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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 3, 12.1 2018

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Physical interpretation of quantum theories

The basic assumption of any quantum theory is that there is a duality between a pure (isolated) system, which can be completely characterized by a ket $|\,\psi\,\rangle$ in the Hilbert space, $\mathcal{H}$, and a (complete and ideal) measurement, which can detect if the system is in a state $|\,\psi\,\rangle$, or not. Such an experiment only produces a probabilistic result, a conditional probability that we observe the system to be in state $|\,\phi\,\rangle$ given that the system was in state $|\,\psi\,\rangle$:

$$P(\phi|\psi) = \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle\langle\psi|\psi\rangle} \tag{3.1}$$

If $|\,\phi\,\rangle$ and $|\,\psi\,\rangle$ are normalized to 1, the denominator drops out. This will be assumed in the following. From the basic property $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ (eq. (2.3)), we have the identity:

$$P(\phi|\psi) = |\langle\phi|\psi\rangle|^2 = |\langle\psi|\phi\rangle|^2 = P(\psi|\phi)\,. \tag{3.2}$$

We therefore have the non-intuitive result that in any quantum theory the probability of finding a system prepared in state $|\,\psi\,\rangle$ to be in state $|\,\phi\,\rangle$ is the same as the probability to measure it to be in state $|\,\psi\,\rangle$ if it was prepared to be in state $|\,\phi\,\rangle$. This equality immediately leads to the principles of *microreversibility* and *detailed balance*, which belong to the foundations of equilibrium statistical mechanics.

When discussing quantum measurements, it has been traditional, since the early days of quantum mechanics, to add Niels Bohr's postulate of the **Collapse of the wave function**, which states that:

> *After a measurement the system will be in the state selected by the measurement process, independently of its state before the measurement.*

Thus, if a system prepared in an arbitrary state $|\,\psi\,\rangle$ is measured to be in state $|\,\phi\,\rangle$, it *will* be in that state, with probability 1, immediately after the measurement. Although supported by countless experiments, the status of this postulate has always been contentious, because it does not seem compatible with the mathematical structure of the theory, essentially because a collapse of the wavefunction cannot be described by a linear operator on $\mathcal{H}$.

Niels Bohr circumvented this problem by demanding that the measurement equipment must be *classical objects*, describable by classical physics. The point is that the applicability of eq. (3.1) assumes that both the system and measurement apparatus are isolated quantum systems, each completely described by a wavefunction. This assumption is not fulfilled in most experimental situations, in particular not by the measurement apparatus. This would require it to be totally insulated from the surroundings during a measurement, as any contact, *e.g.* thermal or electrical, with the surroundings will invalidate it. And the interaction of a quantum system with a classical system will quite generally effectively lead to the collapse of the wavefunction.

Nevertheless, this situation is not entirely satisfactory, if one considers Quantum Mechanics as the fundamental theory, and Classical Mechanics as an approximation to it. One would think that it should be possible to describe measurements without relying on the classical approximation. And this does indeed turn out to be the case. Just like for a classical system, one must turn to statistical mechanics to describe a quantum system interacting with its surroundings. In the 1980s it was realized that quantum statistical mechanics *predicts* that a non-isolated quantum system interacting with its surroundings will undergo a process called *quantum decoherence*, resulting in the system evolving into an eigenstate with the result of the measurement process as eigenvalue. Thus the *collapse of the wave function* need not be added as a separate postulate, after all.

Furthermore, about the same time it was realized that it is possible to perform *quantum non-demolition experiments*, where the measurement process itself can be described by quantum mechanics. Nevertheless, in most situations the collapse of the wave function gives the correct result, it should just not be regarded as a postulate, but as a theoretical prediction.

Observables

Classical dynamical variables are represented by *Hermitean* (or self-adjoint) operators in quantum mechanics. These are assumed to have a complete set of real eigenvalues, *i.e.* the solutions of the eigenvalue equation:

$$A|\, a_i \,\rangle = a_i|\, a_i \,\rangle \,, \tag{3.3}$$

has infinitely many solutions, $|\, a_i \,\rangle$, which we for convenience have labelled by their eigenvalues, a_i. For the moment we assume that the eigenvalues are discrete, and that some way of labelling degenerate states have been implemented, so that $\langle a_i | a_j \rangle = \delta_{ij}$. That the set $\{|\, a_i \,\rangle\}$ is complete, and hence can be used as a basis for $\mathcal{H}$, means that we have the resolution of the unit, eq. (2.6):

$$\sum_i |\, a_i \,\rangle\langle a_i| = \mathbb{1} \,, \tag{2.6}$$

The expansion of an arbitrary wavefunction in this basis of eigenvectors is of course identical in form to eq. (2.1a):

$$|\, \psi \,\rangle = \mathbb{1}|\, \psi \,\rangle = \sum_i |\, a_i \,\rangle\langle a_i|\psi\rangle = \sum_i \psi_i|\, a_i \,\rangle \,, \qquad \psi_i = \langle a_i|\psi\rangle \,. \tag{2.1c}$$

The probability that we shall obtain the value a_i in an experiment measuring the observable A is then, according to eq. (3.1):

$$P(a_i|\psi) = |\langle a_i|\psi\rangle|^2 = |\psi_i|^2 \,.$$

We note that since:

$$P(a_i|a_j) = |\langle a_i|a_j\rangle|^2 = \delta_{ij}\,, \tag{3.4}$$

a measurement to see if the system is in the state $|\,a_i\,\rangle$ when it was prepared in the state $|\,a_j\,\rangle$ delivers the only sensible answer.

By the usual definition, the expectation value of A in the state $|\,\psi\,\rangle$ is, using eqs. (2.6) and (3.3):

$$\begin{aligned} < A >_\psi &= \sum_i a_i P(a_i|\psi) = \sum_i a_i\,|\langle a_i|\psi\rangle|^2 = \sum_i a_i \langle\psi|a_i\rangle\langle a_i|\psi\rangle \\ &= \sum_i \langle\psi|a_i\rangle\langle a_i|A|\,\psi\rangle = \langle\psi|A|\,\psi\rangle\,. \end{aligned} \tag{3.5}$$

Which operators that are important in describing a physical system, and their interrelations, is of course an experimental issue in the last instance. In this course we shall assume that they can be found from the corresponding classical theory, using canonical quantization, or path integral quantization.

Continuous variables

We know that some important physical variables, like position, x, and momentum, p, have a continuous set of eigenvalues. The expansion in eq. (2.1c) then becomes an integral:

$$|\,\psi\,\rangle = \int \mathrm{d}a\,\psi(a)|\,a\,\rangle\,, \qquad \psi(a) = \langle a|\psi\rangle\,, \tag{2.1$'$}$$

while the completeness relation takes the form

$$\int \mathrm{d}a\,|\,a\,\rangle\langle a| = \mathbb{1}\,. \tag{2.6$'$}$$

The states $\{|\,a\,\rangle\}$ are not physical, in the sense that one can ever precisely create or measure a physical system in such a state. This would require infinite precision. Indeed, the eigenvectors of continuous variables are not normalizable, although the result $\langle a|a'\rangle = 0$ for $a \neq a'$ still holds. The correct normalization condition is instead:

$$\langle a|a'\rangle = \delta(a - a')\,. \tag{2.2$'$}$$

This means that we have extended our formalism to allow distributions. But only normalizable states represent truly physical states, and such states have a straightforward interpretation. From eq. (2.6') we have, if $|\,\psi\,\rangle$ is normalized:

$$1 = \langle\psi|\psi\rangle = \int \mathrm{d}a\,|\langle a|\psi\rangle|^2 = \int \mathrm{d}a|\psi(a)|^2\,. \tag{2.4$'$}$$

This means, of course, that $p(a|\psi) = |\psi(a)|^2$ is the *probability density* of finding a system in the state $|\,\psi\,\rangle$ to have a value of a between a and $a + \mathrm{d}a$. Similarly, the expectation value of A can be written:

$$\langle\psi|A|\,\psi\rangle = \int \mathrm{d}a\,a|\psi(a)|^2 = \int \mathrm{d}a\,a\rho(a)\,. \tag{3.5$'$}$$

The above in particular applies to $a = x$ and $a = p$. One also finds the coordinate representation of p from the canonical commutation relation $[x, p] = \mathrm{i}\hbar$. Let $|\,\xi\,\rangle$ and $|\,\xi'\,\rangle$ be two eigenvectors of x. Then:

$$\begin{aligned} \mathrm{i}\hbar\delta(\xi - \xi') &= \mathrm{i}\hbar\langle\xi|\xi'\rangle = \langle\xi|(xp - px)|\,\xi'\rangle = (\xi - \xi')\langle\xi|p|\,\xi'\rangle \\ \langle\xi|p|\,\xi'\rangle &= \mathrm{i}\hbar\frac{\delta(\xi - \xi')}{\xi - \xi'} = \frac{\hbar}{\mathrm{i}}\delta'(\xi - \xi')\,, \end{aligned} \tag{3.6}$$

where we have used the relation between $\delta(x)$ and $\delta'(x) = -\mathrm{d}\delta(x)/\mathrm{d}x = -\delta(x)/x$ in the last step. It is easy to verify the correctness of this result:

$$p\psi(x) = \langle x|p|\,\psi\rangle = \int \mathrm{d}x'\,\langle x|p|\,x'\rangle\langle x'|\psi\rangle = \frac{\hbar}{\mathrm{i}}\int \mathrm{d}x'\,\delta'(x - x')\psi(x') = \frac{\hbar}{\mathrm{i}}\psi'(x)\,.$$

Interchanging $x \leftrightarrow p$ in this derivation, one also finds the momentum-space representation of x:

$$x\psi(p) = \mathrm{i}\hbar\psi'(p)\,.$$