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Lecture notes for FYS610 Many particle Quantum Mechanics

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Additions and comments to *Quantum Field Theory and the Standard Model* by Matthew D. Schwartz (2014)

The Dirac equation

In 1939 Eugene P. Wigner was able to find and classify the irreducible representation of the Poincaré group. To do this, he had to identify a set of commuting Hermitean operators constructed from the generators of the group, such that all the group generators transform the eigenvectors of these operators among themselves. Such operators are called Casimir Operators. In contrast to the rotation and Euclidean groups, where the Casimir operators can be chosen as $\mathbf{P}^2$ and $\mathbf{J}^2$, this turned out to require representations with an infinite number of basis vectors, so all the unitary representations of the Poincare group are infinitely dimensional, except for the trivial (scalar) representation, if we disregard the discrete transformations of space inversion and time reversals for the moment.

To do this, Wigner first had to find a suitable set of operators which commute with all the generators of the group. One simple such operator is $M = P^2 = {P^0}^2 + \mathbf{P}^2$. Thus the eigenvalues of this operator, the mass squared, m^2, can be used to classify the irreducible representations. Another fairly simple operator is the "energy sign" operator, $\mathcal{E} = P^0/|P^0|$ if $P^2 > 0$. In addition a Casimir operator characterizing the spin of the system is clearly needed, but $\mathbf{J}^2$ does not work, since from eq. (14.16) one has:

$$\begin{aligned}[\mathbf{J}^2, K^i] &= J^j[J^j, K^i] + [J^j, K^i]J^j = \mathrm{i}\epsilon_{jik}\left(J^jK^k + K^kJ^j\right) \\ &= -\mathrm{i}\epsilon_{ijk}\left(2J^jK^k - \mathrm{i}\epsilon_{kjl}K^l\right) = 2\left(K^i - \mathrm{i}(\mathbf{J}\times\mathbf{K})^i\right) \neq 0\,.\end{aligned} \tag{15.1}$$

To find the missing operator(s), we use that a Poincaré transformation is represented by an operator of the form:

$$U(\omega^{\mu\nu}, a) = U(\boldsymbol{\beta}, \boldsymbol{\phi}, a) = e^{-\mathrm{i}a_\mu P^\mu}\, e^{\mathrm{i}\omega_{\mu\nu}M^{\mu\nu}}\,. \tag{14.22}$$

Here $M^{\mu\nu}$ are the six generators of the Lorentz group, parameterized by $\omega^{\mu\nu}$.

$$M^{0k} = -M^{k0} = K^k\,, \quad M^{kl} = \epsilon_{klm}J^m\,; \quad \omega^{0k} = -\omega^{k0} = \frac{1}{2}\beta^k\,, \quad \omega^{kl} = \frac{1}{2}\epsilon_{klm}\phi^m\,. \tag{14.20 − 21}$$

It can be verified that $M^{\mu\nu}$ and $\omega^{\mu\nu}$ indeed transform as tensors under the Lorentz group, so $\omega_{\mu\nu}M^{\mu\nu}$ is a scalar, which is useful when transforming the group parameters $\omega^{\mu\nu}$ between different coordinate systems. In order for eq. (14.22) to be a unitary transformation, the $M^{\mu\nu}$ must be Hermitean. Wigner exploited that it is possible to introduce a covariant spin-vector, called the Pauli–Lubanski, or Bargmann–Wigner, spin vector, which is a genuine 4-vector:

$$\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}P^\nu M^{\alpha\beta}\,, \qquad \omega^\mu = [\mathbf{P}\cdot\mathbf{J}, P^0\mathbf{J} + \mathbf{P}\times\mathbf{K}]\,. \tag{15.2}$$

Here $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor of the Lorentz group, normalized to $\epsilon_{0123} = 1 = -\epsilon^{0123}$. An explicit calculation shows that the square of ω_μ commutes with all the generators of the Poincaré group. Since this is a scalar, it can be evaluated in the rest-frame, where $P^\mu = [m, 0, 0, 0]$. If $m > 0$ we find:

$$W = -\omega_\mu\omega^\mu = m^2\mathbf{J}^2 = m^2 s(s+1)\,, \tag{15.3}$$

where we have used that in the rest frame we can use the Lie algebra of the rotation group to find the irreducible representations.

Wigner then showed that the operators M, W and $\mathcal{E}$ suffices to characterize *all* unitary irreducible representations of the Poincare group. All these representations, except for one, is infinite-dimensional, because each of them contain base vectors with arbitrary continuous momentum labels. They fall into 6 distinct classes as follows:

I $M = m^2 > 0$, $\mathcal{E} = \epsilon = 1$. This class describes massive particles. They can further be classified by the spin, $W = m^2 s(s+1)$, with s integer for true representation, half integer for projective representations, a property inherited from the rotation group. Furthermore, there are $2s + 1$ possible eigenvalues for J^3.

II $M = m^2 > 0$, $\mathcal{E} = \epsilon - 1$. This class is isomorphous to class I, and can be mapped to it by an additional discrete transformation. It may be used to describe antiparticles.

III $M = m^2 = 0$, $\mathcal{E} = \epsilon = 1$, $P^\mu \neq 0$. This class describes massless particles. $W = 0$, but is not needed. The irreducible representations have only 2 spin states, with eigenvalues of $J^3 = \pm s$, except for $s = 0$, which has only one spin state. Again an integer s yields a true representation, a half integer s a projective representations.

IV $M = m^2 = 0$, $\mathcal{E} = \epsilon = -1$, $P^\mu \neq 0$. Isomorphous to class III (cf. class II).

V $P^\mu = 0$. This class describes vacuum. It only has a trivial (scalar) representation.

VI $M = m^2 < 0$. This class describes tachyons. This class describes particles which always move faster than light. It does not seem to be relevant for physics.

One can now continue to construct the representation and the group generators. We shall not do this, because many different equivalent ways of writing them is possible, corresponding to the possibility of making unitary transformations on the basis states. However, for the classes I–IV one can always choose a representation so that $\mathbf{J}$ retains its well-known form $\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{X}\times\mathbf{P} + \mathbf{S}$, where $\mathbf{X} = \mathrm{i}\boldsymbol{\nabla}_{\mathbf{p}}$. One fairly simple form for $\mathbf{K}$ is Shirkov's form:

$$\mathbf{K} = P^0\,\mathbf{X} - \mathbf{P}\,t + \epsilon\frac{\mathbf{P}\times\mathbf{S}}{\omega_P + m} \tag{15.4}$$

The momentum derivatives appearing through X in these expressions show that these operators connect states of different momenta, which is what makes the representation infinite-dimensional. The first part of eq. (15.4) is analogous to $\mathbf{L} = \mathbf{X}\times\mathbf{P}$. Indeed, if we extend the canonical quantization procedure $p^i = \mathrm{i}\partial^i = -\mathrm{i}\partial_i = -\mathrm{i}\partial/\partial x^i$, to the time domain, with $\mathrm{i}\partial_0 = P^0$, which is nothing but the Schrödinger equation when applied to a wavefunction, we see that the spatial part of $M^{\mu\nu}$ takes the form

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \,. \tag{15.5}$$

which indeed is the correct generator for Lorentz transformations of classical space time functions.

These unitary representattions of the Poincaré group can be used to construct theories of free fields for any spin. However, such explicit constructions have proven of limited usefulness, because it has proven impossible extend this to *renormalizable* quantum field theories of spin higher than 1 (or 3/2 if we include supersymmetric theories). Of the useful theories, we have already discussed the spin-0 Klein–Gordon theory in some detail, including the massless case, while the massless spin-1 theory is just the quantized Maxwell theory. The massive spin-1 theory is discussed in some detail in *Schwartz*, ch. 8. Also the important spin-$\frac{1}{2}$ Dirac theory had was found before Wigner's analysis of the Poincaré group. Thus, although the form of these theories can be deduced from group theory, and this is indeed more or less what *Schwartz* does, it is in practice more efficient to guess the form, and verify with group theory that it really is a relativistic theory of well defined spin.

This is more or less what Paul A. M. Dirac did in 1928. He was really searching for a relativistic wave equation which, unlike the Klein–Gordon equation, does not have negative energy solutions. He realised, along with several others, that although this poses no problem for a free theory, the introduction of interactions would lead to predictions of transitions between positive and negative energy states, which appeared to destroy the stability of the theory, and seemed to make no sense at the time. Dirac therefore tried to find a theory with only a first order time derivative, just like the non-relativistic Schrödinger equation. To be relativistically invariant, it should then also be linear in the momenta. He realized that this could be obtained by allowing for wave-functions which had several components, *i.e.* are vectors, ψ, in some finite-dimensional internal vector space, describing some internal degree of freedom. He thus found the **Dirac equation**, written in Hamiltonian form as:

$$\mathrm{i}\partial_t \psi = H_D \psi = (\alpha^i P^i + \beta\, m)\psi = (\frac{1}{\mathrm{i}}\alpha^i \partial_i + \beta\, m)\psi \,, \tag{15.6}$$

where α^i and β are Hermitean matrices. In order for the Klein-Gordon equation to be satisfied for each component of ψ, he required ($P^0 = H_D$):

$$\begin{aligned} {P^0}^2 \psi &= (\alpha^i P^i + \beta^m)^2 \psi \qquad (15.7) \\ &= \left[\frac{1}{2} \left(\alpha^i \alpha^j + \alpha^j \alpha^i \right) P^i P^j + \left(\alpha^i \beta + \beta \alpha^i \right) P^i + \beta^2 m^2 \right] \psi = \left[\mathbf{P}^2 + m^2 \right] \psi \,. \end{aligned}$$

Here we have symmetrized the product of α's in front of $P^i P^j$, because the antisymmetric part cancels automatically. For this equation to be satisfied for any ψ, the matrices α^i and β must satisfy the anti-commutator algebra:

$$\{\alpha^i, \alpha^j\} = \alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta_{ij} \,, \qquad \{\alpha^i, \beta\} = \alpha^i \beta + \beta \alpha^i = 0 \,, \qquad \beta^2 = 1 \,. \tag{15.8}$$

As a matter of fact, it has turned out that the Dirac equation is not a fully satisfactory *quantum mechanical* equation, although it has proven quite useful in relativistic atomic and nuclear physics. Indeed, it did not really even get rid of the negative energy solutions. However, as a field equation it has proven extremely successful, including describing leptons and quarks in the Standard Model.

Instead of the Hamiltonian form of eq. (15.6), the Dirac equation can also be written in a covariant form:

$$(\gamma^\mu P_\mu - m) = (\mathrm{i}\gamma^\mu \partial_\mu - m)\psi(x) = 0\,. \tag{15.9}$$

where the Dirac gamma-matrices are defined as:

$$\gamma^0 = \beta\,, \qquad \gamma^i = \beta\alpha^i\,, \tag{15.10a}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_n\,, \tag{15.10b}$$

with n as the number of components of ψ.

It remains to find matrices that satisfy the algebras of eqs. (15.8) or (15.10). If we restrict ourselves to three space dimensions, eq. (15.10) reduces to:

$$\{\gamma^i, \gamma^j\} = -2\delta_{ij}\mathbb{1}_n\,, \tag{15.11}$$

This algebra is satisfied by the matrices $\gamma^i = \mathrm{i}\sigma^i$ for $n = 2$, according to the anti-commutator of the Pauli matrices σ^i, see eq. (13.16). But this algebra cannot be extended by a fourth matrix for $n = 2$, there is no fourth 2×2 matrix that anti-commute with the Pauli matrices. But Dirac found a solution for $n = 4$, a solution that can be written in block form as:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{15.12}$$

This representation, which was long the most popular one, is by far not unique, it turns out that all solutions of eq (15.10b) are unitarily equivalent. Modern textbooks, including *Schwartz*, prefer another representation, the Weyl, or chiral, representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{15.13}$$

Now it turns out that for any set of matrices satisfying eqs. (15.10) (it turns out that n must be a multiple of 4), if one defines:

$$S^{\mu\nu} = \frac{\mathrm{i}}{4}\left[\gamma^\mu, \gamma^\nu\right]\,, \tag{15.14}$$

one can verify that $S^{\mu\mu}$ satisfies the Lie algebra of the Lorentz group, written in the form of eq. (14.22). In the Weyl representation, the generators for boosts and rotations are, respectively:

$$\begin{aligned} S^{0i} &= \frac{\mathrm{i}}{4}\left[\gamma^i, \gamma^0\right] = \frac{-i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} = -S^{0i\,\dagger}\,, \\ S^{ij} &= \frac{\mathrm{i}}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \frac{1}{2}\epsilon_{ijk}S^k = S^{ij\,\dagger}\,. \end{aligned} \tag{15.15}$$

WE see that S^{ij} is Hermitan, but S^{0i} is anti-Hermitean. Here $\breve{g}^k$ are the spin matrices of the Dirac theory. It is evident that this representation of the Lorentz group is reducible, since the generators are in block diagonal form. It is also clear that each block is 2-dimensional, so $2s+1=2$, or $s=\frac{1}{2}$, so the Dirac equation describes spin-$\frac{1}{2}$ particles. Finally, and this looks like a serious problem, the boost generators, S^{0i} are not Hermitean — if they were, we would have a counter example to our statement that there are no non-trivial finite-dimensional unitary representation of the Lorentz group.

Internal vectors ψ, transforming according to the spin-$\frac{1}{2}$ representation of the Lorentz group, are called spinors. Those transforming according to this doubled 4-dimensional representation are called Dirac-spinors. Under a finite Lorentz transformation, Λ, they transform as $\psi \to \psi'$, where, in agreement with eq. (11.8b):

$$\psi(\Lambda x) = D(\Lambda)\psi(x)\,, \qquad D(\Lambda) = e^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu}}\,, \tag{15.16}$$

The operators must transform with the inverse transformation matrix, cf. eq(12.1), so;

$$\gamma'^{\mu} = D^{-1}(\Lambda)\gamma^{\mu}D(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}\,, \tag{15.17}$$

which can be verified by an explicit calculation, *e.g.* using the Hausdorff–Campbell–Baker formula, eq. (14.7). This means that γ^μ is indeed a 4-vector and hence $\gamma^\mu x_\mu$ is a Lorentz scalar, and since the Dirac equation is trivially translation invariant, it is Poincaré invariant.

To find a Lagrangian for the Dirac equation, we need to form a Lorentz scalar from the spinors. Now S^{ij} is Hermitean, so $(D(R)\psi)^\dagger = \psi^\dagger D^{-1}(R)$ for a rotation, and $\psi^\dagger\psi$ is a scalar under rotation, But this is not so under a boost, Λ, since S^{0i} is anti-Hermitean, so $\psi^\dagger\psi$ is not Lorentz invariant. This problem is solved by introducing $\overline{\psi} = \psi^\dagger\gamma^0$. From eqs. (15.13) and (15.15), we find that $\gamma^0 S^{ij} = S^{ij}\gamma^0$, but $\gamma^0 S^{0i} = -S^{0i}$. We then have:

$$\left(\omega_{\mu\nu}S^{\mu\nu}\gamma^0\right)^\dagger\gamma_0 = \left(\omega_{ij}S^{ij} - 2\omega_{0i}S^{0i}\right)\gamma^0 = \gamma^0\left(\omega_{ij}S^{ij} + 2\omega_{0i}S^{0i}\right) = \gamma_0(\omega_{\mu\nu}S^{\mu\nu})\,.$$

We can then commute γ^0 past each term in the series expansion for $D(\Lambda)$. Remembering the factor i in the exponent of $D(\Lambda)$ and that $\Lambda^{-1}(\omega^{\mu\nu}) = \Lambda(-\omega^{\mu\nu})$, we find:

$$\overline{\psi'}\psi' = \psi^\dagger D(\Lambda)^\dagger\gamma_0 D(\Lambda)\psi = \psi^\dagger\gamma^0 D(\Lambda^{-1})D(\Lambda)\psi = \overline{\psi}\psi\,. \tag{15.18}$$

We can then write down the free Lagrangian for the Dirac field:

$$\mathcal{L} = \overline{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi\,, \tag{15.19}$$

Variation with respect to $\overline{\psi}$ (or $\psi^\dagger$) immediately yields the Dirac equation, eq. (5.9), as the Euler–Lagrange equations. Variation with respect to ψ yields the Hermitean conjugate equation:

$$-i\partial_\mu\overline{\psi}\gamma^\mu - m\overline{\psi} = 0\,.$$