisfied at all times. Taking time derivatives, and using the Schrödinger equation and its Hermitean conjugate, we find ($U = U(t, t_0)$):

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{S}\langle\psi(t)|A^S|\,\phi(t)\rangle^S\right) \\
&= \frac{\partial}{\partial t}\left({}^{S}\langle\psi(t)|\right) A^S|\,\phi(t)\,\rangle^S + {}^{S}\langle\psi(t)|\frac{\partial A^S}{\partial t}|\,\phi(t)\rangle^S + {}^{S}\langle\psi(t)|A^S\frac{\partial}{\partial t}|\,\psi(t)\,\rangle^S \\
&= {}^{S}\langle\psi(t)|\mathrm{i}(HA^S - A^S H) + \frac{\partial A^S}{\partial t}|\,\phi(t)\rangle^S \\
&= {}^{H}\langle\psi|\frac{\mathrm{d}A^H(t)}{\mathrm{d}t}|\,\phi\rangle^H = {}^{S}\langle\psi(t)|U\frac{\mathrm{d}A^H(t)}{\mathrm{d}t}U^\dagger|\,\phi(t)\rangle^S \,.
\end{aligned}$$

Since this is true for any states $|\phi\rangle$ and $|\phi\rangle$, it is an operator identity, so:

$$\frac{\mathrm{d}A^H(t)}{\mathrm{d}t} = U^\dagger \left(\frac{1}{\mathrm{i}}[A^S, H] + \frac{\partial A^S}{\partial t} \right) U = \frac{1}{\mathrm{i}}[A^H, H] + \frac{\partial A^H}{\partial t} ,$$

where we have exploited that unitary transformations preserve operator identities. This is nothing but Heisenberg's equation of motion for A^H.

Finally, we introduce the interaction picture, which will be used to develop perturbation theory and the Feynman diagrams. This picture is intermediate between the Heisenberg and the Schrödinger pictures. We assume that the Hamiltonian can be split into two parts, called the *unperturbed Hamiltonian*, H_0, and the *interaction*, H_I or V. We shall follow *Schwartz*, and use the latter designation:

$$H = H_0 + V .$$

We assume that we can solve the unperturbed problem, with $V = 0$, exactly. In relativistic quantum field theory H_0 is almost always taken to be the time independent Hamiltonian for free fields, so that we can use our Fock-space construction as a convenient basis. If we start with the Schrödinger picture, the interaction picture wavefunction, which we might denote with a superscript I, but which Schwartz denotes by a subscript 0, we define, similarly to eq. (8.7):

$$|\psi(t)\rangle^I = |\psi(t)\rangle_0 = U_0^\dagger(t, t_0)|\psi(t)\rangle^S , \tag{8.8}$$

At $t = t_0$ all three pictures coincide. The interaction picture time evolution operator $U_0(t, t_0)$ evolves according to (cf. eq. (8.3)):

$$\mathrm{i}\frac{\partial}{\partial t}U_0(t, t_0) = U_0(t, t_0)H_0 , \qquad U_0(t_0, t_0) = \mathbb{I} . \tag{8.9}$$

In the following we shall assume that H_0 is time independent, in which case we have:

$$U_0(t, t_0) = U_0(t - t_0) = e^{-\mathrm{i}H_0(t-t_0)} . \tag{8.10}$$

From eqs. (8.8) and (8.9) it immediately follows that $|\psi\rangle_0$ satisfies the Schrödinger equation, with V only as Hamiltonian:

$$\begin{aligned}
\mathrm{i}\frac{\partial}{\mathrm{d}t}|\psi(t)\rangle_0 &= \mathrm{i}\frac{\partial U_0^\dagger(t,t_0)}{\partial t}|\psi(t)\rangle^S + U_0^\dagger(t,t_0)\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle^S \\
&= \left(-H_0U_0^\dagger(t,t_0) + U_0^\dagger(t,t_0)(H_0+V)\right) U_0(t,t_0)U_0^\dagger(t,t_0)|\psi(t)\rangle^S \; = V_0|\psi(t)\rangle_0 ,
\end{aligned} \tag{8.11}$$

where

$$V_0(t, t_0) = U^\dagger(t, t_0)V(t)U(t, t_0) \tag{8.12}$$

is the interaction in the interaction picture. Here we have used $[H_0, U_0^\dagger] = 0$, which follows immediately from eq. (8.12). Thus, if the interaction vanishes, the wavefunctions are constant, and the interaction picture and the Heisenberg picture coincide. The remaining time-evolution is taken care of by the operators, which satisfy the interaction picture version of the Heisenberg equation:

$$\frac{\mathrm{d}A_0(t)}{\mathrm{d}t} = \frac{1}{\mathrm{i}}[A_0, H_0] + \frac{\partial A_0}{\partial t} , \tag{8.13}$$

where $A_0 = U_0^\dagger A^s U_0$. This is shown in the same manner as for the Heisenberg picture.