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Lecture notes for FYS610 Many particle Quantum Mechanics

Note 17, 4.4 2017

Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

Quantization of the Dirac field

In Lecture Notes 6 and 7 we have discussed the quantization of Fermion fields, and we know from introductory quantum mechanics courses that spin-$\frac{1}{2}$particles must be quantized as Fermions. Still, it is instructive to see *why* this is necessary. From the Dirac Lagrangian,

$$\mathcal{L}[\psi, \partial_\mu \psi] = \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi\,, \tag{15.19}$$

one finds the conjugate field momentum density $\pi^r(x) = \mathrm{i}\psi^{r\,\dagger}(x)$, and the field Hamiltonian:

$$H = \int \mathrm{d}^3x\,[\pi(x)\psi(x) - \mathcal{L}] = \int \mathrm{d}^3x\,\bar{\psi}[-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m]\psi = \int \mathrm{d}^3x\,\psi^\dagger H_D \psi\,, \tag{17.1}$$

where H_D is the one-particle Dirac Hamiltonian:

$$H_D = -\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\beta = \gamma^0(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,. \tag{15.6}$$

In order to obtain the creation and annihilation operators, we expand $\psi(x)$ in a complete set of solution of our one-particle equation, which now is the Dirac equation, *i.e.* in terms of plane waves times basis spinors $u^r(p), v'^s(p)$. Working in the Schrödinger picture, where the operators for free particles are time independent, it suffices to make the expansion at $t_0 = 0$, and we can disregard the sign of the energy phase, *i.e.* the dependence of the sign of p^0, so $v'^r(\mathbf{p}) = v^r(-\mathbf{p})$. Thus we expand:

$$\psi^S(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \sum_{r=1}^{2} \left(a^r_{\mathbf{p}} u^r(\mathbf{p}) + b^r_{-\mathbf{p}} v^r(-\mathbf{p})\right)\,, \tag{17.2}$$

where $a^s_{\mathbf{p}}$ and $b^s_{\mathbf{p}}$ are Hermitean operators. We now *postulate* that they satisfy the standard commutation relations for Bosonic creation and annihilation operators:

$$\begin{aligned} &[a^r_{\mathbf{p}}, a^s_{\mathbf{q}}{}^\dagger] = [b^r_{\mathbf{p}}, b^s_{\mathbf{q}}{}^\dagger] = (2\pi)^3\delta^3(\mathbf{x}-\mathbf{y})\,\delta_{rs}\,, \qquad [a^r_{\mathbf{p}}, b^s_{\mathbf{q}}{}^\dagger] = [b^r_{\mathbf{p}}, a^s_{\mathbf{q}}{}^\dagger] = 0\,, \\ &[a^r_{\mathbf{p}}, a^s_{\mathbf{q}}] = [b^r_{\mathbf{p}}, b^s_{\mathbf{q}}] = [a^r_{\mathbf{p}}{}^\dagger, a^s_{\mathbf{q}}{}^\dagger] = [b^r_{\mathbf{p}}{}^\dagger, b^s_{\mathbf{q}}{}^\dagger] = 0\,. \end{aligned} \tag{17.3 ?}$$

We can then calculate the basic field commutator essentially as done for the Klein–Gordon equation in *Schwartz* eqs. (2.92-3):

$$\begin{aligned}
&\frac{1}{\mathrm{i}}[\psi^r(\mathbf{x}),\pi^s(\mathbf{y})] = [\psi^r(\mathbf{x}),\psi^{s\dagger}(\mathbf{y})] \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\frac{1}{\sqrt{4\omega_p\omega_q}}e^{\mathrm{i}(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&\times \sum_{r,s}\left([a^r_{\mathbf{p}},a^{s\,\dagger}_{\mathbf{q}}]u^r(\mathbf{p})\bar{u}^s(\mathbf{p}) + [b^r_{-\mathbf{p}},b^s_{-\mathbf{q}}{}^\dagger]v^r(-\mathbf{p})\bar{v}^s(-\mathbf{p})-\right)\gamma^0 \qquad (17.4\,?) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_p}e^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left[(\gamma^0\omega_p - \boldsymbol{\gamma}\cdot\mathbf{p}+m\mathbb{1}_4)+(\gamma^0\omega_p+\boldsymbol{\gamma}\cdot\mathbf{p}-m\mathbb{1}_4)\right]\gamma^0 \\
&= \delta^3(\mathbf{x}-\mathbf{y})\,\mathbb{1}_4\,.
\end{aligned}$$

where we have used $u^{s\dagger}(\mathbf{q}) = \bar{u}^s(\mathbf{q})\gamma^0$ and the spin sums in eqs. (16.19). This is the result expected from canonical quantization.

In a similar manner we find the mode sum for the field Hamiltonian, using eqs. (16.18-9) (we skip the details):

$$H = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\sum_{s=1}^{2}\omega_p\left(N^{s+}_{\mathbf{p}} - N^{s-}_{\mathbf{p}}\right)\,, \qquad N^{s+}_{\mathbf{p}} = a^{s\,\dagger}_{\mathbf{p}}a^s_{\mathbf{p}}\,, \quad N^{s-}_{\mathbf{p}} = b^{s\,\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}\,. \qquad (17.5\,??)$$

But there is something terribly wrong with this expression, because N^{s+} and N^{s-} are positive operators:

$$\langle\psi|N^{-s}_{\mathbf{p}}|\,\psi\rangle = \langle\psi|b^{s\,\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}|\,\psi\rangle = |b^s_{\mathbf{p}}|\,\psi\,\rangle|^2 \geq 0\,,$$

and the same for $N^{s+}_{\mathbf{p}}$. Indeed, $a^s_{\mathbf{p}}$ and $b^s_{\mathbf{p}}$ satisfy the harmonic oscillator algebra, see Lecture Notes 4, and hence $N^{s+}_{\mathbf{p}}$ and $N^{s-}_{\mathbf{p}}$ have eigenvalues n^+ and n^-, respectively, where $n^\pm$ are integers. Thus the Hamiltonian of eq. (17.5 ??) have positive *and negative* eigenvalues of arbitrarily large absolute magnitudes. This might not be a problem in a free particle theory, we may just introduce a rule stating that $N^{s+} - N^{s-} \geq 0$ for all physically accessible states. Since $[H, N^{s\pm}_{\mathbf{p}} = 0]$, this constraint is conserved by the dynamics. But this no longer works for an interacting theory. Then we would get transitions to arbitrary negative energy states, and we would be able to extract infinitely much energy from any spin-$\frac{1}{2}$system, which is, of course totally unsatisfactory. We actually have got back the original problem of the Klein–Gordon theory.

Dirac solved this problem in an imaginative way. At the time, only three elementary particles were known, the electron, the proton and the photon. Of these, photons obey Bose-Einstein statistics, and make no problems. But both electrons and protons were known to be Fermions, and hence obey the Pauli principle. To implement this, Dirac invented the particle-hole formalism which we discussed in Lecture Note 7, and then introduced what we now know as the *Fermi level* as the vacuum state $|\,0\,\rangle$ defined by $N^{+s}_{\mathbf{p}}|\,0\,\rangle = 0$ for all $\mathbf{p}$ and s. He then postulated that all negative energy states are filled in this state, which became known as the *Dirac sea*. The rest follows our discussion. He predicted that excitations of the Dirac sea would appear as pairs of negatively and positively charged particles, the positive particles being holes in the Dirac sea. And when Carl Anderson discovered the positron on 2 August 1932, his luck was made. Actually, positrons had been seen in several earlier experiments, but these were not correctly interpreted.

Although one can indeed construct the correctly quantized Dirac theory from the approach of Dirac, following closely the steps of Lecture Notes 7, we shall just represent the modern approach, and *postulate* the basic commutation relations:

$$\begin{aligned} \{a^r_{\mathbf{p}}, a^{s\,\dagger}_{\mathbf{q}}\} &= \{b^r_{\mathbf{p}}, b^{s\,\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^3(\mathbf{x}-\mathbf{y})\delta_{rs}\,, \\ \{a^r_{\mathbf{p}}, a^s_{\mathbf{q}}\} &= \{b^r_{\mathbf{p}}, b^s_{\mathbf{q}}\} = \{a^{r\,\dagger}_{\mathbf{p}}, a^{s\,\dagger}_{\mathbf{q}}\} = \{b^{r\,\dagger}_{\mathbf{p}}, b^{s\,\dagger}_{\mathbf{q}}\} = 0\,. \end{aligned} \tag{17.3}$$

Furthermore, as we already know, the *annihilation* of a negative energy particle in the Dirac sea is the same as *creating* a hole, which we shall now identify as an *antiparticle*. WE could have done this by introducing new creation and annihilation operators for the negative energy particles, as we did in eqs. (7.7), but we save some rewriting by noting that if we substitute $b^s_{\mathbf{q}} \leftrightarrow b^{s\,\dagger}_{\mathbf{q}}$ the anticommutator algebra in eq. (17.3) is unchanged. After this switch, we define the vacuum state as:

$$a^r_{\mathbf{p}}|\,0\,\rangle = 0; \qquad b^r_{\mathbf{p}}|\,0\,\rangle = 0\,. \tag{17.4}$$

and the one-particle states by:

$$|\,\mathbf{p}, r\,\rangle = \sqrt{2\omega_p}\,a^{r\,\dagger}_{\mathbf{p}}|\,0\,\rangle = 0\,, \qquad |\,\tilde{\mathbf{p}}, r\,\rangle = \sqrt{2\omega_p}\,b^{r\,\dagger}_{\mathbf{p}}|\,0\,\rangle = 0\,, \tag{17.5}$$

where the tilde over the $\mathbf{p}$ denotes an antiparticle. These states have the same normalization as for the spin-0 states:

$$\langle \mathbf{p}, r|\mathbf{q}, s\rangle = 2\omega_p(2\pi)^3\delta(\mathbf{p}-\mathbf{q})\,, \qquad \langle \tilde{\mathbf{p}}, r|\mathbf{q}, s\rangle = 0\,. \tag{17.6}$$

We also note that from eq. (17.3) it follows that $\left(a^r_{\mathbf{p}}\right)^2 = \left(b^r_{\mathbf{p}}\right)^2 = 0$, which implements the Pauli principle:

$$a^r_{\mathbf{p}}|\,\mathbf{p}, r\,\rangle = \sqrt{2\omega_p}\left(a^r_{\mathbf{p}}\right)^2|\,0\,\rangle = 0\,, \qquad b^r_{\mathbf{p}}|\,\tilde{\mathbf{p}}, r\,\rangle = \sqrt{2\omega_p}\left(b^r_{\mathbf{p}}\right)^2|\,0\,\rangle = 0\,. \tag{17.7}$$

We can now write the field operator in the Heisenberg picture. It turns out to be slightly more convenient to work with $\bar{\psi} = \psi^\dagger\gamma^0$. Changing the integral over $\mathbf{p} \to -\mathbf{p}$ in the antiparticle part, we find:

$$\begin{aligned} \psi(x) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{r=1}^{2}\left(a^r_{\mathbf{p}}u^r(p)\,e^{-\mathrm{i}p\cdot x} + b^{r\,\dagger}_{\mathbf{p}}v^r(p)\,e^{+\mathrm{i}p\cdot x}\right)\,, \\ \bar{\psi}(x) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{r=1}^{2}\left(b^r_{\mathbf{p}}\bar{v}^r(p)\,e^{-\mathrm{i}p\cdot x} + a^{r\,\dagger}_{\mathbf{p}}\bar{u}^r(p)\,e^{+\mathrm{i}p\cdot x}\right)\,. \end{aligned} \tag{17.2$'$}$$

The most fundamental change to the theory is that the canonical commutation relations are replaced by *anticommutator* relations for fermions. They are calculated along the same lines as before:

$$\{\psi^a(\mathbf{x}), \psi^{b\dagger}(\mathbf{y})\} = \delta^3(\mathbf{x}-\mathbf{y})\,\delta_{ab}\,, \qquad \{\psi^a(\mathbf{x}), \psi^b(\mathbf{y})\} = \{\psi^{a\dagger}(\mathbf{x}), \psi^{b\dagger}(\mathbf{y})\} = 0\,. \tag{17.8}$$

Here a and b runs over $1\dots 4$, *i.e.* both over particle and antiparticle spins.

We can also find the Hamiltonian:

$$H = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \sum_{s=1}^{2} \omega_p \left(N_{\mathbf{p}}^{s+} + N_{\mathbf{p}}^{s-}\right) , \qquad N_{\mathbf{p}}^{s+} = a_{\mathbf{p}}^{s\,\dagger} a_{\mathbf{p}}^{s} , \quad N_{\mathbf{p}}^{s-} = b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{p}}^{s} . \tag{17.9}$$

As shown in Lecture Notes 6, for Fermions the number operators $N^s\pm$ only take on the values $n_{\mathbf{p}}^{\pm} \in \{0, 1\}$. Hence we now have that all the eigenvalues of H are positive, and the state can be interpreted as that of a collection of an integer number of particles and antiparticles. The momentum operator can be calculated in the same way:

$$\mathbf{P} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \sum_{s=1}^{2} \mathbf{p} \left(N_{\mathbf{p}}^{s+} + N_{\mathbf{p}}^{s-}\right) . \tag{17.10}$$

It remains to find the propagators. We shall work with $\bar{\psi}$, but since $\psi^\dagger = \bar{\psi}\gamma^0$, it is trivial to find propagators involving $\psi^\dagger$, if needed. Before proceeding, we shall introduce the standard notation:

$$\not{A} = \gamma^\mu A_\mu , \qquad \not{A}\not{A} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} A_\mu A_\nu = g^{\mu\nu} A_{\mu\nu} = A^2 , \tag{17.11}$$

for any 4-vector operator A^μ where we have used eq. (15.10b) and the symmetry. We then find, using eqs. (16.19), (17.3) and (17.4):

$$\begin{aligned}
\langle 0|\psi_a(x)\bar{\psi}_b(y)|\,0\rangle &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_p} \sum_s u_a^s(p)\bar{u}_b^s(p)\, e^{-p\cdot(x-y)} \\
&= (\mathrm{i}\not{\partial} + m)_{ab} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, e^{-p\cdot(x-y)} , \\
\langle 0|\bar{\psi}_b(y)\psi_a(x)|\,0\rangle &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_p} \sum_s v_a^s(p)\bar{v}_b^s(p)\, e^{-p\cdot(y-x)} \\
&= -(\mathrm{i}\not{\partial} + m)_{ab} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, e^{-p\cdot(y-x)} .
\end{aligned} \tag{17.12}$$

Here $(a, b \in \{1\dots 4\})$ are arbitrary spinor indices, which may or may not coincide with the summation indices s.

Just like for the Klein–Gordon equation, we can now construct various Green's functions from these vacuum expectation values. The retarded Greens function (cf. Lecture Note 9) can be defined as:

$$S_R^{ab}(x-y) = \theta(x^0 - y^0)\langle 0|\{\psi_a(x), \bar{\psi}_b(y)\}|\,0\rangle . \tag{17.13}$$

This is indeed a Green's function for the Dirac equation:

$$\begin{aligned}
&(\mathrm{i}\not{\partial}_x - m) S_R(x-y) \\
&= \mathrm{i}\gamma^0\delta(x^0 - y^0)\langle 0|\{\psi_a(x), \bar{\psi}_b(y)\}|\,0\rangle + \theta(x^0 - y^0)\langle 0|\{[(\mathrm{i}\not{\partial} - m)\psi(x)], \bar{\psi}(y)\}|\,0\rangle \\
&= \mathrm{i}\gamma^0\delta(x^0 - y^0)\langle 0|\{\psi_a(x^0, \mathbf{x}), {\psi_b}^\dagger(x^0, \mathbf{y})\gamma^0\}|\,0\rangle + 0 \\
&= \mathrm{i}\gamma^0\delta(x^0 - y^0)\,\delta^3(\mathbf{x} - \mathbf{y})\gamma^0 = \mathrm{i}\delta^4(x-y)\,\mathbb{1}_4 .
\end{aligned}$$

Here we have used that $\psi(x)$ is a solution of the Dirac equation and the basic anti-commutator, eq. (17.8). It is clear that S_R only propagate in the forward light cone, $x^0 > y^0$, and hence is the retarded Green's function. Furthermore, a now straightforward calculation shows that:

$$S_R(x-y) = (\mathrm{i}\partial\!\!\!/ + m) D_R(x-y)\,, \tag{17.14a}$$

where $D_R(x)$ is the retarded Klein–Gordon Green's function of eq. (9.2). Thus, the momentum space Green's function is simply:

$$\tilde{S}_R(p) = (p\!\!\!/ + m)\tilde{D}_R(p) = \mathrm{i}\frac{p\!\!\!/ + m}{p^2 - m^2} = \frac{\mathrm{i}}{p\!\!\!/ - m}\,. \tag{17.14b}$$

Here it is understood that the poles in the complex p^0 planes have a small negative imaginary part, which reflects the $\theta(x^0 - y^0)$ in eq. (17.13).

The Feynman propagator follows similarly, and one finds

$$S_F(x-y) = \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|\,0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\bar{\psi}(y)\psi(x)|\,0\rangle & \text{for } x^0 < y^0 \end{cases} = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|\,0\rangle\,. \tag{17.15}$$

where the Fermionic time-ordered product includes an extra minus sign whenever two fermionic operators are commuted. This rule is applied recursively for products of several fermionic operators. Proceeding exactly as in the Bosonic case, one finds that the Feynman propagator in momentum space is:

$$\tilde{S}_F(p) = (p\!\!\!/ + m)\tilde{D}_R(p) = \mathrm{i}\frac{p\!\!\!/ + m}{p^2 - m^2 + \mathrm{i}\epsilon} = \frac{\mathrm{i}}{p\!\!\!/ - m + \mathrm{i}\epsilon}\,, \tag{17.16}$$

with poles at $p^0 = \pm(\omega_p + \mathrm{i}\epsilon)$.