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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 18, 5.4 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Further symmetries of the Dirac field

We have already encountered 3 sets of independent Dirac matrices which play an important role in the Dirac theory, consisting of $\mathbb{1}_4, \gamma^\mu$ and $S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\sigma^{\mu\nu}$. These transform as a scalar, vector and an antisymmetric tensor under (proper) Lorentz transformations, respectively. But there are two more such sets. One only contains the scalar:

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\,. \tag{18.1}$$

One easily verifies that

$$\gamma^{5\dagger} = \gamma^5\,, \qquad \left(\gamma^5\right)^2 = \mathbb{1}_4\,, \qquad \{\gamma^5, \gamma^\mu\} = 0\,. \tag{18.2}$$

From the last equation it follows that $[\gamma^5, S^{\mu\nu}] = 0$, which means that γ^5 commutes with all Lorentz transformation operators, proving that the Dirac representation is reducible, since states with different eigenvalues of γ^5 transform without mixing. But from $\left(\gamma^5\right)^2 = \mathbb{1}_4$ it follows that the eigenvalues are ± 1. Indeed, in the Weyl basis γ^5 is diagonal:

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}\,. \tag{18.3}$$

Indeed, from the representation of eq. (16.1):

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}\,, \tag{16.1}$$

we see that the left-handed and right-handed components of the Dirac-spinor are eigenfunctions of γ^5, with eigenvalues -1 and $+1$, respectively.

The last set of independent Dirac matrices contains the matrices $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$. One can show that by using $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$, all other products of gamma matrices can be reduced to a member of one of these sets. Altogether we then have 16 basic matrices in the Dirac algebra, 2 scalars with together 8 matrices and 1 antisymmetric tensor, with 6 independent components.

From the five sets of Dirac matrices, we can construct local density operators which transform simply under Lorentz transformation by squeezing them between $\bar\psi$ and ψ, which together transform as a Lorentz scalar, as we know. In addition to the scalar $\bar\psi\psi$ the most important are the two vector current densities:

$$j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)\,, \qquad j^{5\mu}(x) = \bar\psi(x)\gamma^\mu\gamma^5\psi(x)\,, \tag{18.3}$$

It is easily shown that j^μ is conserved if $\psi(x)$ satisfies the Dirac equation:

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi = (\mathrm{i}m\bar{\psi})\psi + \bar{\psi}(-\mathrm{i}m\psi) = 0\,. \tag{18.4}$$

When we couple this to the electromagnetic field $\pm ej^\mu$ will become the electric current density of a field of particles of charges $\pm e$. We also note that $\psi^\dagger\psi = j^0$, so this quantity will become the charge density of the field, not a probability density, as one might expect in analogy with non-relativistic quantum mechanics. It is easily verified from *Schwartz* eq. (3.23) that j^μ is the Noether current for the internal symmetry of the Dirac Lagrangian:

$$\psi(x) \to \psi'(x) = e^{\mathrm{i}\alpha}\psi(x)\,, \tag{18.5}$$

where α is an arbitrary constant.

The other current density, $j^{5\mu}$, is called the **axial current**. This has a divergence:

$$\begin{aligned}\partial_\mu ju\mu &= (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi \\ &= (\mathrm{i}m\bar{\psi})\gamma^5\psi - \bar{\psi}(-\mathrm{i}m\psi) = 2\mathrm{i}m\,\bar{\psi}\gamma^5\psi\,,\end{aligned} \tag{18.6}$$

where we have used the last of eqs. (18.2). Thus $j^{5\mu}$ is conserved only if $m = 0$. In cases where the mass can be (almost) neglected, like in neutrino physics, it is often convenient the linear combinations:

$$j^\mu_L = \bar{\psi}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi\,, \qquad j^\mu_R = \bar{\psi}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi\,, \tag{18.7}$$

These are (proportional to) the current densities of left-handed and right-handed particles, respectively, and are separately conserved in the massless limit. In this limit, also $\gamma^{5\mu}$ is a Noether current, for the internal transformation:

$$\psi(x) \to \psi'(x) = e^{\mathrm{i}\alpha\gamma^5}\psi(x)\,, \tag{18.8}$$

Under such a transformation, the Lagrangian density becomes, from eq. (15.19):

$$\begin{aligned}\mathcal{L}[\psi', \partial_\mu\psi'] &= \bar{\psi}'^\dagger(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi' = \mathrm{i}\psi^\dagger e^{-\mathrm{i}\alpha\gamma^5}\gamma^0\gamma^\mu e^{\mathrm{i}\alpha\gamma^5}\partial_\mu\psi - m\psi^\dagger e^{-\mathrm{i}\alpha\gamma^5}\gamma^0 e^{\mathrm{i}\alpha\gamma^5}\psi \\ &= \mathcal{L}[\psi, \partial_\mu\psi] - m\bar{\psi}\left(e^{2\mathrm{i}\alpha\gamma^5} - 1\right)\psi\,.\end{aligned}$$

where we have used that from eq. (18.2) it follows that:

$$e^{-\mathrm{i}\alpha\gamma^5}\gamma^\mu = \gamma^\mu e^{\mathrm{i}\alpha\gamma^5}\,, \tag{18.9}$$

as we see by expanding the exponential function and using ${\gamma^5}^2 = \mathbb{1}_4$. Thus the chiral transformation of eq. (18.8) is a symmetry of the classical Lagrangian for $m = 0$. It is worth noting, though, that it is one of the relatively rare cases where quantum effects actually break the classical symmetry, so $j^{5\mu}$ is, after all, *not* conserved in the quantum theory. Such a phenomenon is called an *anomaly*, in the present case the *axial*, or *Adler–Bell–Jackiw*, anomaly.

For the discrete symmetries, parity, P, time reversal, T, and particle-antiparticle, or *charge conjugation*, C, see *Schwartz* sec. 11.4-6.