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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 20, 19.4 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Quantum electrodynamics (QED)

In the previous lecture we studied the Yukawa coupling. What other possible couplings can fermions have? It is, of course tempting to try $\mathcal{L}_I = g/n!\bar{\psi}\psi\phi^n$, coupling a fermion-anti-fermion pair to n scalars. But it turns out that no such coupling is renormalizable for $n > 1$ in four space-time dimensions, although they can be in lower dimensions. Furthermore, it turns out that all couplings with more than one $\bar{\psi}\psi$ pair are also non-renormalizable. Thus a renormalizable interaction Lagrangian involving fermions much must one or more of the 16 basic Dirac bilinear forms combined with gamma-matrices: (see Lecture Note 18):

$$\bar{\psi}\psi\,; \qquad \bar{\psi}\gamma^5\psi\,; \qquad j^\mu = \bar{\psi}\gamma^\mu\psi\,; \qquad j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi\,; \qquad \bar{\psi}\sigma^{\mu\nu}\psi\,.$$

The first gives rise to the Yukawa-coupling to a scalar field. The same does the second, with the only difference that it couples to a *pseudoscalar* field, *i.e.* a field of negative parity. The antisymmetric tensor $\bar{\psi}\sigma^{\mu\nu}\psi$ could couple to some other antisymmetric tensor field. One obvious candidate is the electromagnetic field tensor, $F^{\mu\nu}$. Indeed, $\mathcal{L}_I = g\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ describes a theory of a fermion with an arbitrary magnetic moment, unrelated to its charge, as shown by Pauli already in 1941. But also this theory is non-renormalizable. This is perhaps fortunate, as it would otherwise add another set of constants of nature to the Standard Model. Also note that no traceless *symmetric* tensor can be formed from the gamma matrices. This reflects the fact that such a tensor transforms according to the spin-2 representation of the rotation group, while two spin-$\frac{1}{2}$Dirac particles can only couple directly to $s = 0$ and $s = 1$, according to the Clebsh–Gordan series, eq. (13.26-7). Thus, there is no direct coupling to gravitons. Actually, the description of spinors in general relativity is somewhat subtle.

It remains to consider couplings to vector (and pseudo-vector) currents. From Classical Mechanics we know that the interaction between a particle of charge q and an external field can be implemented by the simple rule of *minimal substitution*, $p^\mu \to p^\mu + qA^\mu$, where $A^\mu = [A^0 = \Phi, \mathbf{A}]$ is the electromagnetic 4-vector potential. After quantization, $p_\mu \to \mathrm{i}\partial_\mu$, the Dirac equation, eq. (15.9), becomes (note that *Schwartz* mostly considers electrons as his particles, so he uses $q = -e$ with $e > 0$):

$$(\mathrm{i}\gamma^\mu\partial_\mu + q\gamma^\mu A_\mu - m)\psi(x) = (\mathrm{i}\not{D} - m)\psi = 0\,, \tag{20.1}$$

where we have introduced the *covariant derivative* $D^\mu = \partial^\mu - iqA^\mu$. The Dirac Hamiltonian can be similarly written as:

$$H = \boldsymbol{\alpha}\cdot(\mathbf{p} + q\boldsymbol{A}) + \beta m + q\Phi\,, \tag{20.2}$$

with $\Phi = A^0$. This can be used in *relativistic quantum mechanics* as the basic Hamiltonian for an electron moving in an external field A^μ. It is heavily used in relativistic atomic physics, where the electrons of an atom are treated as moving in the classical Coulomb field of a heavy nucleus.

Eq. (20.1) is the Euler–Lagrange equation of the Dirac theory with a Lagrangian density:

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_I = \bar{\psi}(\mathrm{i}\partial_\mu \gamma^\mu - m)\psi + q\bar{\psi}\gamma^\mu \psi A_\mu = \bar{\psi}(\mathrm{i}\not{D} - m)\psi \tag{20.3}$$

We see that we can write the interaction simply as $\mathcal{L}_I = qj^\mu A_\mu$, where j^μ is the conserved Noether current of the Dirac theory, arising from its global phase invariance. The charge q is often absorbed in the definition of j^μ, with or without its sign. Thus *Schwartz* uses $J^\mu = ej^\mu = -qj^\mu$. It is instructive to "square" this equation, similarly to what we did in the free case. Since:

$$[D^\mu, D^\nu] = -iq(\partial^\mu A^\nu - \partial^\nu A^\mu) = -iqF^{\mu\nu} \neq 0\,, \tag{20.4}$$

we use the relation:

$$\begin{aligned} \not{O}_1 \not{O}_2 &= \frac{1}{4}\left(\{O_1^\mu, O_2^\nu\}\{\gamma^\mu, \gamma^\nu\} + [O_{1\mu}, O_{2\nu}][\gamma_1^\mu, \gamma_2^\nu]\right) \\ &= \frac{1}{2}\left(O_1 \cdot O_2 + O_2 \cdot O_1\right) - \frac{\mathrm{i}}{2}[O_{1\mu}, O_{2\nu}]\sigma^{\mu\nu}\,, \end{aligned} \tag{20.5}$$

valid for any vector operators Q_1^μ and O_2^ν. With $O_1 = (\mathrm{i}\not{D} + m)$ and $O_2 = (\mathrm{i}\not{D} - m)$, one finds:

$$\begin{aligned} 0 = (\mathrm{i}\not{D} + m)(\mathrm{i}\not{D} - m)\psi &= \left((\mathrm{i}\not{D})^2 - m^2 + \frac{\mathrm{i}}{2}[D_\mu, D_\nu]\sigma^{\mu\nu}\right)\psi \\ &= \left((\mathrm{i}\partial^\mu + qA^\mu)^2 - m^2 + \frac{q}{2}F_{\mu\nu}\sigma^{\mu\nu}\right)\psi\,. \end{aligned} \tag{20.6}$$

If we drop the term involving $\sigma^{\mu\nu}$ we have the Klein-Gordon equation for a charged spin-0 particle in an electromagnetic field (see below). For a Dirac particle the additional term is identical in form to the one which appears in a Pauli term. Thus, the Dirac Hamiltonian predicts that charged fermions have an intrinsic *magnetic moment.* In the non-relativistic limit one can show that this coupling to a magnetic field is given by a potential energy:

$$V_s = \frac{q}{2m}(\mathbf{L} + 2\mathbf{S}) \cdot \mathbf{B},. \tag{20.7}$$

The term with $\mathbf{L}$ is the coupling of the magnetic field to the orbital angular momentum, describing the precession of a particle in a magnetic field, which is also present for a spin-0 particle. The term involving $\mathbf{S}$ is the spin precession. The coefficient $g = 2$ for this term is a surprise of the Dirac theory, called the *gyromagnetic ratio.* This coefficient is further modified in QED by radiative corrections. The lowest order correction can be calculated fairly straightforwardly in perturbation theory. The result is famous: $g = 2 + \frac{\alpha}{2\pi}$, where, reinstating units, $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}$ is the *fine structure constant* (see *Schwartz* ch. 17).

If we add the Lagrangian density of the free electromagnetic field (see *Schwartz* sec. (3.4)) to $\mathcal{L}$ in eq. (20.3), we obtain the full QED Lagrangian density:

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_D + \mathcal{L}_I + \mathcal{L}_{\text{EM}}\,, \qquad \mathcal{L}_{\text{EM}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)^2\,. \tag{20.8}$$

However, based on what we have learned about the representations of the Poincaré group, we see that something strange is going on here. The irreducible representations of that group are characterized by mass and spin. Since there is no mass term in $\mathcal{L}_{\text{EM}}$, it must describe massless particles, which is excellent, because the photon mass is known experimentally to be below $10^{-18}\,\text{eV}$ or $2 \cdot 10^{-54}\,\text{kg}$. But a massless particle of spin larger than zero should only have two spin components (see Lecture Notes 15), and A_μ seems to have four degrees of freedom. As a matter of fact, A^0 transforms as a scalar under rotations, which corresponds to spin-0 particle. Thus this must somehow be removed. But even $\mathbf{A}$ needs a constraint in order to eliminate the $m_s = 0$ component of the field. This turns out to be closely related to the property of *gauge invariance* of the electromagnetic field.

The explicit presence of the vector potential A^μ in the field equations is at first glance already quite disturbing, because we know that it is not unique. Under a *gauge transformation*:

$$A^\mu \quad \to \quad A'^\mu = A^\mu + \frac{1}{q}\partial^\mu \alpha\,, \tag{20.9a}$$

with an arbitrary real function $\alpha(x)$, $F^{\mu\nu}$, *i.e.* the physical fields $\mathbf{E}$ and $\mathbf{B}$, remain unchanged. The crucial observation is that this transformation can be compensated for by a *local* phase transformation on the fields:

$$\psi(x) \quad \to \quad \psi'(x) = e^{\mathrm{i}\alpha(x)}\psi(x)\,. \tag{20.9b}$$

Under this combined transformation we have:

$$\begin{aligned}(\mathrm{i}\partial^\mu + qA'^\mu)\psi'(x) &= \mathrm{i}\partial^\mu\left(e^{\mathrm{i}\alpha(x)}\psi(x)\right) + q\left(A^\mu + \frac{1}{q}\partial^\mu\alpha(x)\right)e^{\mathrm{i}\alpha(x)}\psi(x) \\ &= e^{\mathrm{i}\alpha(x)}\left(\mathrm{i}\partial^\mu + qA^\mu\right)\psi(x)\,.\end{aligned} \tag{20.9c}$$

Therefore,

$$\bar{\psi}'(x)(\mathrm{i}\partial^\mu + qA'^\mu)\psi'(x) = \bar{\psi}(x)(\mathrm{i}\partial^\mu + qA^\mu)\psi(x)\,,$$

so the Lagrangian is indeed invariant under local gauge transformation.

The same construction goes through for a *complex* scalar field. The Lagrangian is

$$\mathcal{L} = (D_\mu\phi)^*(D_\mu\phi) - m^2\phi^*\phi\,. \tag{20.10}$$

Eqs. (20.9) remain valid, so this Lagrangian density is gauge invariant. It leads to the Euler-Lagrange equation:

$$[(p^\mu + qA^\mu)^2 - m^2]\phi = [(\mathrm{i}\partial^\mu + qA^\mu)^2 - m^2]\phi(x) = 0\,. \tag{20.11}$$

Eq. (20.10) together with the Lagrangian for the free electromagnetic field constitutes the Lagrangian for *scalar electrodynamics.*

The technically correct canonical quantization procedure of the electromagnetic field is actually extremely tricky, indeed the standard field theory textbook of *Peskin and Schroeder* simply skip the issue, preferring the path integral approach, which has the additional advantage that it applies also to non-Abelian gauge theories. Here we shall follow suit, just mentioning some of the more important points. The first is that since $\mathcal{L}$ does not contain $\partial_0 A^0$ at all, which means that the scalar potential has no conjugate momentum, and so is no dynamical field at all, and cannot be quantized. Thus it is basically just an auxillary classical field. Now we can actually choose to work in a gauge with $A^0 = 0$, which is called an *axial gauge*. This in no way prevents the existence of electric fields, they are simply given by $\mathbf{E} = -\partial_0 \mathbf{A}$. One then just quantize $\mathbf{A}$, but one must still get rid of one degree of freedom, which can be done. Another common gauge choice is *Coulomb gauge*. In this one chooses $\alpha(x)$ such that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, and treat A_0 as an auxillary field, without dynamics. This choice actually works very well for radiation interacting with essentially non-relativistic systems, and is the gauge of choice for many applications in atomic and condensed matter physics, including laser physics. But its lack of nice Lorentz transformation properties makes the Feynman rules unwieldy and unsuitable for studies of the basic structure of QED and application in fields like high energy physics.

For the latter class of problems, the *Lorentz gauge* is mostly preferred, which is defined by the covariant constraint:

$$\partial_\mu A^\mu = \partial^0 A^0 + \boldsymbol{\nabla} \cdot \mathbf{A} = 0\,. \tag{20.12}$$

To get an idea how this might work, assume that we expand the photon field in this base in terms of the *two* polarization modes which the group theory tells us are all there is.

$$A_\mu(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \sum_{i=1}^{2} \left[\epsilon^i_\mu(k) a^i_k e^{-\mathrm{i}p\cdot x} + \epsilon^{i*}_\mu(k) a^{i\,\dagger}_k e^{\mathrm{i}p\cdot x} \right]\,. \tag{20.13}$$

Here $k^0 = |\mathbf{k}| = \omega_k$ for a massless field, and $\epsilon^i_\mu(k)$, $k = 1, 2$ are called *polarization vectors*. We see that A^μ is real even if the ϵ^i_μ's are complex. They are chosen both to be orthonormal and space-like, and to satisfy the Lorentz gauge condition:

$$e^{i*} \cdot e^j = e^{i\,*}_\mu e^{j\,\mu} = -\delta_{ij}\,, \qquad k \cdot \epsilon^i = k^\mu \epsilon^i_\mu(k) = 0\,. \tag{20.14}$$

If one performs a Lorentz transformation on A^μ, the polarization vectors are mixed, not only among themselves, but also with k^μ, but do not get time-like component along an time-like basis vector ϵ^0 with $\epsilon^0 \cdot \epsilon^0 > 0$. If they did, it would be possible to find some vector which is transformed from being space-like to being time-like by the transformation. Thus under a Lorentz transformation, Λ, one has:

$$\epsilon^i_\mu \to {\epsilon'}^{\,i}_\mu = c^{ij}(\Lambda)\epsilon^j_\mu + c^i(\Lambda) k_\mu\,. \tag{20.15}$$

The Lorentz transformations preserve $k \cdot e^i = 0$ since $k^2 = 0$.

Now, consider the matrix element of a process creating or absorbing a single photon. In follows from the LSZ formula, just as in with eq. (19.2), that this must have the form

$$\mathcal{M} = \epsilon_\mu \mathcal{M}^\mu\,, \tag{20.16}$$

where ϵ is some linear combination or ϵ^1 and ϵ^2. But under a Lorentz transformation, also $\mathcal{M}^\mu$ will transform as a four-vector, with the same transformation law as ϵ^μ. That means that the transformed $\mathcal{M}$ has the general structure:

$$\mathcal{M}' = \epsilon''_\mu \mathcal{M}'^\mu + c(\Lambda) k_\mu \mathcal{M}'^\mu ,$$

where ϵ'' is a linear combination of ϵ'^1 and ϵ'^2. But if we had made the calculation directly in the primed coordinate system, the last term would not have appeared. The solution to this dilemma is to assume that we have a consistency condition, which reads:

$$k_\mu \mathcal{M}^\mu = 0 , \tag{20.17}$$

for all processes. This is called a *Ward identity*, and has to be satisfied by any matrix element of a physical process. It remains valid also in the case of several photons in the initial and/or final state. This correctness of this result can indeed be proven rigorously.

For completeness, we note that a gauge field construction based on the pseudo-vector current $j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$ does not work, because it corresponds to a phase transformation $e^{\mathrm{i}\gamma^5\alpha}$, which cannot be undone by any transformation on a pseudo-vector field B^μ, as this carries no spin indices.