UNIVERSITETET I STAVANGER
Institutt for matematikk og naturvitenskap

Lecture notes for FYS610 Many particle Quantum Mechanics

Note 19, 18.4 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Feynman rules for spinors

We have now established the momentum space Feynman propagator for spinors:

$$S_F(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|\,0\rangle\,; \tag{17.15}$$

$$\tilde{S}_F(p) = \mathrm{i}\frac{\not{p}+m}{p^2-m^2+\mathrm{i}\epsilon} = \frac{\mathrm{i}}{\not{p}-m+\mathrm{i}\epsilon}\,. \tag{17.16}$$

It remains to establish the Feynman rules from the LSZ formula. In deriving that the starting point was the mode expansion of the field operator, *Schwartz* eq. (6.7), which for fermions reads:

$$\psi(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{r=1}^{2}\left(a^r_{\mathbf{p}}u^r(p)\,e^{-\mathrm{i}p\cdot x} + b^{r\,\dagger}_{\mathbf{p}}v^r(p)\,e^{+\mathrm{i}p\cdot x}\right)\,. \tag{17.2$'$}$$

The only difference from the Klein-Gordon case is the presence of the spinors $u^r(p)$, and of course that the creation and annihilation operators now anticommute. By going through the derivation, taking these changes into account, one finds after some work that the LSZ formula, *Schwartz* eq. (6.19), is modified to:

$$\begin{aligned}
&\langle p_1', s_1'; \ldots ; p_m', s_m'|S|\,p_1, s_1; \ldots ; p_n, s_n\rangle \\
&= \left[\mathrm{i}\,u^{s_1}(p_1)\int \mathrm{d}^4x_1 e^{-\mathrm{i}p_1x_1}S_F^{-1}(x_1)\right]\ldots\left[\mathrm{i}\,u^{s_n}(p_n)\int \mathrm{d}^4x_n e^{-\mathrm{i}p_nx_n}S_F^{-1}(x_n)\right] \\
&\times\left[\mathrm{i}\,\bar{u}^{s_1'}(p_1')\int \mathrm{d}^4x_1' e^{-\mathrm{i}p_1'x_1'}S_F^{-1}(x_1')\right]\ldots\left[\mathrm{i}\,\bar{u}^{s_m'}(p_m')\int \mathrm{d}^4x_m' e^{-\mathrm{i}p_m'x_m'}S_F^{-1}(x_m')\right] \\
&\times\left[\mathrm{i}\int(\text{Other fields})\;\ldots\right] \\
&\times\langle\Omega|T\left\{\psi^{s_1}(x_1)\ldots\psi^{s_n}(x_n)\bar{\psi}^{s_1'}(x_1')\ldots\bar{\psi}^{s_m'}(x_m')[\text{Other fields}\;\ldots]\right\}|\,\Omega\rangle
\end{aligned} \tag{19.1}$$

The only important differences are the replacing of the inverse propagators by Dirac propagators, which will just cancel the propagators on the external legs when calculating the Feynman diagrams, and the presence of the spinors for the external states. In addition, as we have already noted, the definition of the time-ordered product must be changed, so that every time two fermionic operators are interchanged, the matrix element is changing sign. To have a specific phase convention, operators with equal time arguments should not be interchanged. LSZ in the form of eq. (19.1) generalizes to all kinds of quantum fields: One should replace the propagators with those appropriate to the free theory, and insert free particle momentum-space wavefunctions appropriate for the asymptotic free states, taking care of spins and internal quantum numbers.

Wick's theorem remain valid with the obvious modification that also in the definition of normal ordering an extra minus sign is introduced when two fermion operators are interchanged. One can then prove Wick's theorem as before, except that we have to keep track of signs. We can then go to the interaction picture as before, arriving at *Schwartz* eq. (7.64), expanding the time ordered matrix element in eq. (19.1) in powers of the interaction Lagrangian and taking the Fourier transform, and so establish the Feynman series in momentum space. Doing this, we find that the basic formula for cross sections in *Schwartz* sec. (5.1) remain unchanged, with the matrix element $\mathcal{M}$ as a sum terms calculated from all the relevant Feynman diagrams. As before bubble diagrams should not be included, and disconnected diagrams can be neglected.

The Feynman rules for bosons ad fermions are almost identical:

1) Each particle line contributes an appropriate Feynman propagator for its kind. It is convenient to give different types of particles typographically different lines.

2 Each vertex contributes a factor, derived from the interaction Lagrangian, $\mathcal{L}_I$.

3) Each external lines contributes a momentum space wavefunction instead of a propagator. For spin 0 particles, this is just a 1, for incoming fermions of momentum p and spin s it is $u^s(p)$, for incoming anti-fermions $\bar{v}^s(p)$, for outgoing particles $\bar{u}^s(p)$, for outgoing anti-particles $v^s(p)$.

4) Impose momentum conservation at each vertex. Other Noether currents may also be conserved by $\mathcal{L}_I$.

5) Integrate over each undetermined momentum.

6) Each closed fermion loop contributes an extra factor (-1). Interchange of external fermions also induces a minus sign for each interchange.

7) Divide by the symmetry factor (essentially always 1 for fermions).

It is customary and convenient to give the lines of particles which are not identical from their own anti-particles an arrow pointing in the direction of the particle flow. This coincides with the direction of the external momenta for a particle, while an antiparticle gets an arrow in the opposite direction. Theories with distinct particles and antiparticles always have an associated conserved current density, j^μ, and letting the arrow follow the current makes it easy to verify charge conservation by inspection.

Rule 6, which is very important, is the only remnant of the fact that fermions obey anticommutator relations. Additional minus signs will crop up when one actually evaluate products of gamma matrices in accordance with the Feynman rules.

A new rule in the above set is rule 3, which only concern external lines. The external propagators are eliminated by the LSZ formula, as before. But the asymptotic wavefunctions (spinors for the Dirac theory) are not. But they are always the same for all diagrams contributing to a given process, since the external states are fixed. We can therefore always write:

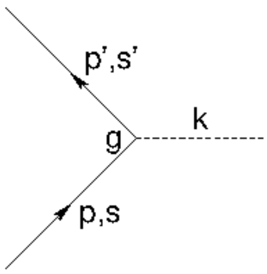
$$\mathcal{M} = \bar{u}^{s_1'}(p_1') \ldots \bar{u}^{s_m'}(p_m') \mathcal{M}^{s_1' \ldots s_m' ; s_1 \ldots s_m}(p_1', \ldots p_m'; p_1, \ldots p_n) u^{s_1}(p_1) \ldots u^{s_m}(p_m) \,, \tag{19.2}$$

with an obvious generalization if we have several kinds of external (anti-)particles. Here $\mathcal{M}^{s_1'\cdots}(p_1',\ldots)$ does not depend on the external wavefunctions.

We now must consider what kinds of interaction terms one can introduce into the Dirac Lagrangian. The simplest coupling we can have is the *Yukawa coupling* to a neutral spin-0 Klein-Gordon boson ϕ. The Lagrangian density for this is:

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_{KG} + \mathcal{L}_I = \overline{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + g\bar{\psi}\psi\phi\,. \tag{19.3}$$

Note that the g in this formula is not identical to the g in the ϕ^3 theory in *Schwartz* sec. 7.4, it does not even have the same physical dimension. This gives rise to a single vertex:



$$= \quad g\delta_{ss'}\,, \tag{19.4}$$

where, as is usual, we have not explicitly written the momentum-conserving delta-function. Note that the condition $s' = s$, which we would have obtained if we had actually carried out the detailed calculations, but which is obvious from spin conservation, since the Klein-Gordon boson is spinless.

We may then calculate the lowest order tree diagram for fermion-fermion scattering. One could think of the fermions as protons, the boson as a π^0 particle, similar to Yukawa's original application. Following sec. 7.4 in *Schwartz*, we have three possible diagrams, the s-channel, the t-channel and the u-channel. But in the present case, the s-channel does not contribute, because there is no vertex with two incoming fermions, or two outgoing ones, for that matter. But the t-channel and the u-channels contribute. Note that unless we write out the spinor indices explicitly, we have to be carful and place the spinors next to the appropriate gamma-matrices, so that the standard rules of matrix manipulations applies. With $p_1 = p$, $p_2 = p'$, $p_3 = q = p - k$, $p_4 = q' = p' + k$, so $k = p - q$, we have for the t-channel.

$$\mathrm{i}\mathcal{M}_t = (\mathrm{i}g)^2\bar{u}^s(q)u^s(p)\frac{\mathrm{i}}{(p-q)^2 - \mu^2 + \mathrm{i}\epsilon}\bar{u}^{s'}(q')u^{s'}(p')\,. \tag{19.5}$$

Going to the center of mass frame, we have $p = p_0 = (\omega_p^0, \mathbf{p})$ and $p' = \bar{p}_0 = (\omega_p^0, -\mathbf{p})$, and similarly for q and q'. We could no evaluate $\bar{u}^s()u^s(p)$ from the formula of Lecture Notes 16, but the calculation is lengthy and the result is not terribly rewarding. But in the non-relativistic limit, we have from eq. 16.8:

$$u^s(p) \longrightarrow \sqrt{m}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix} \qquad \Longrightarrow \qquad \bar{u}^s(p)u^{s'}(p') \longrightarrow 2m\delta_{ss'}\,. \tag{19.6}$$

In this limit we also have $p^0 \approx q^0 \approx m$, so $(p-q)^2 \to -(\mathbf{p}-\mathbf{q})^2$. Thus we find:

$$\mathrm{i}\mathcal{M}_t \quad \longrightarrow \quad \frac{\mathrm{i}4mg^2}{(\mathbf{p}-\mathbf{q})^2+\mu^2} \,. \tag{19.7}$$

We see that in this limit $\mathcal{M}$ is the same as for the scattering of scalar particles, if we rescale the $4mg^2 \to g^2$. But in the fermionic case we must also add the u-channel result, which is obtained by interchanging $(q,s) \leftrightarrow (q',s')$ in eq. (19.5). In addition, because of the operator orders of the final states are different in the u-channel, there is an extra minus sign in $\mathcal{M}_u$. The cross section then follows from as in *Schwartz* eq. (7.92):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(pp \to pp) = \frac{1}{64\pi^2 E_{\mathrm{CM}}^2} \left|\mathcal{M}_t + \mathcal{M}_u\right|^2 .$$

If the incoming particles are unpolarized, this crosss-section must be averaged over the incoming spin states s and s'. For processes where spin flips are allowed, but not det4ected, one must aslo sum over all spin final states.

Since $\mathcal{M}_t$ to the lowest order corresponds to amplitude of first order perturbation theory, which by Fermi's golden rule is proportional to the Fourier transform of the interaction potential, we see that eq. (19.7) reproduce the Yukawa potential:

$$V(r) \propto \int \mathrm{d}^3\mathbf{k}\mathcal{M}_t(\mathbf{k}) \propto \frac{e^{-r/\mu}}{r} \,.$$

At this point, it is worth pointing out the similarity between the fermion mass term and the coupling term in eq. (19.3). The former follows from the latter by the substitution $g\phi \to m$. This is the foundation of *Higg's mechanism* for generating fermion masses, *e.g.* for quarks and leptons in the Standard Model. It is based on the observation that classical solutions of the Euler–Lagrange equations remain solutions of the field Heisenberg equations after quantization, unless an anomaly (in the technical sense) appears. Thus if the classical theory has a non-trivial classical solution of lower energy than the trivial one, $\phi = 0$ for a single scalar field, so has the quantum theory. The simplest example is the simple anharmonic oscillator, with a Lagrangian density ($\lambda > 0$):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{24}\phi^4 \,. \tag{19.8}$$

Note the sign of the ϕ^2-term! The corresponding Hamiltonian density is:

$$\mathcal{H} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \mathcal{V}(\phi) \,, \qquad \mathcal{V}(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{24}\phi^4 \,, \tag{19.9}$$

If the ϕ-field has more than one component, say n, and $\mathcal{V} = -\frac{1}{2}\boldsymbol{\phi}^2 + \frac{\lambda}{24}\boldsymbol{\phi}^4$, with $\boldsymbol{\phi}^2 = \sum_{a=1}^n \phi_i^2$, this is called a *Mexican hat* potential. In this case $\mathcal{L}$ has the n-dimensional rotation group, $\mathrm{O}(n)$, as an internal symmetry group. The Euler-Lagrange equation derived from eq. (19.8) is:

$$\Box\phi = \mathcal{V}'(\phi) = -\mu^2\phi + \frac{\lambda}{6}\phi^3 \,. \tag{19.10}$$

This still has the constant solution $\phi = 0$ with a vanishing classical energy density, $\mathcal{H}(0) = 0$. But there are two more constant solutions, with a *lower* energy density:

$$\phi = \phi_0 = \pm\sqrt{\frac{6\mu^2}{\lambda}}\,, \qquad \mathcal{H}(\phi_0) = -\frac{3}{2}\frac{\mu^4}{\lambda} < 0\,. \tag{19.11}$$

Thus these two states are possible classical ground states of the system, while the solution with $\phi = 0$ is actually a local maximum of the action. This will general remain so also after quantization, if the difference in energy density is not very small, since the quantum fluctuations only raise the energy of any state. Note that only classical quantities are involved in these solutions, so this vacuum energy density is independent of $\hbar$. Either of these states are a possible ground state, but one cannot predict which. Note that there is no *Higgs boson* involved in this description, the solution is essentially classical.

Picking one of the vacuum solutions, perturbation theory is then based on one of these states as the ground state, and developed as before in terms of a shifted field, $\chi = \psi - \psi_0$. Expressing $\mathcal{L}$ in terms of χ one then quantize the quadratic part as a normal Klein-Gordon field. The ensuing bosons are the Higgs bosons. The remaining terms give raise to a perturbation series and Feynman rules, as before. Coupling such a scalar theory to Dirac fermions with a Yukawa coupling then simply yields a theory with a modified effective fermion mass $m \to m \pm g\phi_0$, which works even if $m = 0$. There is no problem with $m < 0$ in the Dirac theory, the sign in the mass term can be removed by interchanging the interpretation of the upper and lower components of the Dirac spinor.

Note that calculating any finite number of Feynmann diagrams will not give any hint of the existence of the other energy minimum, at $-\phi_0$, of the theory. This is an example showing that a quantum field theory may have solutions which do not appear in a perturbative treatment. It may happen, however, that the summation of a suitable infinite class of diagrams may reveal them. This is similar to what we saw in Lecture Note 10 that summing an infinite number of diagarams often gives rise to a renormalization of the mass.

The Higgs mechanism is how the quarks and leptons acquire mass in the Standard Model. The reason is that for the underlying non-abelian gauge theory to be renormalizable, all basic fermions must be massless. We also note that $\mathcal{L}$ in eq. 19.8 has an obvious symmetry, $\phi \leftrightarrow -\phi$, which becomes an $\mathrm{O}(n)$ symmetry if ϕ has several components. This symmetry is broken by the vacuum solutions ϕ_0. This situation, when the ground state of a theory is broken by the ground state solutions of the theory, so they have less symmetry than the Lagrangian, is called *spontaneous symmetry breaking*.