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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 4, 20.1 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Algebraic quantization of the harmonic oscillator

In this section we shall fill in some details of the algebraic approach to quantizing the harmonic oscillator given in *Schwartz*, secs. 2.2.1 and 1.3. For the solution obtained by solving the Schrödinger equation in coordinate space, see *Brandsden & Joachin*, sec. 2.4.

The one-dimensional linear harmonic oscillator is defined by the Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \; = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \,. \tag{4.1}$$

where p is the canonical momentum conjugate to the coordinate x, so in the quantized theory one has the canonical commutator relation:

$$[x, p] = \mathrm{i}\hbar \,. \tag{4.2}$$

For a massive bead hanging in an elastic massless spring m is the mass, k the spring constant and $\omega = \sqrt{k/m}$ the classical angular frequency of the oscillations, but a Hamiltonian of this form shows up in many different situations, and the particular interpretation of m and k plays no role in the following.

The trick to find the eigenvalues of H without solving a differential equation, is to introduce what are known as the **annihilation** and **creation** operators, also called *raising* and *lowering* operators, or *ladder* operators, a and $a^\dagger$, defined as:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\mathrm{i}p}{m\omega}\right) \qquad \Longleftrightarrow \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\mathrm{i}p}{m\omega}\right) \,. \tag{4.3}$$

The reason for their names will become clear in the following. The definitions are easily inverted:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \,, \qquad p = \frac{1}{\mathrm{i}}\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger) \,. \tag{4.4}$$

The commutator between a and $a^\dagger$ is very simple:

$$[a, a^\dagger] = \frac{m\omega}{2\hbar}\left([x,x] + \frac{\mathrm{i}}{m\omega}(-[x,p] + [p,x]) + \frac{1}{m^2\omega^2}[p,p]\right) = 1 \,. \tag{4.5}$$

We also introduce an Hermitean operator, which we shall call the **number** operator, N, as:

$$N = N^\dagger = a^\dagger a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{i}{m\omega}[x,p] + \frac{p^2}{m^2\omega^2}\right) = \frac{p^2}{2m\omega\hbar} + \frac{m\omega}{2\hbar}x^2 - \frac{1}{2}\,.$$

Thus H can be written simply:

$$H = \hbar\omega\left(N + \frac{1}{2}\right)\,. \tag{4.6}$$

This means that the eigenvectors of H and N are the same, and can be labelled by the eigenvalues of N, which we call n. We see already from eq. (4.1) that H is the sum of two positive definite operators, so all the eigenvalues of H must be positive. But the above formula allows us to obtain a sharper result. We note from the definition of N that for any normalized eigenstate $|\,n\rangle$ of N:

$$n = \langle n|N|\,n\rangle = \langle n|a^\dagger a|\,n\rangle = \|a|\,n\rangle\|^2 \geq 0\,, \tag{4.7}$$

with equality if and only if $a|\,n\rangle = 0$, since the norm of a non-vanishing state is strictly positive. But this means that any eigenvalue E_n of H satisfies, using eq. (4.6):

$$E_n = \langle n|H|\,n\rangle = \hbar\omega\left(n + \frac{1}{2}\right) \geq \frac{1}{2}\hbar\omega\,. \tag{4.8}$$

To find the possible values of n, we calculate the commutators between N and $a^\dagger$ and a from eq. (4.5) and commutator properties:

$$[N,a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a,a^\dagger] + [a^\dagger,a^\dagger] = a^\dagger \quad\Longleftrightarrow\quad [N,a] = [a^\dagger,N]^\dagger = -a\,. \tag{4.9}$$

From this it follows that if $N|\,n\rangle = n|\,n\rangle$, then:

$$Na^\dagger|\,n\rangle = (a^\dagger N + a^\dagger)|\,n\rangle = (n+1)a^\dagger|\,n\rangle = (n+1)C_n|\,n+1\rangle\,, \tag{4.10a}$$

because clearly $a^\dagger|\,n\rangle$ is an eigenvector of N with eigenvalue $n+1$, but not necessarily normalized. The reason for the name *raising* operator is now clear. There might, of course, be several eigenvectors of N with this eigenvalue, and we shall come back to this point.

Similarly to eq. (4.10a) we also find why a is called the *lowering* operator:

$$Na|\,n\rangle = (aN - a)|\,n\rangle = (n-1)a|\,n\rangle = (n-1)C'_n|\,n-1\rangle\,, \tag{4.10b}$$

telling us that for any eigenvalue n the state $a|\,n\rangle$ is an eigenvector of N with eigenvalue $n-1$. It thus seems that we can construct eigenvectors of N with arbitrarily small values of n. But according to eq. (4.7), N has no negative eigenvalues. The only way to avoid a contradiction is that there is a state with eigenvalue $n = 0$, *i.e.* a state $|\,0\rangle$ such that:

$$a|\,0\rangle = 0 \qquad\Longleftrightarrow\qquad N|\,0\rangle = 0 \qquad\Longleftrightarrow\qquad E_0 = \frac{1}{2}\hbar\omega \tag{4.11}$$

This is the **ground state**, the state of lowest energy. This property of a is the reason for the name *annihilation* operator.

It is easy to prove that this ground state is unique. From eq. (4.3) in coordinate representation we find that the wavefunction $\psi_0(x) = \langle x|0\rangle$ satisfies:

$$\langle x|a|\,0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|x + \frac{\mathrm{i}}{m\omega}p|\,0\rangle = \sqrt{\frac{1}{2\hbar\omega m}}\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega\, x\right)\psi_0(x) = 0\,. \tag{4.12}$$

This is a first order ordinary differential equation for $\psi_0(x)$, which has a unique solution containing a single integration constant, which is just the normalization constant. Thus the ground state is unique, and one finds the normalized solution:

$$\psi_0(x) = \sqrt{\frac{2\pi\hbar}{m\omega}}e^{-\frac{m\omega}{2\hbar}x^2} \tag{4.13}$$

From $|\,0\rangle$ we can iteratively build a tower of states with integer eigenvalues n, using eq. (4.10a). One only finds a single state for each n, so the energy spectrum has no degeneracies. Thus, starting with $|\,0\rangle$, we have an infinite tower of eigenstates of N and H, one for each n: $a|\,0\rangle = C_0|\,1\rangle, a|\,1\rangle = C_1|\,2\rangle$, etc. This explains the name *creation* operator.

It remains to find the normalization constants. Assuming the eigenstates to be normalized, we find from eqs. (4.10a) and (4.5):

$$\begin{aligned}
C_n^2 &= \langle n+1|C_n^2|\,n+1\rangle = \langle n|aa^\dagger|\,n\rangle = \langle n|a^\dagger a + 1|\,n\rangle = \langle n|N+1|\,n\rangle = n+1\,,\\
C_n &= \sqrt{n+1}\,.
\end{aligned}$$

Using induction, this yields:

$$|\,n\rangle = \frac{1}{C_{n-1}}a^\dagger|\,n-1\rangle = \ldots = \frac{1}{\sqrt{(n!)}}(a^\dagger)^n|\,0\rangle\,. \tag{4.14}$$

Using this formula and eq. (4.3) in coordinate representation, the wavefunctions for any energy eigenstate can be calculated.

Creation and annihilation operators for more general systems.

Although creation and annihilation operators are specially suited to find the energy eigenstates for an harmonic oscillator, they can be introduced for any Hamiltonian with discrete eigenvalues, provided none of them are infinitely degenerate. We assume for simplicity that these single particle states are labelled by an index i:

$$H|\,i\rangle = E_i|\,i\rangle \tag{4.15}$$

We also assume the states $|\,i\rangle$ to be normalized. In practice i may actually be a set of quantum numbers, like n, l, m, m_s for an electron in a central field. Furthermore, we assume that we have introduced a fixed order for the index i, such that the E_i is a non-decreasing function of i:

$$E_0 \le E_1 \le E_2 \le \ldots\,,$$

$|\,0\rangle$ being the single particle ground state. Since the eigenstates of H form a complete set and an operator is defined by its action on such a basis, or, equivalently, by the value of all its matrix elements in the basis, we can *define* the creation operator, $a^\dagger$, as an operator with matrix elements:

$$\langle m|a^\dagger|\,n\rangle = \sqrt{n+1}\,\delta_{m,n+1}\,, \tag{4.16a}$$

Using the completeness relation this can also be written:

$$a^\dagger|\,n\rangle = \sum_m |\,m\rangle\langle m|a^\dagger|\,n\rangle = \sqrt{n+1}\,|\,n+1\rangle \tag{4.16b}$$

For the Hermitean conjugate, a we then have:

$$\begin{aligned} \langle m|a|\,n\rangle &= \langle n|a^\dagger|\,n\rangle^* = \sqrt{m+1}\,\delta_{n,m+1} = \sqrt{n}\,\delta_{m,n-1} \\ a|\,n\rangle &= \sum_m |\,m\rangle\langle m|a|\,n\rangle = \sqrt{n}\,|\,n-1\rangle\,. \end{aligned} \tag{4.17}$$

It is easy to verify that $a^\dagger$, a and $N = a^\dagger a$ satisfy the basic commutation relation $[a, a^\dagger] = 1$, $N|\,n\rangle = n|\,n\rangle$ and even $[H, N] = 0$.

There is, however, one important difference to the harmonic oscillator case, namely that we do not in general have any useful relationship between N and H, or equivalently, a simple formula to calculate E_n from n. But, as we shall see, there are other situations where this is the case.